Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 397–428

ON EMBEDDINGS BETWEEN CLASSICAL LORENTZ SPACES

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(*communicated by B. Opic*)

Abstract. Let $p \in (0,\infty)$, let v be a weight on $(0,\infty)$ and let $\Lambda^p(v)$ be the classical Lorentz space, determined by the norm $\|f\|_{\Lambda^p(v)} := (\int_0^\infty (f^*(t))^p v(t)\,dt)^{1/p}$. When $p \in (1,\infty)$, this space is known to be a Banach space if and only if v is non-increasing, while it is only equivalent to a Banach space if and only if $\Lambda^p(v) = \Gamma^p(v)$, where $\|f\|_{\Gamma^p(v)} := (\int_0^\infty (f^{**}(t))^p v(t)\,dt)^{1/p}$. We may thus conclude that, for $p \in (1,\infty)$, the space $\Lambda^p(v)$ is equivalent to a Banach space if and only if the norm of a function f in it can be expressed in terms of f^{**}. We study the question whether an analogous assertion holds when $p = 1$. Motivated by this problem, we consider general embeddings between four types of classical and weak Lorentz spaces, namely, $\Lambda^p(v)$, $\Lambda^{p,\infty}(v)$, $\Gamma^p(v)$, $\Gamma^{p,\infty}(v)$, where $\Lambda^{p,\infty}(v)$ and $\Gamma^{p,\infty}(v)$ are certain weak analogues of the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$, respectively. We present a unified approach to these embeddings, based on rearrangement techniques. We survey all the known results and prove new ones. Our main results concern the embedding $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ which had not been characterized so far. We apply our results to the characterization of associate spaces of classical and weak Lorentz spaces and we give a characterization of fundamental functions for which the endpoint Lorentz space and the endpoint Marcinkiewicz space coincide.

1. Introduction

Let $(\mathcal{R},\mu)$ be a totally σ–finite measure space with a non-atomic measure μ, and let $\mathcal{M}(\mathcal{R},\mu)$ be the set of all extended complex–valued μ-measurable functions on $\mathcal{R}$. For $f \in \mathcal{M}(\mathcal{R},\mu)$, let $f_*(t) = \mu(\{x \in \mathcal{R}; |f(x)| > t\})$, $t > 0$, be the *distribution function* of f. The *non-increasing rearrangement* of f is defined by

$$f^*(t) = \inf\{s > 0; f_*(s) \leqslant t\}, \quad t \in [0,\mu(\mathcal{R})). \tag{1}$$

We further denote

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \quad t \in [0,\mu(\mathcal{R})).$$

Mathematics subject classification (2000): 46E30, 26D15.

Key words and phrases: Classical Lorentz spaces, continuous embeddings, weighted inequalities for non-increasing functions.

The research of M. Carro and J. Soria was partially supported by DGICYT PB97-0986 and CIRIT 1997SGR 00185. The research of L. Pick was partially supported by the Centre de Recerca Matematica at Barcelona, the research grant No. 201/97/0744 of the Grant Agency of the Czech Republic, and MSM 113200007. The research of V. D. Stepanov was partially supported by the Centre de Recerca Matematica at Barcelona, the INTAS project No. 94–881, and the Russian FFR Grant No. 97-01-00604. We wish to thank the referee and B. Opic for helpful comments.

Throughout the paper we assume that $\mu(\mathcal{R}) = \infty$.

When v is a non-negative measurable function on $(0,\infty)$, we say that v is a *weight*. We denote $V(t) = \int_0^t v(s)\,ds$.

DEFINITION 1.1. Let $p \in (0,\infty)$ and let v be a weight. We define four types of function spaces by

$$(2)\quad \begin{cases} \Lambda^p(v) &= \left\{ f \in \mathcal{M}(R,\mu);\, \|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p\, v(t)\, dt\right)^{1/p} < \infty \right\};\\ \Lambda^{p,\infty}(v) &= \left\{ f \in \mathcal{M}(R,\mu);\, \|f\|_{\Lambda^{p,\infty}(v)} := \sup\limits_{0<t<\infty} f^*(t) V^{1/p}(t) < \infty \right\};\\ \Gamma^p(v) &= \left\{ f \in \mathcal{M}(R,\mu);\, \|f\|_{\Gamma^p(v)} := \left(\int_0^\infty (f^{**}(t))^p\, v(t)\, dt\right)^{1/p} < \infty \right\};\\ \Gamma^{p,\infty}(v) &= \left\{ f \in \mathcal{M}(R,\mu);\, \|f\|_{\Gamma^{p,\infty}(v)} := \sup\limits_{0<t<\infty} f^{**}(t) V^{1/p}(t) < \infty \right\}. \end{cases}$$

The spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ are called *classical Lorentz spaces*. The spaces $\Lambda^{p,\infty}(v)$, $\Gamma^{p,\infty}(v)$ are called *weak Lorentz spaces*. Of course, $\|\cdot\|$ in (2) is not always a norm (consider, for example, the cases when $p \in (0,1)$), and, indeed, not even a quasinorm (cf. [8, Corollary 2.2]).

Perhaps the most familiar examples of classical and weak Lorentz spaces are the spaces $L_{p,q}$, defined as

$$L_{p,q} = \left\{ f \in \mathcal{M}(R,\mu);\, \|f\|_{p,q} := \|f^*(t) t^{\frac{1}{p}-\frac{1}{q}}\|_q < \infty \right\},$$

where $p, q \in (0,\infty]$ and $\|\cdot\|_q$ is the usual Lebesgue norm. Then $L_{p,q} = \Lambda^q(v)$ with $v(t) = t^{\frac{q}{p}} - 1$ when $q \in (0,\infty)$ and $L_{p,\infty} = \Lambda^{p,\infty}(v)$, where $v(t) = 1$.

By Fubini's theorem,

$$(3)\qquad \Gamma^1(v) = \Lambda^1(\tilde{v}), \quad \text{where } \tilde{v}(t) = \int_t^\infty s^{-1} v(s)\, ds.$$

Similarly, for $p \in (0,\infty)$,

$$\Lambda^{p,\infty}(v) = \Lambda^{1,\infty}(\tilde{v}) \quad \text{and} \quad \Gamma^{p,\infty}(v) = \Gamma^{1,\infty}(\tilde{v}), \quad \text{where } \tilde{v}(t) = \tfrac{1}{p} V(t)^{\frac{1}{p}-1} v(t).$$

Moreover, it is readily seen that, for every $p \in (0,\infty)$ and every weight v,

$$(4)\qquad \begin{cases} \Gamma^p(v) \hookrightarrow \Lambda^p(v),\\ \Lambda^p(v) \hookrightarrow \Lambda^{p,\infty}(v),\\ \Gamma^{p,\infty}(v) \hookrightarrow \Gamma^{p,\infty}(v). \end{cases}$$

(Here, as usual, $X \hookrightarrow Y$ stands for the continuous embedding, that is, $\|f\|_Y \leqslant C\|f\|_X$ for some $C > 0$ and every $f \in X$.)

The spaces $\Lambda^p(v)$ were introduced by Lorentz in 1951 in [17]. The spaces $\Gamma^p(v)$ were first used by Sawyer in [19]. The weak Lorentz spaces were introduced in [8] and further investigated in [9], [10] and [7].

Lorentz [17] proved that, for $p \geqslant 1$, $\|f\|_{\Lambda^p(v)}$ is a norm if and only if v is non-increasing. The class of weights for which $\|f\|_{\Lambda^p(v)}$ is merely *equivalent* to a norm, that is, $\Lambda^p(v)$ is equivalent to a Banach space, is however considerably wider. For $p \in (1,\infty)$, $\Lambda^p(v)$ is equivalent to a Banach space if and only if

$$t^p \int_t^\infty s^{-p} v(s)\, ds \leqslant C \int_0^t v(s)\, ds \tag{5}$$

for some C and all $t > 0$ ([19, Theorem 4]). When v satisfies (5), we say that $v \in B_p$. On the other hand, $\Lambda^1(v)$ is equivalent to a Banach space if and only if

$$\frac{1}{t}\int_0^t v(s)\, ds \leqslant \frac{C}{s}\int_0^s v(y)\, dy \qquad \text{for} \quad 0 < s \leqslant t \tag{6}$$

([7, Theorem 2.3]). When (6) is true, we say that $v \in B_{1,\infty}$. It is worth noticing that it is an essentially weaker requirement on v than $v \in B_1$. In this sense, (6) is not the limiting case of (5) when $p \to 1+$. On the other hand, Ariño and Muckenhoupt [1] established that, for $p \in (0,\infty)$, $v \in B_p$ if and only if $\Lambda^p(v) = \Gamma^p(v)$. Sawyer ([19, Theorem 4]) then showed that $v \in B_p$ if and only if

$$\left(\int_0^t v(s)\, ds\right)^{1/p} \left(\int_0^t \left(\frac{1}{s}\int_0^s v(y)\, dy\right)^{1-p'} ds\right)^{1/p'} \leqslant Ct \quad \text{for all} \quad t > 0,$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$. On integration by parts one can prove that this is equivalent to

$$\left(\int_0^t \left(\frac{1}{s}\int_0^s v(y)\, dy\right)^{-p'} v(s)\, ds\right)^{1/p'} \left(\int_0^t v(s)\, ds\right)^{1/p} \leqslant Ct \quad \text{for all} \quad t > 0. \tag{7}$$

But now, observe that (6) *is* the limiting case of (7) when $p \to 1_+$ (even though not of (5)).

When $(\mathcal{R},\mu)$ is further endowed with an appropriate metric structure (so that the notion of a ball makes sense), the embeddings between classical and weak Lorentz spaces can be expressed in terms of boundedness of the Hardy–Littlewood maximal operator. Let us consider the case when $\mathcal{R} = \mathbb{R}^n$ (with some $n \in \mathbb{N}$) and μ is the Lebesgue measure. Then the *Hardy–Littlewood maximal operator* M is defined for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(x)|\, d\mu(x), \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all balls B containing x, and it is a well–known fact (see for example [4, Chapter 3, Theorem 3.8]), that there are constants $c, C > 0$ such that

$$cf^{**}(t) \leqslant (Mf)^*(t) \leqslant Cf^{**}(t) \quad \text{for all } f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } t \in (0,\infty).$$

Using this inequality, we see at once that the embedding $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$ is equivalent to the boundedness of M on $\Lambda^p(v)$. Similarly, the embedding $\Lambda^p(v) \hookrightarrow \Gamma^{p,\infty}(v)$ is equivalent to the "weak boundedness" of M on $\Lambda^p(v)$, that is, to the boundedness of M from $\Lambda^p(v)$ into $\Lambda^{p,\infty}(v)$.

The above-mentioned result of [1] states that $v \in B_p$ if and only if M is bounded on $\Lambda^p(v)$ whenever $p \in (1,\infty)$. However, it turns out ([9, Theorem 3.3], [10, Theorem 3.9]) that, for $p \in (1,\infty)$, $v \in B_p$ is also equivalent to the seemingly weaker statement $M : \Lambda^p(v) \to \Lambda^{p,\infty}(v)$. As for the case when $p = 1$, it was proved in [7] that (6) holds if and only if $M : \Lambda^1(v) \to \Lambda^{1,\infty}(v)$. From this point of view, the picture is complete and consistent; we may conclude that $\Lambda^p(v)$ is equivalent to a Banach space if and only if $p \in [1,\infty)$ and M is bounded from $\Lambda^p(v)$ into $\Lambda^{p,\infty}(v)$ (or, in other words, if and only if $\Lambda^p(v)$ is continuously embedded into $\Gamma^{p,\infty}(v)$). Now, for $p \in (1,\infty)$, this is further equivalent to the embedding $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$, hence (since $f^* \leqslant f^{**}$) to the identity $\Lambda^p(v) = \Gamma^p(v)$. Consequently, for $1 < p < \infty$, $\Lambda^p(v)$ is equivalent to a Banach space if and only if the norm in $\Lambda^p(v)$ can be equivalently rephrased in terms of f^{**}. These considerations however cannot be extended to the case $p = 1$ and we are left with this case open.

It is our aim to investigate when the norm on $\Lambda^1(v)$ can be expressed in terms of f^{**}, in particular whether this is true whenever $\Lambda^1(v)$ is equivalent to a Banach space (or, which is the same, whenever $\Lambda^1(v) \hookrightarrow \Gamma^{1,\infty}(v)$). It is evident that this is so for example when either $\Lambda^1(v) = \Gamma^1(v)$ or when $\Lambda^1(v) = \Gamma^{1,\infty}(v)$. The former identity however requires a very strong restriction on v, namely $v \in B_1$, while the latter has not been characterized yet.

Motivated by this problem, we first consider general embeddings among all possible pairs of the spaces given by (2) with possibly different exponents and weights. For the sake of completeness we give a summary of all the known results, and we prove new ones. We present a unified approach to the characterization of embeddings, based on rearrangement techniques. Our main contribution is a variety of new results concerning the embeddings of type $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$. As a special case we give a characterization of weights v for which $\Gamma^{1,\infty}(v) \hookrightarrow \Lambda^1(v)$ and in turn of those for which $\Lambda^1(v) = \Gamma^{1,\infty}(v)$, solving the above-mentioned problem. As a corollary we obtain a result of independent interest: when φ is a fundamental function of an r.i. space X and the endpoint Lorentz space Λ_φ coincides with the endpoint Marcinkiewicz space M_φ (see the definitions in Section 8), then necessarily X is equivalent to one of the spaces L^1, L^∞, $L^1 \cap L^\infty$ or $L^1 + L^\infty$.

The Lorentz spaces with power and logarithmic weights have been widely used in many areas of analysis, especially in the theory of function spaces and interpolation (see [4], [5], [3]). However, the theory of classical and weak Lorentz spaces is so far incomplete even in the sense of characterization of all embeddings between them. The first result on $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$, $p \in (1,\infty)$, due to Boyd [6], had an implicit form. An explicit characterization was later obtained in [1] and [19]. These fundamental papers opened an extensive research. Further results on embeddings and mapping properties of linear operators on classical and weak Lorentz spaces can be found in the recent papers [2], [7], [8], [9], [10], [11], [12], [13], [15], [22], [23], [24], [25].

The paper is organized as follows: in Section 2 we present some useful general observations on rearrangement-invariant lattices and Halperin's level functions. In Sections 3, 4, 5 and 6 we summarize criteria for embeddings between classical and weak Lorentz spaces of all types. Most of these results are known. It is one of our goals to provide the reader with a useful comprehensive reference tool. In Section 7 we state and prove our main results concerning the embeddings $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$. In Section 8 we use the embedding results to clarify the relations between $\Lambda^1(v)$ and $\Gamma^{1,\infty}(v)$. In Section 9 we characterize the associate spaces of classical and weak Lorentz spaces.

Everywhere below, v and w are weights. We shall throughout denote $V(t) = \int_0^t v(s)\,ds$ and $W(t) = \int_0^t w(s)\,ds$ for $t > 0$. By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$. We shall use the symbol $f\downarrow$ to denote that f is non-increasing on $(0,\infty)$. Embedding constants and other important quantities are denoted as $A_{(12)}$, $A_{(13)}, \dots$, where the subscript indicates the label of the formula where they are introduced.

2. Rearrangement-invariant lattices

DEFINITION 2.1. Let X be a linear set of functions from $\mathcal{M}(\mathcal{R},\mu)$ containing characteristic functions of sets of finite measure, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f \in \mathcal{M}(\mathcal{R},\mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$. If $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$, and moreover $0 \leqslant f \leqslant g$ implies $\|f\|_X \leqslant \|g\|_X$, then we say that X is a *rearrangement-invariant (r.i.) lattice*. We further define

$$X' = \left\{ f \in \mathcal{M}(\mathcal{R},\mu);\ \int_{\mathcal{R}} |f(x)g(x)|\,d\mu(x) < \infty \text{ for every } g \in X \right\},$$

and

$$\|f\|_{X'} = \sup\left\{ \int_{\mathcal{R}} |f(x)g(x)|\,d\mu(x);\ \|g\|_X \leqslant 1 \right\}.$$

If, in particular, X is a *Banach function space*, (cf. [4, Chapter 1]), then X' is its *associate space*.

DEFINITION 2.2. Let X be an r.i. lattice.

(i) For each finite $t \in [0,\mu(\mathcal{R})]$ let E be any subset of $\mathcal{R}$ such that $\mu(E) = t$, and $\varphi_X(t) = \|\chi_E\|_X$. The function φ_X so defined is called the *fundamental function* of X. Since μ is non-atomic, for every finite $t \in (0,\mu(\mathcal{R})]$ there is a μ–measurable subset E of $\mathcal{R}$ such that $\mu(E) = t$, and therefore $(\chi_E)^* = \chi_{(0,t)}$.

(ii) Set

$$\psi_X(t) = \sup\left\{ \int_0^t f^*(s)\,ds; \|f\|_X \leqslant 1 \right\}, \quad t \in (0,\infty), \tag{8}$$

and

$$\varrho_X(t) := \sup_{f \not\equiv 0} \frac{f^*(t)}{\|f\|_X}, \quad t \in (0,\infty).$$

REMARKS 2.3. (i) If X is a Banach function space, then $\psi_X = \varphi_{X'}$ and X' is its associate space.

(ii) For any r.i. lattice X and every $t \in (0,\infty)$,

$$\varrho_X(t) = \frac{1}{\varphi_X(t)}.$$

Indeed, let $t > 0$ and let $E \subset \mathcal{R}$ be such that $\mu(E) = t$. Testing the supremum in the definition of ϱ_X on $f = \chi_E$, we get $\varrho_X(t) \geqslant \frac{1}{\varphi_X(t)}$. Conversely, let $E_t := \{|f| \geqslant f^*(t)\}$. Then $\mu(E_t) \geqslant t$, and therefore $\|f\|_X \geqslant \|f\chi_{E_t}\|_X \geqslant f^*(t)\|\chi_{E_t}\|_X \geqslant f^*(t)\varphi_X(t)$. Hence, $\varrho_X(t) \leqslant \frac{1}{\varphi_X(t)}$.

For "bad enough" weight v, the classical or weak Lorentz spaces may be trivial, that is, equal to $\{0\}$. If such a space is not trivial, then it is an r.i. lattice. Our next lemma gives necessary and sufficient conditions for classical and weak Lorentz spaces to be non-trivial and formulas for fundamental functions in these cases.

LEMMA 2.4. *Let $p \in (0,\infty)$ and let v be a weight.*

(i) *The spaces $\Lambda^p(v)$ and $\Lambda^{p,\infty}(v)$ are non-trivial if and only if v is integrable near zero. In such case,*

$$\varphi_{\Lambda^p(v)}(t) = \varphi_{\Lambda^{p,\infty}(v)}(t) = V^{1/p}(t), \quad t \in (0,\infty). \tag{9}$$

(ii) *The space $\Gamma^p(v)$ is non-trivial if and only if v is integrable near zero and $\int_t^\infty s^{-p}v(s)\,ds < \infty$ for every $t > 0$. In such case,*

$$\varphi_{\Gamma^p(v)}(t) = \left(V(t) + t^p \int_t^\infty \frac{v(s)}{s^p}\,ds\right)^{1/p}, \quad t \in (0,\infty). \tag{10}$$

(iii) *The space $\Gamma^{p,\infty}(v)$ is non-trivial if and only if v is integrable on every interval $(0,a)$, $a > 0$, and $\limsup_{t\to\infty} t^{-1}V^{1/p}(t) < \infty$. In such case,*

$$\varphi_{\Gamma^{p,\infty}(v)}(t) = t \sup_{s \geqslant t} \frac{V^{1/p}(s)}{s}, \quad t \in (0,\infty). \tag{11}$$

Proof. All the assertions follow easily from definitions and (4). □

EXAMPLE 2.5. If $p \in (0,\infty)$ and $v(t) = t^{p-1}$, then $\Gamma^p(v)$ is trivial but $\Gamma^{p,\infty}(v)$, $\Lambda^p(v)$, and $\Lambda^{p,\infty}(v)$ are non-trivial. More precisely,

$$\Gamma^p(v) = \{0\}, \quad \Gamma^{p,\infty}(v) = L_1, \quad \Lambda^p(v) = L_{1,p} \quad \text{and} \quad \Lambda^{p,\infty}(v) = L_{1,\infty}.$$

This example also shows that there is no general inclusion between the spaces $\Lambda^p(v)$ and $\Gamma^{p,\infty}(v)$ (here $\Lambda^p(v) \hookrightarrow \Gamma^{p,\infty}(v)$ when $0 < p \leqslant 1$, but $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^p(v)$ when $1 < p < \infty$, and both these embeddings are proper in general).

Some embeddings can be characterized in quite a general setting by means of the functions ψ and ϱ.

THEOREM 2.6. *Let* X *be a rearrangement-invariant lattice,* $p,q \in (0,\infty)$ *and let* v,w *be weights.*

(i) *The embedding* $\Lambda^{p,\infty}(v) \hookrightarrow X$ *holds if and only if*

$$A_{(12)} := \|V^{-1/p}\|_X < \infty, \tag{12}$$

and the optimal constant C *of the embedding equals* $A_{(12)}$.

(ii) *The embedding* $X \hookrightarrow \Gamma^{q,\infty}(w)$ *holds if and only if*

$$A_{(13)} := \sup_{t>0} \frac{W^{1/q}(t)\psi_X(t)}{t} < \infty, \tag{13}$$

and the optimal constant C *of the embedding equals* $A_{(13)}$.

(iii) *The embedding* $X \hookrightarrow \Lambda^{q,\infty}(w)$ *holds if and only if*

$$A_{(14)} := \sup_{t>0} W^{1/q}(t)\varrho_X(t) = \sup_{t>0} \frac{W^{1/q}(t)}{\varphi_X(t)} < \infty, \tag{14}$$

and the optimal constant C *of the embedding equals* $A_{(14)}$.

Proof. (i) This is a particular case of a more general result in [22, Proposition 2.7].

(ii) Changing the suprema, we get

$$C = \sup_{\|f\|_X \leqslant 1} \sup_{t>0} f^{**}(t)W^{1/q}(t) = \sup_{t>0} \frac{W^{1/q}(t)}{t} \sup_{\|f\|_X \leqslant 1} \int_0^t f^*(s)\,ds = A_{(13)}.$$

(iii) Again, changing the suprema and using Remark 2.3 (ii), we obtain

$$C = \sup_{f \not\equiv 0} \sup_{t>0} \frac{f^*(t)W^{1/q}(t)}{\|f\|_X} = \sup_{t>0} W^{1/q}(t)\varrho_X(t) = A_{(14)}.$$

The proof is complete. □

COROLLARY 2.7. *If* X *is a rearrangement-invariant Banach function space,* $q \in (0,\infty)$ *and* w *is a weight, then*

$$X \hookrightarrow \Gamma^{q,\infty}(w) \qquad \text{if and only if} \qquad X \hookrightarrow \Lambda^{q,\infty}(w). \tag{15}$$

Proof. The “only if” part is a direct consequence of (4). The “if” part follows from Theorem 2.6 (ii) and (iii) combined with the relation

$$\frac{\psi_X(t)}{t} = \varrho_X(t), \tag{16}$$

which is satisfied whenever X is a Banach function space (cf. [4, Chapter 2, Theorem 5.2]). □

REMARKS 2.8. (i) In general, (16) and (15) are false. For example, take $p \in (0,1)$, $q \in (0,\infty)$, $w(t) = t^{\frac{q}{p}-1}$, and $X = L_p$. Then $\psi_X(t) = \infty$ for every $t > 0$, thus $A_{(13)} = \infty$, but $A_{(14)} < \infty$.

(ii) If $\Lambda^{q,\infty}(w)$ is a Banach function space, then ([22, Theorem 3.1]), $\Gamma^{q,\infty}(w) = \Lambda^{q,\infty}(w)$. In particular, in such case (15) is true.

(iii) Let X be an r.i. lattice. There is a close connection between $\|g\|_{X'}$ and the optimal constant C of the embedding $X \hookrightarrow \Lambda^1(g^*)$. Indeed, (cf. [4, Chapter 2, Theorem 2.7])

$$(17) \qquad \|g\|_{X'} = \sup_{f\not\equiv 0} \frac{\int_{\mathcal{R}} |f(x)g(x)|\, d\mu(x)}{\|f\|_X} = \sup_{f\not\equiv 0} \frac{\int_0^\infty f^*(t)g^*(t)\, dt}{\|f\|_X} = C.$$

We shall find it useful to write the norm in $\Gamma^{p,\infty}(v)$ in a slightly modified way. To this end we shall need the following definition (cf. [4, Chapter 2, Definition 5.6]).

DEFINITION 2.9. Let φ be a non-negative non-decreasing function on $[0,\infty)$ such that $\varphi(t) = 0$ if and only if $t = 0$, and $\varphi(t)/t$ is non-increasing on $(0,\infty)$. Then φ is said to be *quasiconcave*. We shall denote by Ω the cone of all quasiconcave functions, and $\Omega_0 = \{\varphi \in \Omega;\ \lim_{x\to 0_+} \varphi(x) = 0\}$.

DEFINITION 2.10. Let $p \in (0,\infty)$ and let v be a weight. We denote

$$(18) \qquad \mathcal{V}_p(t) = \sup_{s\geqslant t} \frac{V^{1/p}(s)}{s}, \qquad t > 0.$$

REMARKS 2.11. (i) Let us recall ([5, Lemma 5.4.3]) that $\varphi \in \Omega$ if and only if there is a constant $\lambda \geqslant 0$ and a non-increasing function ψ such that $\varphi(t) \leqslant \lambda + \int_0^t \psi(s)\, ds \leqslant 2\varphi(t)$ for $t > 0$. Similarly, $\varphi \in \Omega_0$ if and only if there is a non-increasing function ψ such that $\varphi(t) \leqslant \int_0^t \psi(s)\, ds \leqslant 2\varphi(t)$.

(ii) If $\mathcal{V}_p(t) < \infty$ for every $t > 0$, then $\frac{1}{\mathcal{V}_p} \in \Omega$. Moreover,

$$t\mathcal{V}_p(t) = \sup_{s>0} \min\{1, \tfrac{t}{s}\} V^{1/p}(s),$$

hence $t\mathcal{V}_p(t)$ is the least quasiconcave majorant of $V^{1/p}(t)$. Therefore, $t\mathcal{V}_p(t) \in \Omega_0$.

(iii) It follows readily from (18) that there exists a decomposition $(0,\infty) = E \cup \bigcup_{j=1}^\infty (a_j, b_j)$, such that

$$(19) \qquad \mathcal{V}_p(t) = \frac{V^{1/p}(t)}{t}\chi_E(t) + \sum_{j=1}^\infty c_j \chi_{(a_j,b_j)}(t), \qquad c_j = \frac{V^{1/p}(b_j)}{b_j}.$$

As long as $a_j \neq 0$, we also have

$$c_j = \frac{V^{1/p}(a_j)}{a_j}.$$

(iv) The introduction of $\mathcal{V}_p(t)$ allows an alternative version of $\|f\|_{\Gamma^{p,\infty}(v)}$, namely,

$$\|f\|_{\Gamma^{p,\infty}(v)} = \sup_{t>0} \left(\int_0^t f^*(s)\,ds \right) \mathcal{V}_p(t) = \sup_{t>0} t\mathcal{V}_p(t) f^{**}(t). \tag{20}$$

Indeed, we have

$$\begin{aligned} \|f\|_{\Gamma^{p,\infty}(v)} &\leqslant \sup_t \left(\int_0^t f^*(s)\,ds \right) \sup_{s \geqslant t} \frac{V^{1/p}(s)}{s} \\ &= \sup_s \frac{V^{1/p}(s)}{s} \sup_{t \leqslant s} \int_0^t f^*(s)\,ds = \|f\|_{\Gamma^{p,\infty}(v)}. \end{aligned}$$

(v) It follows from [4, Chapter 2, Proposition 5.10] that there exist a $\lambda \geqslant 0$ and a non-increasing function v_p such that

$$\frac{1}{2} \left(\lambda + \int_0^t v_p(s)\,ds \right) \leqslant \frac{1}{\sup_{s \geqslant t} s^{-1} V^{1/p}(s)} \leqslant \lambda + \int_0^t v_p(s)\,ds.$$

Consequently,

$$\sup_{t>0} \frac{\int_0^t f^*(s)\,ds}{\lambda + \int_0^t v_p(s)\,ds} \leqslant \|f\|_{\Gamma^{p,\infty}(v)} \leqslant 2 \sup_{t>0} \frac{\int_0^t f^*(s)\,ds}{\lambda + \int_0^t v_p(s)\,ds}. \tag{21}$$

Let us now recall the notion of Halperin's level function, introduced in [14, Definition 3.2]. We shall follow a slightly modified approach of [20, Section 4]. Let w be a weight on $(0,\infty)$, satisfying $\limsup_{t\to\infty} \frac{W(t)}{t} < \infty$. Then there exists a function w°, called the *level function* of w, and a partition of $\mathbb{R}^+ = E \cup F$, $E \cap F = \emptyset$, where $F = \bigcup_{j=1}^\infty I_j \cup [a,\infty)$, I_j are disjoint intervals of finite measure and $a = \sup \left\{ s; s \in E \cup \left(\bigcup_{j=1}^\infty I_j \right) \right\}$ (with $a = \infty$ as a possibility), such that w° is decreasing, and

$$w^\circ(x) = \begin{cases} w(x), & x \in E; \\ \frac{1}{|I_j|} \int_{I_j} w(x)\,dx, & x \in I_j,\ j \in \mathbb{N}; \\ \limsup\limits_{t\to\infty} \frac{W(t)}{t} =: w^\circ(\infty), & x \in (a,\infty) \end{cases}$$

(thus, in particular, w° is decreasing on E). Moreover,

$$\int_0^x w(t)\,dt \leqslant \int_0^x w^\circ(t)\,dt,\ x > 0, \qquad \text{(with equality for } x \in E\text{)}, \tag{22}$$

and

$$\lim_{x\to\infty} \frac{\int_0^x w(t)\,dt}{\int_0^x w^\circ(t)\,dt} = 1.$$

It was shown in [20, Section 4] that w° coincides almost everywhere with the derivative of the least concave majorant of W.

The following observation will be useful in the sequel.

PROPOSITION 2.12. *Let $q \in (0,1]$. Let $\mathcal{N}$ be a functional defined on locally integrable functions on $(0,\infty)$, with the property*

$$\int_0^t f(s)\,ds \leqslant \int_0^t g(s)\,ds, \quad f,g\downarrow,\ t>0 \qquad \Rightarrow \qquad \mathcal{N}(f) \leqslant \mathcal{N}(g).$$

For a weight w on $(0,\infty)$, satisfying $\limsup_{t\to\infty}\frac{W(t)}{t}<\infty$, define

$$\beta(w) := \sup_{f\downarrow} \frac{\|f\|_{\Lambda^q(w)}}{\mathcal{N}(f)}.$$

Then

$$\beta(w) = \beta(w^\circ),$$

where w° is the level function of w.

Proof. First, $\beta(w) \leqslant \beta(w^\circ)$ follows from (22) and the Hardy's lemma (cf. [4, Chapter 2, Proposition 3.6]). As for the converse inequality, let $\mathbb{R}^+ = E \cup \bigcup_j I_j \cup [a,\infty)$ be the partition of $(0,\infty)$ from the definition of w°. We shall assume that $a<\infty$. The proof needs only trivial modifications when $a=\infty$. Let f be a non-increasing function on $(0,\infty)$ and let $t>a$. Then we define

$$f_t(x) = \begin{cases} f(x), & x \in E; \\ \frac{1}{|I_j|}\int_{I_j} f(t)\,dt, & x \in I_j,\ j \in \mathbb{N}; \\ \frac{1}{t-a}\int_a^t f(s)\,ds, & x \in [a,t); \\ 0, & x \in [t,\infty). \end{cases} \tag{23}$$

It is easy to see that f_t is non-increasing for any $t>a$, and that

$$\int_0^s f_t(y)\,dy \leqslant \int_0^s f(y)\,dy, \qquad s>0. \tag{24}$$

It follows immediately from (24) that for every $t>0$

$$\mathcal{N}(f_t) \leqslant \mathcal{N}(f). \tag{25}$$

We now claim that

$$\int_0^\infty f^q(y) w^\circ(y)\,dy \leqslant \sup_{t>a} \int_0^\infty f_t^q(y) w(y)\,dy. \tag{26}$$

To prove (26), note first that

$$\limsup_{s\to\infty} \frac{W(s)}{s} = \limsup_{s\to\infty} \frac{1}{s-a}\int_a^s w(y)\,dy,$$

whence, by Hölder's inequality (recall $q \leqslant 1$),

$$\begin{aligned}
(27) \qquad \int_a^\infty f^q(y)w^\circ(y)\,dy &= \limsup_{t\to\infty} \frac{W(t)}{t}\left(\int_a^\infty f^q(t)\,dt\right)\\
&= \limsup_{t\to\infty}\left(\frac{1}{t-a}\int_a^t f^q(s)\,ds\right)\left(\int_a^t w(s)\,ds\right)\\
&\leqslant \sup_{t>a}\left(\frac{1}{t-a}\int_a^t f(s)\,ds\right)^q\left(\int_a^t w(s)\,ds\right)\\
&= \sup_{t>a}\int_a^\infty f_t^q(s)w(s)\,ds.
\end{aligned}$$

Now, by the definition of w°, (23), and (27),

$$\int_0^\infty f^q(s)w^\circ(s)\,ds = \int_E f^q(s)w(s)\,ds + \sum_j \frac{1}{|I_j|}\int_{I_j} f^q(s)\,ds\int_{I_j} w(s)\,ds$$
$$+\int_a^\infty f^q(s)w^\circ(s)\,ds \leqslant \sup_{t>a}\int_0^\infty f_t^q(s)w(s)\,ds,$$

and (26) follows. Finally, combining (26), (25), and the fact that f_t is non-increasing, we get

$$\begin{aligned}
\beta(w^\circ) &\leqslant \sup_{f\downarrow}\sup_{t>a}\frac{\left(\int_0^\infty (f_t)^q(s)w(s)\,ds\right)^{1/q}}{\mathcal{N}(f)} \leqslant \sup_{f\downarrow}\sup_{t>a}\frac{\left(\int_0^\infty (f_t)^q(s)w(s)\,ds\right)^{1/q}}{\mathcal{N}(f_t)}\\
&\leqslant \sup_{g\downarrow}\frac{\left(\int_0^\infty g^q(s)w(s)\,ds\right)^{1/q}}{\mathcal{N}(g)} = \beta(w),
\end{aligned}$$

as desired. □

EXAMPLE 2.13. Setting, for $p \in (0,\infty)$, either

$$\mathcal{N}(f) = \left(\int_0^\infty\left(\frac{1}{t}\int_0^t f(s)\,ds\right)^p v(t)\,dt\right)^{1/p}$$

or

$$\mathcal{N}(f) = \sup_{0<t<\infty}\left(\frac{1}{t}\int_0^t f(s)\,ds\right)V^{1/p}(t),$$

we obtain from Proposition 2.12 that, for $q \in (0,1)$, $\Gamma^p(v) \hookrightarrow \Lambda^q(w)$ is equivalent to $\Gamma^p(v) \hookrightarrow \Lambda^q(w^\circ)$, and the same for $\Gamma^p(v)$ replaced by $\Gamma^{p,\infty}(v)$. Moreover, the corresponding optimal embedding constants coincide.

3. Embeddings of type $\Lambda \hookrightarrow \Lambda$

THEOREM 3.1. (the case $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$) *Let $p, q \in (0,\infty)$ and let v, w be weights.*

(i) *Let* $0 < p \leqslant q < \infty$. *Then the inequality*

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant C\left(\int_0^\infty (f^*(t))^p\, v(t)\, dt\right)^{1/p}, \tag{28}$$

holds if and only if

$$A_{(29)} := \sup_{t>0} W^{1/q}(t) V^{-1/p}(t) < \infty, \tag{29}$$

and the optimal constant C *in* (28) *equals* $A_{(29)}$.

(ii) *Let* $0 < q < p < \infty$ *and let* r *be given by* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Then* (28) *holds if and only if*

$$\begin{aligned} A_{(30)} :&= \left(\int_0^\infty \left(\frac{W(t)}{V(t)}\right)^{r/p} w(t)\, dt\right)^{1/r} \\ &= \left(\frac{q}{r}\frac{W^{r/q}(\infty)}{V^{r/p}(\infty)} + \frac{q}{p}\int_0^\infty \left(\frac{W(t)}{V(t)}\right)^{r/q} v(t)\, dt\right)^{1/r} < \infty, \end{aligned} \tag{30}$$

and $C \approx A_{(30)}$.

These results can be found in [19, Remark (i), p. 148] for $1 < p,\ q < \infty$, and in [23, Proposition 1] for all values $0 < p,\ q < \infty$. Part (i) also follows from a more general result in [8, Corollary 2.7].

THEOREM 3.2. (the case $\Lambda^p(v) \hookrightarrow \Lambda^{q,\infty}(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights. The inequality*

$$\sup_{t>0} f^*(t) W^{1/q}(t) \leqslant C\left(\int_0^\infty (f^*(t))^p\, v(t)\, dt\right)^{1/p}, \tag{31}$$

holds if and only if $A_{(29)} < \infty$, *and moreover the optimal constant* C *in* (31) *equals* $A_{(29)}$.

Proof. This follows from Theorem 2.6 (iii) with $X = \Lambda^p(v)$ combined with (9). $\square$

THEOREM 3.3. (the case $\Lambda^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights. The inequality*

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant C \sup_{t>0} f^*(t) V^{1/p}(t) \tag{32}$$

holds if and only if

$$A_{(33)} := \left(\int_0^\infty V^{-q/p}(t) w(t)\, dt\right)^{1/q} < \infty, \tag{33}$$

and the optimal constant C *in* (32) *equals* $A_{(33)}$.

Proof. This follows from Theorem 2.6 (i) with $X = \Lambda^q(w)$. A direct proof is an easy exercise. $\square$

THEOREM 3.4. (the case $\Lambda^{p,\infty}(v) \hookrightarrow \Lambda^{q,\infty}(w)$) *Let $p,q \in (0,\infty)$ and let v,w be weights. Then the inequality*

$$\sup_{t>0} f^*(t)W^{1/q}(t) \leqslant C \sup_{t>0} f^*(t)V^{1/p}(t) \tag{34}$$

holds if and only if $A_{(29)} < \infty$, and moreover the optimal constant C in (34) equals $A_{(29)}$.

Proof. The assertion immediately follows from Theorem 2.6 (i) or (iii) and (9). □

4. Embeddings of type $\Lambda \hookrightarrow \Gamma$

THEOREM 4.1. (the case $\Lambda^p(v) \hookrightarrow \Gamma^q(w)$) *Let $p,q \in (0,\infty)$ and let v,w be weights.*

(i) *Let $1 < p \leqslant q < \infty$ Then the inequality*

$$\left(\int_0^\infty (f^{**}(t))^q w(t)\,dt\right)^{1/q} \leqslant C\left(\int_0^\infty (f^*(t))^p v(t)\,dt\right)^{1/p} \tag{35}$$

holds if and only if $A_{(29)} < \infty$ and

$$A_{(36)} := \sup_{t>0}\left(\int_t^\infty \frac{w(s)}{s^q}\,ds\right)^{1/q}\left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}\,ds\right)^{1/p'} < \infty, \tag{36}$$

and the optimal constant C in (35) satisfies $C \approx A_{(29)} + A_{(36)}$.

(ii) *Let $0 < p \leqslant 1$, $0 < p \leqslant q < \infty$. Then (35) holds if and only if $A_{(29)} < \infty$ and*

$$A_{(37)} := \sup_{t>0} t\left(\int_t^\infty \frac{w(s)}{s^q}\,ds\right)^{1/q} V^{-1/p}(t) < \infty, \tag{37}$$

and $C \approx A_{(29)} + A_{(37)}$.

(iii) *Let $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$. Then (35) holds if and only if $A_{(30)} < \infty$ and, for r given by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$,*

(38)

$$A_{(38)} := \left(\int_0^\infty \left[\left(\int_t^\infty \frac{w(s)}{s^q}\,ds\right)^{1/q}\left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}\,ds\right)^{(q-1)/q}\right]^r \frac{v(t)t^{p'}}{V^{p'}(t)}\,dt\right)^{1/r}$$
$$\approx \left(\int_0^\infty \left[\left(\int_t^\infty \frac{w(s)}{s^q}\,ds\right)^{1/p}\left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}\,ds\right)^{1/p'}\right]^r \frac{w(t)}{t^q}\,dt\right)^{1/r} < \infty,$$

and $C \approx A_{(30)} + A_{(38)}$.

(iv) *Let* $1 = q < p < \infty$. *Then* (35) *holds if and only if* $A_{(30)} < \infty$ *and*

(39)
$$A_{(39)} := \left(\int_0^\infty \left(\frac{W(t) + t\int_t^\infty \frac{w(s)}{s}\,ds}{V(t)} \right)^{p'-1} \int_t^\infty \frac{w(s)}{s}\,ds\,dt \right)^{1/p'}$$
$$\approx \frac{W(\infty)}{V^{1/p}(\infty)} + \left(\int_0^\infty \left(\frac{W(t) + t\int_t^\infty \frac{w(s)}{s}\,ds}{V(t)} \right)^{p'} v(t)\,dt \right)^{1/p'} < \infty,$$

and $C \approx A_{(30)} + A_{(39)}$.

(v) *Let* $0 < q < p = 1$. *Then* (35) *holds if and only if* $A(5) < \infty$ *and* $A_{(40)} < \infty$, *where*

(40)
$$A_{(40)} := \left(\int_0^\infty \left(\int_t^\infty \frac{w(s)}{s^q}\,ds \right)^{q/(1-q)} \left(\operatorname*{ess\,inf}_{0<s<t} \frac{V(s)}{s} \right)^{q/(q-1)} \frac{w(t)}{t^q}\,dt \right)^{(1-q)/q},$$

and $C \approx A_{(30)} + A_{(40)}$.

(vi) *Let* $0 < q < p < 1$. *If* $A_{(30)} < \infty$ *and*

(41)
$$A_{(41)} := \left(\int_0^\infty \left(\int_t^\infty \frac{w(s)}{s^q}\,ds \right)^{r/q} V^{-r/p}(t) t^{r-1}\,dt \right)^{1/r} < \infty,$$

then (35) holds. Moreover, $A_{(30)} \lesssim C \lesssim A_{(30)} + A_{(41)}$.

Case (i) is shown in [19, Theorem 2], case (ii) independently in [9, Proposition 2.6b] and [23, Theorem 3b] (for a particular case see also [16, Theorem 2.2]), case (iii) in [19, Theorem 2] for $1 < q < p < \infty$ and in [23, Theorem 3a] for $0 < q < 1 < p < \infty$, case (iv) follows from (3) and Theorem 3.1, case (v) can be found in [21, Theorem 4.1], and finally case (vi) is treated in [23, Proposition 2], see also [12, Theorem 6].

THEOREM 4.2. (the case $\Lambda^p(v) \hookrightarrow \Gamma^{q,\infty}(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights.*

(i) *Let* $0 < p \leqslant 1$. *The inequality*

(42)
$$\sup_{t>0} f^{**}(t) W^{1/q}(t) \leqslant C \left(\int_0^\infty (f^*(t))^p\, v(t)\,dt \right)^{1/p}$$

holds if and only if

(43)
$$A_{(43)} := \sup_{0<s\leqslant t<\infty} \frac{W^{1/q}(t)s}{tV^{1/p}(s)} < \infty,$$

and the optimal constant C *in* (42) *equals* $A_{(43)}$.

(ii) *Let* $1<p<\infty$. *Then* (42) *holds if and only if*

$$A_{(44)} := \sup_{t>0} \frac{W^{1/q}(t)}{t} \left(\int_0^t \left(\frac{s}{V(s)} \right)^{p'-1} ds \right)^{1/p'} < \infty, \tag{44}$$

and $C \approx A_{(44)}$.

Proof. Both assertions follow from (a slightly modified version of) [9, Theorem 3.3] with $k(x,t) = \frac{1}{x}\chi_{(0,x)}(t)$. $\square$

REMARKS 4.3. (i) By Theorem 3.1,

$$\psi_{\Lambda^p(v)}(t) = \sup_{f\not\equiv 0} \frac{\int_0^\infty f^*(s)\chi_{(0,t)}(s)\,ds}{\|f\|_{\Lambda^p(v)}} \begin{cases} = \sup\limits_{0<s\leqslant t} \frac{s}{V^{1/p}(s)} & \text{if } 0<p\leqslant 1; \\ \approx \left(\int_0^t \left(\frac{s}{V(s)} \right)^{p'-1} ds \right)^{1/p'} & \text{if } 1<p<\infty. \end{cases}$$

Applying Theorem 2.6 (ii), we get a new proof of Theorem 4.2.

THEOREM 4.4. (the case $\Lambda^{p,\infty}(v) \hookrightarrow \Gamma^q(w)$) *Let* $p,q \in (0,\infty)$ *and let* v,w *be weights. The inequality*

$$\left(\int_0^\infty (f^{**}(t))^q\, w(t)\,dt \right)^{1/q} \leqslant C \sup_{t>0} f^*(t) V^{1/p}(t) \tag{45}$$

holds if and only if

$$A_{(46)} := \left(\int_0^\infty \left(\frac{1}{t} \int_0^t V^{-1/p}(s)\,ds \right)^q w(t)\,dt \right)^{1/q} < \infty, \tag{46}$$

and moreover the optimal constant C *in* (45) *equals* $A_{(46)}$.

This result can be found in [22, Theorem 4.1 (i)] (cf. also Theorem 2.6 (i) with $X = \Gamma^q(w)$).

THEOREM 4.5. (the case $\Lambda^{p,\infty}(v) \hookrightarrow \Gamma^{q,\infty}(w)$) *Let* $p,q \in (0,\infty)$ *and let* v,w *be weights. The inequality*

$$\sup_{t>0} f^{**}(t) W^{1/q}(t) \leqslant c \sup_{t>0} f^*(t) V^{1/p}(t) \tag{47}$$

holds if and only if

$$A_{(48)} := \sup_{t>0} \frac{W^{1/q}(t)}{t} \int_0^t V^{-1/p}(s)\,ds < \infty, \tag{48}$$

and moreover the optimal constant C *in* (47) *equals* $A_{(48)}$.

This result can be found in [22, Theorem 4.1 (ii)]. It also follows from Theorem 2.6 (i) with $X = \Gamma^{q,\infty}(w)$ or Theorem 2.6 (ii) with $X = \Lambda^{p,\infty}(v)$ combined with an easy observation that $\psi_X(t)$ from (8) equals $\int_0^t V^{-1/p}(s)\,ds$.

5. Embeddings of type $\Gamma \hookrightarrow \Lambda$

THEOREM 5.1. (the case $\Gamma^p(v) \hookrightarrow \Lambda^q(w)$) *Let* $p, q \in (0, \infty)$ *and let* v, w *be weights.*

(i) *If* $0 < p \leqslant q < \infty$, $1 \leqslant q < \infty$, *then the inequality*

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant C \left(\int_0^\infty (f^{**}(t))^p\, v(t)\, dt\right)^{1/p}, \tag{49}$$

holds if and only if

$$A_{(50)} := \sup_{t>0} \frac{W^{1/q}(t)}{\left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds\right)^{1/p}} < \infty, \tag{50}$$

and moreover the optimal constant C *in* (49) *is equal to* $A_{(50)}$.

(ii) *If* $0 < q < p = 1$, *then* (49) *holds if and only if*

$$A_{(51)} := \left(\int_0^\infty \left(\frac{W(t)}{V(t) + t \int_t^\infty s^{-1} v(s)\, ds}\right)^r w(t)\, dt\right)^{1/r} < \infty, \qquad \frac{1}{r} = \frac{1}{q} - 1, \tag{51}$$

and $C \approx A_{(51)}$.

(iii) *Let* v *satisfy the non-degeneracy conditions* (cf. [13])

$$\int_0^\infty \frac{v(s)}{(s+1)^p}\, ds < \infty, \qquad \int_0^1 \frac{v(s)}{s^p}\, ds = \int_1^\infty v(s)\, ds = \infty. \tag{52}$$

Moreover, let $q = 1 < p < \infty$. *Then* (49) *holds if and only if*

(53)

$$A_{(53)} := \left(\int_0^\infty \left(\int_0^t w^\circ(s)\, ds\right)^{p'} \times \sum_k t \left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds\right)^{-p'} \delta_{\mu_k}(t)\, dt\right)^{1/p'} < \infty,$$

where δ_{μ_k} *are Dirac measures situated at* μ_k, *and* $C \approx A_{(53)}$, *where the discretizing sequence* $\{\mu_k\}$ *is defined for some* $a > 1$ *by* $\mu_0 = 1$, *and*

$$\mu_{k+1} = \inf\left\{t; \min\left\{\frac{\varrho_{v,p}(\mu_k)}{\varrho_{v,p}(t)}, \frac{t\varrho_{v,p}(t)}{\mu_k \varrho_{v,p}(\mu_k)}\right\} = a\right\}, \quad k \geqslant 0,$$

$$\mu_{k-1} = \sup\left\{t; \min\left\{\frac{\varrho_{v,p}(t)}{\varrho_{v,p}(\mu_k)}, \frac{\mu_k \varrho_{v,p}(\mu_k)}{t\varrho_{v,p}(t)}\right\} = a\right\}, \quad k \leqslant 0,$$

and

$$\varrho_{v,p}(t) = \left(\int_0^\infty \frac{v(s)}{(s+t)^p}\, ds\right)^{1/p}, \qquad \varrho_{w,q}(t) = \left(\int_0^\infty \frac{w(s)}{(s+t)^q}\, ds\right)^{1/q}.$$

The assertion (i) is proved in [18, Theorem 3.2] (for $1 \leqslant p = q < \infty$), [16, Theorem 2.1] (for $1 \leqslant p \leqslant q < \infty$), and [24, p. 473] (for $0 < p \leqslant q < \infty$, $1 \leqslant q < \infty$). The proof with the best constant can be found in [15, Theorem 3.2 (a) and (c)] or [2], where a multidimensional case is treated.

The assertion (ii) follows from (3) and Theorem 3.1 (ii) with $0 < q < 1 = p$, and v replaced by $\int_t^\infty s^{-1} v(s)\, ds$.

When w itself is non-increasing, then the proof of (iii) can be found in [13, Theorem 2.2a]. For general w, the result follows from Proposition 2.12.

THEOREM 5.2. (the case $\Gamma^p(v) \hookrightarrow \Lambda^{q,\infty}(w)$) *Let $p,q \in (0,\infty)$ and let v,w be weights. The inequality*

$$\sup_{t>0} f^*(t) W^{1/q}(t) \leqslant C \left(\int_0^\infty (f^{**}(t))^p\, v(t)\, dt \right)^{1/p} \tag{54}$$

holds if and only if $A_{(50)} < \infty$ (cf. (50)). *Moreover, the optimal constant C in* (54) *equals $A_{(50)}$.*

Proof. This follows from Theorem 2.6 (iii) with $X = \Gamma^p(v)$, and (10). ☐

THEOREM 5.3. *(the case $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^{q,\infty}(w)$). Let $p,q \in (0,\infty)$ and let v,w be weights. The inequality*

$$\sup_{t>0} f^*(t) W^{1/q}(t) \leqslant C \sup_{t>0} f^{**}(t) V^{1/p}(t) \tag{55}$$

holds if and only if

$$A_{(56)} := \sup_{t>0} \frac{W^{1/q}(t)}{t \sup_{s \geqslant t} s^{-1} V^{1/p}(s)} < \infty. \tag{56}$$

Moreover, the optimal constant C in (55) *equals $A_{(56)}$.*

Proof. This follows from Theorem 2.6 (iii) with $X = \Gamma^{p,\infty}(v)$, and (11). ☐

The missing case $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ will be treated separately in Section 7 below.

6. Embeddings of type $\Gamma \hookrightarrow \Gamma$

THEOREM 6.1. (the case $\Gamma^p(v) \hookrightarrow \Gamma^q(w)$) *Let $p,q \in (0,\infty)$ and let v,w be weights. Assume that v satisfies* (52).

(i) *Let $0 < p \leqslant q < \infty$. Then the inequality*

$$\left(\int_0^\infty (f^{**}(t))^q\, (t)\, w(t) dt \right)^{1/q} \leqslant C \left(\int_0^\infty (f^{**}(t))^p\, v(t)\, dt \right)^{1/p} \tag{57}$$

holds if and only if

$$A_{(58)} := \sup_{t>0} \frac{\left(W(t) + t^q \int_t^\infty s^{-q} w(s)\, ds \right)^{1/q}}{\left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds \right)^{1/p}} < \infty, \tag{58}$$

and the optimal constant C *in* (57) *equals* $A_{(58)}$ *except the case* $0 < p \leqslant q < 1$ *when* $C \approx A_{(58)}$.

(ii) *Let* $0 < q < p < \infty$. *Then (57) holds if and only if*

$$A_{(59)} := \left(\sum_{k \in \mathbb{Z}} \left(\frac{\varrho_{w,q}(\mu_k)}{\varrho_{v,p}(\mu_k)} \right)^r \right)^{1/r} < \infty, \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{p}, \tag{59}$$

where $\varrho_{v,p}$, $\varrho_{w,q}$, *and* $\{\mu_k\}$ *are defined as in* Theorem 5.1 (iii). *Moreover,* $C \approx A_{(59)}$.

This result can be found in [13, Theorem 3.2]. The particular case $q \geqslant 1$ of (i) was also treated in [24, Theorem 3.3]. A proof with a sharp constant is given in [15, Theorem 3.7].

THEOREM 6.2. (the case $\Gamma^p(v) \hookrightarrow \Gamma^{q,\infty}(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights. The inequality*

$$\sup_{t>0} f^{**}(t) W^{1/q}(t) \leqslant C \left(\int_0^\infty (f^{**}(t))^p \, v(t)\, dt \right)^{1/p} \tag{60}$$

holds if and only if $A_{(50)} < \infty$ (cf. (50)). *Moreover, the optimal constant* C *in* (60) *satisfies* $C \approx A_{(50)}$.

Proof. The lower bound, $C \geqslant A_{(50)}$, follows from Theorem 5.2 and (4). As for the upper bound, for every $t > 0$ we have

$$\begin{aligned} f^{**}(t) W^{1/q}(t) &\lesssim A_{(50)} \left(f^{**}(t) V^{1/p}(t) + \int_0^t f^*(y)\, dy \left(\int_t^\infty \frac{v(s)}{s^p}\, ds \right)^{1/p} \right) \\ &\lesssim A_{(50)} \left(f^{**}(t) V^{1/p}(t) + \left(\int_t^\infty (f^{**}(s))^p \, v(s)\, ds \right)^{1/p} \right). \end{aligned}$$

Taking the supremum over $t > 0$ and using (4) we get $C \lesssim A_{(50)}$. □

REMARK 6.3. For $0 < p \leqslant 1$, the upper bound in Theorem 6.2 follows also from Theorem 5.1 (i) and Remark 4.3 (ii). Yet another proof can be obtained from Theorem 2.6 (ii) and Theorem 5.1 (i).

THEOREM 6.4. (the case $\Gamma^{p,\infty}(v) \hookrightarrow \Gamma^q(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights. The inequality*

$$\left(\int_0^\infty (f^{**}(t))^q \, w(t)\, dt \right)^{1/q} \leqslant C \sup_{t>0} f^{**}(t) V^{1/p}(t) \tag{61}$$

holds if and only if

$$A_{(62)} := \left(\int_0^\infty \frac{w(t)}{\left(t \sup_{s \geqslant t} s^{-1} V^{1/p}(s) \right)^q}\, dt \right)^{1/q} < \infty. \tag{62}$$

Moreover, the optimal constant C *in* (61) *satisfies* $C \approx A_{(62)}$.

Proof. The upper bound $C \leqslant A_{(62)}$ follows from (20).

Conversely, let f be any function such that $f^* = v_p$. Then, by (21), $\|f\|_{\Gamma^{p,\infty}(v)} \leqslant 2$, and, by (61),

$$A_{(62)} \leqslant \max\{1, 2^{\frac{q-1}{q}}\} \left[2C + \lambda \left(\int_0^\infty \frac{w(t)}{t^q}\, dt\right)^{1/q}\right].$$

If $\lambda = 0$, we are done. Assume that $\lambda > 0$. For arbitrary fixed $a > 0$, take any f such that $f^* = \frac{\lambda}{a}\chi_{(0,a)}$. Now, observe that $\sup_{s\geqslant 0} s^{-1}V^{1/p}(s) \leqslant \frac{2}{\lambda}$, whence

$$\|f\|_{\Gamma^{p,\infty}(v)} \leqslant \int_0^\infty f^*(y)\, dy \sup_{s\geqslant 0} s^{-1}V^{1/p}(s) \leqslant 2.$$

Thus, using the estimate $f^{**}(t) \geqslant \frac{\lambda}{t}\chi_{(a,\infty)}(t)$ and (61), we finally obtain

$$\lambda \left(\int_a^\infty \frac{w(t)}{t^q}\, dt\right)^{1/q} \leqslant C,$$

and the result follows on letting $a \to 0_+$. □

THEOREM 6.5. (the case $\Gamma^{p,\infty}(v) \hookrightarrow \Gamma^{q,\infty}(w)$) *Let* $p, q \in (0,\infty)$ *and let* v, w *be weights. The inequality*

$$\sup_{t>0} f^{**}(t) W^{1/q}(t) \leqslant C \sup_{t>0} f^{**}(t) V^{1/p}(t) \tag{63}$$

holds if and only if

$$A_{(64)} := \sup_{t>0} \frac{W^{1/q}(t)}{t \sup_{s\geqslant t} s^{-1}V^{1/p}(s)} < \infty. \tag{64}$$

Moreover, the optimal constant C *in* (63) *satisfies* $C \approx A_{(64)}$.

Proof. The upper bound follows directly from the definition of $A_{(64)}$ similarly as in (20). To get the lower bound we test (63) with $\chi_{(0,a)}$, $a > 0$. For $0 < p \leqslant 1$, the assertion follows also from Theorem 2.6 (ii). □

7. Embeddings of type $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$

This section is devoted to the inequality

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant C \sup_{t>0} f^{**}(t) V^{1/p}(t), \qquad p, q \in (0,\infty). \tag{65}$$

By (20) and (21), (65) is equivalent to

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant C \sup_{t>0} \left(\int_0^t f^*(s)\, ds\right) \mathcal{V}_p(t), \tag{66}$$

where $\mathcal{V}_p(t)$ is from (18), and also to

$$\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q} \leqslant \widetilde{C} \sup_{t>0} \frac{\int_0^t f^*(s)\, ds}{\lambda + \int_0^t v_p(s)\, ds}, \tag{67}$$

where λ and v_p are from (21). This could be rewritten as

$$C \approx \sup\left\{ \left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q};\ \int_0^t f^*(s)\, ds \leqslant \lambda + \int_0^t v_p(s)\, ds \right\}. \tag{68}$$

7a) The case $q = 1$. We first observe that the quantity

$$A_{(69)} := \sup_{t>0} \frac{W(t)}{t\mathcal{V}_p(t)} \approx \sup_{t>0} \frac{W(t)(\lambda + \int_0^t v_p(s)\, ds)}{t}, \tag{69}$$

is a lower bound for the optimal constant C in (66) with $q = 1$. Indeed, $A_{(69)} \lesssim C$ follows on testing (66) with $\chi_{(0,a)}$, $a > 0$. In particular, (66) implies $A_{(69)} < \infty$. Therefore,

$$\frac{W(t)}{t} \leqslant A_{(69)}\mathcal{V}_p(t), \qquad t > 0,$$

and, as $\mathcal{V}_p$ is non-increasing, also

$$\limsup_{t\to\infty} \frac{W(t)}{t} < \infty.$$

This guarantees the existence of w° (cf. Section 2 above).

THEOREM 7.1. (the case $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^1(w)$) *Let* $p \in (0,\infty)$ *and let* v, w *be weights. The inequality* (65) *holds with* $q = 1$ *for every* $f \in \Gamma^{p,\infty}(v)$ *if and only if* $A_{(69)} < \infty$, *and*

$$A_{(70)} := \int_0^\infty w^\circ(t) v_p(t)\, dt < \infty. \tag{70}$$

Moreover, the optimal constant C *in* (65) *satisfies* $C \approx A_{(69)} + A_{(70)}$.

Proof. By Proposition 2.12 and Example 2.14,

$$C = \sup \frac{\int_0^\infty f^*(t) w^\circ(t)\, dt}{\|f\|_{\Gamma^{p,\infty}(v)}}.$$

We claim that

$$C \approx \int_0^{\|w^\circ\|_\infty} \frac{1}{\mathcal{V}_p(w_*^\circ(y))}\, dy. \tag{71}$$

Indeed, using the formula for the distribution function (cf. e.g. [4, Chapter 2, Proposition 1.8]), and [9, Theorem 2.1], we get

$$\begin{aligned} C &= \sup\left\{\int_0^\infty f^*(s)w^\circ(s)\,ds;\ \int_0^t f^*(s)\,ds \leqslant \frac{1}{\mathcal{V}_p(t)} \text{ for all } \quad t\in(0,\infty)\right\} \\ &= \sup\left\{\int_0^\infty \left(\int_0^{w^\circ_*(s)} f^*(y)\,dy\right)ds;\ \int_0^t f^*(s)\,ds \leqslant \frac{1}{\mathcal{V}_p(t)} \text{ for all } \quad t\in(0,\infty)\right\} \\ &\leqslant \int_0^{\|w^\circ\|_\infty} \frac{1}{\mathcal{V}_p(w^\circ_*(y))}\,dy. \end{aligned}$$

Conversely, let $r>0$ and set

$$g_r(t) = \begin{cases} \frac{t}{r\mathcal{V}_p(r)}, & t\leqslant r \\ \frac{1}{\mathcal{V}_p(t)}, & t>r. \end{cases}$$

Then, since g_r is a quasi-concave function, there exists a non-increasing function f_r so that $g_r(t)\approx\int_0^t f_r(s)\,ds$. Since, obviously, $\int_0^t f_r(s)\,ds \lesssim \frac{1}{\mathcal{V}_p(t)}$, we get, for every $r>0$,

$$\begin{aligned} C &\geqslant \int_0^\infty f_r(t)w^\circ(t)\,dt = \int_0^\infty\int_0^{w^\circ_*(t)} f_r(s)\,ds\,dt \\ &\approx \int_0^\infty g_r(w^\circ_*(s))\,ds \geqslant \int_0^{w^\circ(r)} \frac{1}{\mathcal{V}_p(w^\circ_*(y))}\,dy, \end{aligned}$$

and (71) follows on letting $r\to 0_+$.

Finally, observe that

$$\begin{aligned} C &\approx \int_0^{\|w^\circ\|_\infty} \frac{1}{\mathcal{V}_p(w^\circ_*(y))}\,dy = \lambda\|w^\circ\|_\infty + \int_0^{\|w^\circ\|_\infty}\left(\int_0^{w^\circ_*(t)} v_p(s)\,ds\right)dt \\ &\approx \lambda\|w^\circ\|_\infty + A_{(70)}. \end{aligned}$$

It remains to show $\lambda\|w^\circ\|_\infty \leqslant A_{(69)}$, or (cf. (69)), in particular, $\|w^\circ\|_\infty \leqslant \sup_t \frac{W(t)}{t}$. This is obvious when, for some $\delta>0$, $(0,\delta)$ is one of the intervals I_j, since then $\|w^\circ\|_\infty = \delta^{-1}\int_0^\delta w(s)\,ds$. If $t_k\to 0_+$ for some sequence t_k in E, then the monotonicity of w° and the equality (22) for $x=t_k$ yields

$$\|w^\circ\|_\infty = \lim_{k\to\infty} w^\circ(t_k) \leqslant \lim_{k\to\infty}\frac{\int_0^{t_k} w^\circ(s)\,ds}{t_k} = \lim_{k\to\infty}\frac{W(t_k)}{t_k} \leqslant \sup_t \frac{W(t)}{t}.$$

The proof is complete. □

REMARKS 7.2. (i) It follows from the proof of Theorem 7.1 that, if w is non-increasing, then

$$\sup\left\{\int_0^\infty f(t)w(t)\,dt;\ \int_0^t f(s)\,ds \leqslant \frac{1}{\mathcal{V}_p(t)} \text{ for all } \quad t\in(0,\infty)\right\} = C,$$

that is, we get the same constant without the restriction to non-increasing f.

(ii) By a simple calculation it can be also shown that the optimal constant C in (65) with $q=1$ satisfies

$$C \approx A_{(69)} + \int_0^\infty \frac{1}{\mathcal{V}_p(t)}\, d\left(-w^\circ(t)\right) \approx A_{(69)} + \int_0^\infty w^\circ(t)\, d\left(\frac{1}{\mathcal{V}_p(t)}\right).$$

7b) The case $q>1$. Analogously to (69), we define

$$A_{(72)} := \sup_{t>0} \frac{W^{1/q}(t)}{t\mathcal{V}_p(t)} \approx \sup_{t>0} \frac{W^{1/q}(t)\left(\lambda + \int_0^t v_p(s)\, ds\right)}{t}, \tag{72}$$

where λ and v_p are from Remark 2.11 (v). It is easy to show that

$$C \geqslant A_{(72)},$$

where C is the optimal constant in (65), now with $q>1$. We further introduce the quantities

$$A_{(73)} := \left(\int_0^\infty w(t) v_p^q(t)\, dt\right)^{1/q}, \tag{73}$$

and

$$A_{(74)} := \sup_{\{I_k\}} \left[\sum_{k\in\mathbb{Z}} \left(\frac{1}{|I_k|}\int_{I_k} v_p(s)\, ds\right)^q \int_{I_k} w(s)\, ds\right]^{1/q}, \tag{74}$$

where the supremum is extended over all sequences of disjoint intervals in $\mathbb{R}^+$ of finite measure.

THEOREM 7.3. (the case $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$, $1<q<\infty$) *Let* $q\in(1,\infty)$, $p\in(0,\infty)$ *and let* v,w *be weights. Then the inequality* (65) *holds if and only if* $A_{(72)}<\infty$, $A_{(73)}<\infty$, *and* $A_{(74)}<\infty$. *Moreover, the optimal constant* C *in* (65) *satisfies*

$$C \approx A_{(72)} + A_{(73)} + A_{(74)}.$$

Proof. We start with the lower bounds (recall that $C \gtrsim A_{(72)}$ has already been pointed out above). Let $\{I_k\}$ be a sequence of disjoint intervals in $\mathbb{R}^+$ of finite measures. We set

$$f(x) = \begin{cases} \frac{1}{|I_k|}\int_{I_k} v_p(s)\, ds, & x\in I_k,\ k\in\mathbb{Z},\\ v_p(x), & x\in\mathbb{R}^+ \setminus \bigcup_{k\in\mathbb{Z}} I_k. \end{cases}$$

Since v_p is non-increasing, so is f, and we have $\|f\|_{\Gamma^{p,\infty}(v)} \lesssim 1$, because $\int_0^t f(s)\, ds \leqslant \int_0^t v_p(s)\, ds$, $t>0$. Thus it follows from (67) that $C \approx \widetilde{C} \geqslant A_{(74)}$. To prove

$$C \gtrsim A_{(73)}, \tag{75}$$

we define for $r>0$

$$f_r(x)=\begin{cases}\frac{1}{r}\left(\lambda+\int_0^r v_p(s)\,ds\right), & 0<x<r;\\ v_p(x), & x\geqslant r.\end{cases}$$

Then f_r is non-increasing, and $\sup_{r>0}\|f_r\|_{\Gamma^{p,\infty}(v)}\lesssim 1$. Inserting f_r into (67) we get $C\gtrsim\left(\int_r^\infty w(s)v_p^q(s)\,ds\right)^{1/q}$, and (75) follows on letting $r\to 0_+$.

Now, to prove the upper bound, we use a duality argument. By the change of suprema, Theorem 7.1, and Hölder's inequality,

$$\begin{aligned}C&=\sup_{f\not\equiv 0}\frac{\|f\|_{\Lambda^q(w)}}{\|f\|_{\Gamma^{p,\infty}(v)}}=\sup_{f\not\equiv 0}\frac{\sup\{\int_0^\infty f^*(t)g^*(t)w(t)\,dt;\|g\|_{\Lambda^{q'}(w)}\leqslant 1\}}{\|f\|_{\Gamma^{p,\infty}(v)}}\\ &=\sup\left\{\sup_{f\not\equiv 0}\frac{\int_0^\infty f^*(t)g^*(t)w(t)\,dt}{\|f\|_{\Gamma^{p,\infty}(v)}};\|g\|_{\Lambda^{q'}(w)}\leqslant 1\right\}\\ &\approx\sup\left\{\sup_{t>0}\frac{\left(\int_0^t(s)g^*(s)w(s)\,ds\right)\left(\lambda+\int_0^t v_p(s)\,ds\right)}{t}\right.\\ &\qquad\left.+\int_0^\infty (g^*w)^\circ(t)v_p(t)\,dt;\|g\|_{\Lambda^{q'}(w)}\leqslant 1\right\}\\ &\leqslant\sup_{t>0}\frac{W^{1/q}(t)\left(\lambda+\int_0^t v_p(s)\,ds\right)}{t}+\sup\left\{\int_0^\infty (g^*w)^\circ(t)v_p(t)\,dt;\|g\|_{\Lambda^{q'}(w)}\leqslant 1\right\}.\end{aligned}$$

Moreover, we have $\mathbb{R}^+=E\cup\bigcup_{j\in\mathbb{N}}J_j\cup[a,\infty)$, where J_j are intervals, and

$$(g^*w)^\circ(x)=\begin{cases}g^*(x)w(x), & x\in E,\\ \frac{1}{|J_j|}\int_{J_j}g^*(t)w(t)\,dt, & x\in J_j,\ j\in\mathbb{N},\\ \limsup\limits_{t\to\infty}\frac{1}{t}\int_0^t g^*(s)w(s)\,ds, & x\in[a,\infty).\end{cases}$$

Thus,

$$\begin{aligned}\int_0^\infty (g^*w)^\circ(t)v_p(t)\,dt&=\int_E g^*(t)w(t)v_p(t)\,dt\\ &\quad+\sum_{j\in\mathbb{N}}\frac{1}{|J_j|}\left(\int_{J_j}v_p(t)\,dt\right)\left(\int_{J_j}g^*(t)w(t)\,dt\right)\\ &\quad+\left(\int_a^\infty v_p(s)\,ds\right)\limsup_{t\to\infty}\frac{1}{t}\int_0^t g^*(s)w(s)\,ds\\ &=I+II+III,\end{aligned}$$

say. Let $\|g\|_{\Lambda^{q'}(w)}\leqslant 1$. By Hölder's inequality, $I\leqslant A_{(73)}$. Next, again by the Hölder's inequality (used twice)

$$II\leqslant\sum_{j\in\mathbb{N}}\frac{1}{|J_j|}\left(\int_{J_j}v_p(s)\,ds\right)\left(\int_{J_j}w(s)\,ds\right)^{1/q}\left(\int_{J_j}(g^*)^{q'}(t)w(t)\,dt\right)^{1/q'}\leqslant A_{(74)}.$$

Analogously,

$$III \leqslant \limsup_{t\to\infty} \frac{1}{t}\left(\int_0^t g^*(s)w(s)\,ds\right)\int_a^t v_p(s)\,ds$$
$$\leqslant \limsup_{t\to\infty} \frac{W^{1/q}(t)}{t}\int_a^t v_p(s)\,ds \leqslant A_{(72)},$$

and $C \lesssim A_{(72)} + A_{(73)} + A_{(74)}$ follows. The proof is complete. □

7c) The case $q < 1$. Let $q \in (0,1)$ and $p \in (0,\infty)$. We denote

$$A_{(76)} := \left(\int_0^\infty \left(\frac{W(t)}{t\mathcal{V}_p(t)}\right)^{q/(1-q)} w(t)\,dt\right)^{(1-q)/q}, \tag{76}$$

and

$$A_{(77)} := \left(\int_0^\infty t^{1-q}\mathcal{V}_p^{-q}(t)\,d\left(-w^\circ(t)\right)\right)^{1/q}. \tag{77}$$

THEOREM 7.4. (the case $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$, $0 < q < 1$) *Let* $q \in (0,1)$, $p \in (0,\infty)$ *and let* v, w *be weights. If* $A_{(72)} + A_{(77)} < \infty$, *then* (65) *holds. Conversely, if* (65) *is true, then* $A_{(72)} + A_{(73)} + A_{(74)} + A_{(76)} < \infty$. *Moreover, if* C *is the optimal constant in* (65), *then*

$$A_{(72)} + A_{(73)} + A_{(74)} + A_{(76)} \lesssim C \lesssim A_{(72)} + A_{(77)}. \tag{78}$$

Proof. We start with lower bounds. Note that $A_{(72)} + A_{(73)} + A_{(74)} \lesssim C$ follows from the proof of Theorem 7.3. By Remarks 2.11 (i) and (ii), there exists a non-increasing function ψ such that $t\mathcal{V}_p(t) \approx \int_0^t \psi(s)\,ds$ for all $t > 0$. With ψ so defined, we have by Theorem 4.2 (i),

$$\Lambda^1(\psi) \hookrightarrow \Gamma^{p,\infty}(v). \tag{79}$$

Thus, (65) and (79) imply $\Lambda^1(\psi) \hookrightarrow \Lambda^q(w)$, and $A_{(76)} \lesssim C$ follows from Theorem 3.1 applied to $p = 1$ and $v = \psi$.

For the upper bound, note that $A_{(72)} < \infty$ in particular implies that $\int_0^t w(s)\,ds < \infty$, and hence $\int_0^t w^\circ(s)\,ds < \infty$, for every $t > 0$. Thus, if f is a non-increasing step function, then $\int_0^t f^q(s)w^\circ(s)\,ds < \infty$ for every $t > 0$, and we get on integrating by parts, using also (22) and the Hardy's lemma ([4, Chapter 2, Proposition 3.6])

$$\int_0^\infty f^q(s)w(s)\,ds \leqslant \int_0^\infty f^q(s)w^\circ(s)\,ds$$
$$= \lim_{t\to\infty} w^\circ(t)\int_0^t f^q(s)\,ds + \int_0^\infty \left(\int_0^t f^q(s)\,ds\right) d\left(-w^\circ(t)\right).$$

By the Hölder inequality, (20), and (72),

$$\lim_{t\to\infty} w^\circ(t)\int_0^t f^q(s)\,ds \leqslant \limsup_{t\to\infty} W(t)\left(\frac{1}{t}\int_0^t f(s)\,ds\right)^q \leqslant A^q(27)\|f\|^q_{\Gamma^{p,\infty}(v)}.$$

Analogously,

$$\int_0^\infty \left(\int_0^t f^q(s)\,ds \right) d\left(-w^\circ(t)\right) \leqslant \int_0^\infty t^{1-q} \left(\int_0^t f(s)\,ds \right)^q d\left(-w^\circ(t)\right)$$
$$\leqslant \|f\|^q_{\Gamma^{p,\infty}(v)} \int_0^\infty t^{1-q} \mathcal{V}_p^{-q}(t)\, d\left(-w^\circ(t)\right),$$

and the second inequality in (78) follows. $\square$

Unfortunately, Theorem 7.4 does not give a complete characterization of (65) for $q \in (0,1)$. However, in many particular cases such a characterization is available. We shall collect some particular results. Recall that, by Proposition 2.12, with no loss of generality we may assume that w is non-increasing.

THEOREM 7.5. (Miscellaneous particular cases, $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$, $0 < q < 1$). *Let $q \in (0,1)$, $p \in (0,\infty)$, let v, w be weights, and let C be the optimal constant in (65).*

(i) *If $v_p = 0$, then*

$$C \approx A_{(80)} := \left(\int_0^\infty \left(\frac{W(t)}{t} \right)^{1/(1-q)} w(t)\,dt \right)^{(1-q)/q}. \tag{80}$$

Moreover, we always have $C \gtrsim A_{(80)}$.

(ii) *If $\lambda = 0$ and v_p satisfies $\frac{1}{t}\int_0^t v_p(s)\,ds \approx v_p(t)$, $t > 0$, then $C \approx A_{(73)}$.*

(iii) *If $w \in B_q$, that is,*

$$\int_r^\infty \frac{w(s)}{s^q}\,ds \lesssim \frac{W(r)}{r^q}, \qquad 0 < r < \infty,$$

then

$$C \approx \left(\int_0^\infty \frac{w(t)}{t^q}\,dt \right)^{1/q} + A_{(73)}.$$

(iv) *If wv^{q-1} is non-increasing and $\lambda = 0$, then $C \approx A_{(73)}$.*

(v) *If there exists a non-increasing function g such that $G(t) := \int_0^t g(s)\,ds$ satisfies $G^q(t)\int_t^\infty G^{-q}(s)w(s)\,ds \lesssim W(t)$ for $t > 0$, then $C \approx A_{(73)}$.*

(vi) *If $\lambda = 0$ and $v_p \in A_1$, then $C \approx A_{(73)}$.*

(vii) *If $\lambda = 0$ and v_p satisfies*

$$\int_0^t v_p(s) \log\tfrac{t}{s}\,ds \lesssim \int_0^t v_p(s)\,ds,$$

then, denoting $V_p(t) := \int_0^t v_p(s)\,ds$,

$$C \approx \left(\int_0^\infty \left(\frac{V_p(t)}{t} \right)^q w(t)\,dt \right)^{1/q}.$$

Proof. (i) By (68), the substitution $(f^*)^q = g^*$, and Theorem 3.1 (ii),

$$C \approx \sup\left\{\left(\int_0^\infty (f^*(s))^q\, w(s)\, ds\right)^{1/q};\ \int_0^\infty f^*(t)\, dt \leqslant \lambda\right\}$$
$$= \lambda\left(\sup_g \frac{\int_0^\infty g^*(t)w(t)\, dt}{\|g\|_{1/q}}\right)^{1/q}$$
$$\approx \lambda\left(\int_0^\infty \left(\frac{W(t)}{t}\right)^{q/(1-q)} w(t)\, dt\right)^{(1-q)/q}.$$

(ii) By (68) and the assumption,

$$C \approx \sup\left\{\left(\int_0^\infty (f^*(t))^q\, w(t)\, dt\right)^{1/q};f^* \lesssim v\right\} \approx \left(\int_0^\infty v^q(t)w(t)\, dt\right)^{1/q}.$$

(iii) If $w \in B_q$, then

$$C \approx \sup\left\{\left(\int_0^\infty (f^{**}(t))^q\, w(t)\, dt\right)^{1/q};\ \int_0^t f^*(s)\, ds \leqslant \lambda + \int_0^t v_p(s)\, ds\right\}$$
$$= \left(\int_0^\infty \left(\frac{\lambda + \int_0^t v_p(s)\, ds}{t}\right)^q w(t)\, dt\right)^{1/q}$$
$$\approx \lambda\left(\int_0^\infty \frac{w(t)}{t^q}\, dt\right)^{1/q} + \left(\int_0^\infty v_p^q(t)w(t)\, dt\right)^{1/q}.$$

(iv) If wv^{q-1} is non-increasing, then, by the assumption and [8, Theorem 2.1],

$$C \approx \Bigg(\sup\Bigg\{\int_{\{f\leqslant v_p\}} (f^*(t))^q\, w(t)\, dt$$
$$+ \int_{\{f>v_p\}} (f^*(t))^q\, w(t)\, dt;\ \int_0^t f^*(s)\, ds \leqslant \int_0^t v_p(s)\, ds\Bigg\}\Bigg)^{1/q}$$
$$\leqslant A_{(73)} + \left(\sup\left\{\int_0^\infty f^*(t)v_p^{q-1}(t)w(t)\, dt;\ \int_0^t f^*(s)\, ds \leqslant \int_0^t v_p(s)\, ds\right\}\right)^{1/q}$$
$$\leqslant A_{(73)} + \left(\sup\left\{\int_0^\infty \left(\int_0^{(wv_p^{q-1})_*(y)} f^*(s)\, ds\right) dy;\ \int_0^t f^*(s)\, ds \leqslant \int_0^t v_p(s)\, ds\right\}\right)^{1/q}$$
$$\leqslant A_{(73)} + \left(\int_0^{(wv_p^{q-1})(0+)} \int_0^{(wv_p^{q-1})_*(y)} v_p(s)\, ds\, dy\right)^{1/q} \lesssim A_{(73)}.$$

The lower bound follows from Theorem 7.4.

(v) This is a simple modification of (iv).

(vi) By (68),

$$C \leqslant \sup \left\{ \left(\int_0^\infty (f^{**}(t))^q w(t)\,dt \right)^{1/q} ; \int_0^t f^*(s)\,ds \leqslant \int_0^t v_p(s)\,ds \right\} \tag{81}$$

$$\leqslant \left(\int_0^\infty \left(\frac{V_p(t)}{t} \right)^q w(t)\,dt \right)^{1/q} \lesssim \left(\int_0^\infty v_p^q(t) w(t)\,dt \right)^{1/q},$$

since $v_p \in A_1$. The lower bound follows from Theorem 7.4.

(vii) The function $t^{-1}V_p(t)$ is decreasing. Hence,

$$C \geqslant \frac{\left(\int_0^\infty (t^{-1}V_p(t))^q w(t)\,dt\right)^{1/q}}{\sup_{t>0} \left(\int_0^t s^{-1}V_p(s)\,ds\right)\left(\int_0^t v_p(s)\,ds\right)^{-1}}$$

$$= \frac{\left(\int_0^\infty (t^{-1}V_p(t))^q w(t)\,dt\right)^{1/q}}{\sup_{t>0} \left(\int_0^t v_p(s)\log\frac{t}{s}\,ds\right)\left(\int_0^t v_p(s)\,ds\right)^{-1}}$$

$$\geqslant \left(\int_0^\infty \left(\frac{V_p(t)}{t} \right)^q w(t)\,dt \right)^{1/q}.$$

The upper bound follows from (81). □

8. The single-weight case $\Gamma^{1,\infty}(v) \hookrightarrow \Lambda^1(v)$

Now we return to the problem, when the norm in $\Lambda^1(v)$ can be expressed in terms of f^{**}. By Theorem 7.1 we can characterize the embedding $\Gamma^{1,\infty}(v) \hookrightarrow \Lambda^1(v)$. By Proposition 2.12, we may restrict ourselves to the case when v is non-increasing. By [17], then $\Lambda^1(v)$ is a Banach space, and by Theorem 4.2 (i) it is embedded into $\Gamma^{1,\infty}(v)$. Thus the embedding $\Gamma^{1,\infty}(v) \hookrightarrow \Lambda^1(v)$ is actually equivalent to $\Gamma^{1,\infty}(v) = \Lambda^1(v)$, and consequently, there is an expression of the $\Lambda^1(v)$ norm in terms of f^{**}.

THEOREM 8.1. *Let v be a non-increasing weight. Then the following statements are equivalent:*

$$\int_0^\infty f^*(s)v(s)\,ds \leqslant C \sup_{t>0} f^{**}(t)V(t); \tag{82}$$

$$A_{(83)} := \int_0^\infty \frac{t}{V(t)}\,d(-v(t)) < \infty; \tag{83}$$

$$A_{(84)} := \int_0^\infty \frac{V(t) - tv(t)}{V^2(t)} v(t)\,dt < \infty; \tag{84}$$

$$(85)\qquad \lim_{t\to 0+} v(t) < \infty \quad \text{and} \quad \begin{cases} \text{either} & \lim_{t\to\infty} V(t) < \infty \text{ and } \lim_{t\to\infty} v(t) = 0, \\ \text{or} & \lim_{t\to\infty} V(t) < \infty = \infty \text{ and } \lim_{t\to\infty} v(t) > 0. \end{cases}$$

Moreover, the optimal constant C *in* (82) *satisfies* $C \approx A_{(83)} + 1$.

Proof. We first claim that

$$(86)\qquad A_{(83)} = \begin{cases} A_{(84)} & \text{if} \quad V(\infty) = \infty, \\ A_{(84)} + 1 & \text{if} \quad V(\infty) < \infty. \end{cases}$$

Let $A_{(83)} < \infty$. Then, for $t > 0$,

$$\infty > A_{(83)} = \int_0^\infty \frac{d(V(s) - sv(s))}{V(s)} \geqslant \int_0^t \frac{d(V(s) - sv(s))}{V(s)}$$
$$\geqslant \frac{1}{V(t)} \int_0^t d(V(s) - sv(s)) = \frac{V(t) - tv(t)}{V(t)},$$

since $V(t) - tv(t) = \int_0^t (v(s) - v(t))\, ds \to 0$ as $t \to 0_+$, and therefore, by the absolute continuity of integral,

$$(87)\qquad \lim_{t\to 0_+} \frac{V(t) - tv(t)}{V(t)} = 0, \qquad \lim_{t\to 0_+} \frac{tv(t)}{V(t)} = 1.$$

Let $V(\infty) < \infty$. Then

$$V(t) - tv(t) = \int_0^t (v(s) - v(t))\, ds = \int_0^t \int_s^t d(-v(y))\, ds \;=\; \int_0^t s\, d(-v(s)).$$

Hence $\lim_{t\to\infty}(V(t) - tv(t)) = V(\infty) < \infty$, and consequently

$$(88)\qquad \lim_{t\to\infty} \frac{V(t) - tv(t)}{V(t)} = 1, \qquad \lim_{t\to\infty} \frac{tv(t)}{V(t)} = 0.$$

Integrating by parts we get

$$(89)\qquad A_{(83)} = A_{(84)} + \left[\frac{-tv(t)}{V(t)}\right]_{t=0}^{t=\infty}.$$

By (87), (88) and (89) we have $A_{(83)} = A_{(84)} + 1$.

Now let $V(\infty) = \infty$. Then, assuming that $A_{(84)} < \infty$, we get for $t > 0$,

$$\infty > A_{(84)} = \int_0^\infty (V(s) - sv(s))\, d\left(-\frac{1}{V(s)}\right) \geqslant \int_t^\infty (V(s) - sv(s))\, d\left(-\frac{1}{V(s)}\right)$$
$$\geqslant (V(t) - tv(t)) \left(\frac{1}{V(t)} - \frac{1}{V(\infty)}\right) = \frac{V(t) - tv(t)}{V(t)},$$

since $(V(t) - tv(t))$ is non-decreasing in t, and by the absolute continuity of integral,

$$(90)\qquad \lim_{t\to\infty} \frac{V(t) - tv(t)}{V(t)} = 0, \qquad \lim_{t\to\infty} \frac{tv(t)}{V(t)} = 1.$$

Thus, by (87), $A_{(83)} = A_{(84)}$, proving (86) and in turn the equivalence of (83) and (84).

Since v is non-increasing, we have $\mathcal{V}_p(t) = \mathcal{V}_1(t) = \frac{V(t)}{t}$, whence $A_{(69)} = 1$ and $A_{(70)} = A_{(84)}$. Thus, Theorem 7.1 yields the equivalence of (82) and (84) as well as the best constants relation.

It remains to show that (82) holds if and only if one of the options in (85) takes place. The "if" part is easy to verify. To prove the "only if" part, let $A_{(83)} < \infty$. Then, by (87), $2tv(t) \geqslant V(t)$ for $t \in (0, t_0)$ with some $t_0 > 0$. Hence

$$\infty > A_{(83)} \geqslant \int_0^{t_0} \frac{tv(t)}{V(t)} \frac{d(-v(t))}{v(t)} \geqslant \frac{1}{2} \log \frac{v(0+)}{v(t_0)},$$

and consequently $0 < v(0+) < \infty$. If $V(\infty) < \infty$, then $v(t) \leqslant \frac{V(t)}{t} \to 0$ as $t \to \infty$, and therefore $v(\infty) = 0$. If $V(\infty) = \infty$ and $A_{(84)} < \infty$, then by (90) $2tv(t) \geqslant V(t)$ on (t_1, ∞) for some $t_1 > 0$. Thus,

$$\infty > A_{(83)} \geqslant \frac{1}{2} \int_{t_1}^{\infty} \frac{d(-v(t))}{v(t)} \geqslant \frac{1}{2} \log \frac{v(t_1)}{v(\infty-)},$$

and therefore $0 < v(\infty-) < \infty$. This shows that (82) implies (85), and the proof is complete. $\square$

The following result, of independent interest, follows from Theorem 8.1, more precisely from the equivalence of (82) and (85).

COROLLARY 8.2. *Let φ be a concave non-decreasing function on $[0, \infty)$ such that $\varphi(t) = 0$ if and only if $t = 0$, and $\varphi(t)/t$ is non-increasing on $(0, \infty)$. The endpoint Lorentz space Λ_φ and the endpoint Marcinkiewicz space M_φ are defined as the sets of functions f from $\mathcal{M}(\mathcal{R}, \mu)$ such that*

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t)\, d\varphi(t) < \infty, \qquad \text{and} \qquad \|f\|_{M_\varphi} = \sup_{t>0} f^{**}(t)\varphi(t) < \infty,$$

respectively. Then $\Lambda_\varphi = M_\varphi$ if and only if φ is on $(0, \infty)$ equivalent to one of the following four functions: t, 1, $\max\{1, t\}$ or $\min\{1, t\}$.

9. Associate spaces of classical and weak Lorentz spaces

In this section we apply our embedding results to the characterization of the associate spaces of classical and weak Lorentz spaces.

THEOREM 9.1. *Let $p \in (0, \infty)$ and let v be a weight.*
(i) *Let $X = \Lambda^p(v)$. Then, for $0 < p \leqslant 1$,*

$$\|g\|_{X'} \approx \sup_{t>0} g^{**}(t) \frac{t}{V^{1/p}(t)},$$

and, for $1 < p < \infty$,

$$\|g\|_{X'} \approx \left(\int_0^\infty (g^{**}(t))^{p'} \frac{t^{p'} v(t)}{V^{p'}(t)}\, dt \right)^{1/p'} + V^{-1/p}(\infty) \int_0^\infty g^*(t)\, dt.$$

(ii) *Let* $X = \Lambda^{p,\infty}(v)$. *Then*

$$\|g\|_{X'} \approx \int_0^\infty g^*(t) V^{-1/p}(t)\, dt.$$

(iii) *Let* $X = \Gamma^p(v)$. *Let* v *satisfy (52). Then, for* $0 < p \leqslant 1$,

$$\|g\|_{X'} \approx \sup_{t>0} g^{**}(t) \frac{t}{\left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds\right)^{1/p}},$$

and, for $1 < p < \infty$,

$$\left(\int_0^\infty (g^{**}(t))^{p'} t^{p'} \sum_k t \left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds \right)^{-p'} \delta_{\mu_k}(t)\, dt \right)^{1/p'}$$

(*cf.* Theorem 5.1 *above for the definition of* μ_k *and* δ_{μ_k}).

(iv) *Let* $X = \Gamma^{p,\infty}(v)$. *Then*

$$\|g\|_{X'} \approx \sup_{t>0} g^{**}(t) \frac{1}{\mathcal{V}_p(t)} + \int_0^\infty g^*(t) v_p(t)\, dt,$$

where $\mathcal{V}_p$ *is from* (18) *and* v_p *is from* (21).

Proof. All the results follow from the embedding theorems above (namely Theorems 3.1, 3.3, 5.1 and 7.1), combined with (17). $\square$

REMARKS 9.2. Some particular cases of the results of Theorem 9.1 can be reformulated in terms of function spaces (cf. also [19] and [13]):

(i) If $0 < p \leqslant 1$ and $\frac{t}{V^{1/p}(t)}$ is non-decreasing, then $(\Lambda^p(v))' = \Gamma^{1,\infty}\left(\frac{d}{dt}\left(\frac{t}{V^{1/p}(t)}\right)\right)$.

(ii) If $1 < p < \infty$ and $V(\infty) = \infty$, then ([19, Remark, p. 147]) $(\Lambda^p(v))' = \Gamma^{p'}\left(\frac{t^{p'} v(t)}{V^{p'}(t)}\right)$.

(iii) For $0 < p < \infty$, $(\Lambda^{p,\infty}(v))' = \Lambda^1(V^{-1/p})$.

(iv) If $0 < p \leqslant 1$ and $\frac{t}{\varrho_p(t)}$ is non-decreasing, then $(\Gamma^p(v))' = \Gamma^{1,\infty}\left(\frac{d}{dt}\frac{t}{\varrho_p(t)}\right)$, where $\varrho_p(t) = \left(\int_0^\infty \frac{v(s)}{(s+t)^p}\, ds\right)^{1/p}$.

(v) If $1 < p < \infty$, then $(\Gamma^p(v))' = \Gamma^{p'}(\sigma)$, where

$$\sigma = t^{p'} \sum_k t \left(V(t) + t^p \int_t^\infty s^{-p} v(s)\, ds \right)^{-p'} \delta_{\mu_k}(t).$$

(vi) If $0 < p < \infty$, v is non-increasing and $tV^{-1/p}(t)$ is non-decreasing, then $(\Gamma^{p,\infty}(v))' = \Gamma^{1,\infty}(u) \cap \Lambda^1(u)$, where $u(t) = \frac{d}{dt}\left(\frac{t}{V^{1/p}(t)}\right)$.

REFERENCES

[1] M. ARIÑO AND B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.

[2] S. BARZA, L. E. PERSSON AND J. SORIA, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr **210** (2000), 43–58.

[3] C. BENNETT AND K. RUDNICK, *On Lorentz-Zygmund spaces*, Dissert. Math. **175** (1980), 1–72.

[4] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Academic Press, Princeton, 1988.

[5] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, 1976.

[6] D. W. BOYD, *The Hilbert transform on rearrangement-invariant spaces*, Canad. J. Math. **19** (1967), 599-616.

[7] M. CARRO, A. GARCÍA DEL AMO AND J. SORIA, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), 849–857.

[8] M. CARRO AND J. SORIA, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Analysis **112** (1993), 480–494.

[9] M. CARRO AND J. SORIA, *Boundedness of some integral operators*, Canad. J. Math. **45** (1993), 1155–1166.

[10] M. CARRO AND J. SORIA, *The Hardy-Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. **55** (1997), 146–158.

[11] M. L. GOL'DMAN, *On integral inequalities on a cone of functions with monotonicity properties*, Soviet Math. Dokl. **44** (1992), 581–587.

[12] M. L. GOL'DMAN, *On integral inequalities on the set of functions with some properties of monotonicity*, Function spaces, Differential Operators and Nonlinear Analysis, Teubner Texte Zur Math. 133, Leipzig, 1993, pp. 274–279.

[13] M. L. GOL'DMAN, H. P. HEINIG AND V. D. STEPANOV, *On the principle of duality in Lorentz spaces*, Canad. J. Math. **48** (1996), 959–979.

[14] I. HALPERIN, *Function spaces*, Canad. J. Math. **5** (1953), 273–288.

[15] H. P. HEINIG AND L. MALIGRANDA, *Weighted inequalities for monotone and concave functions*, Studia Math. **116** (1995), 133–165.

[16] S. LAI, *Weighted norm inequalities for general operators on monotone functions*, Trans. Amer. Math. Soc. **340** (1993), 811–836.

[17] G. G. LORENTZ, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.

[18] C. J. NEUGEBAUER, *Weighted norm inequalities for averaging operators of monotone functions*, Publ. Math. **35** (1991), 429–447.

[19] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.

[20] G. SINNAMON, *Spaces defined by level functions and their duals*, Studia Math. **111** (1994), 19–52.

[21] G. SINNAMON AND V. D. STEPANOV, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. **54** (1996), 89–101.

[22] J. SORIA, *Lorentz spaces of weak-type*, Quart. J. Math. Oxford **49** (1998), 93–103.

[23] V. D. STEPANOV, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338** (1993), 173–186.

[24] V. D. STEPANOV, *Integral operators on the cone of monotone functions*, J. London Math. Soc. **48** (1993), 465–487.

[25] V. D. STEPANOV, *On integral operators on the cone of monotone functions, and on imbeddings of Lorentz spaces*, Soviet Math. Dokl. **43** (1991), 620–623.

(Received February 24, 2000)
(Revised December 29, 2000)

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