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A SYSTEM OF GENERALIZED AUXILIARY PROBLEMS PRINCIPLE AND A SYSTEM OF VARIATIONAL INEQUALITIES

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Abstract. The approximation-solvability of a system of nonlinear variational and quasivariational inequalities $(SNVQVI)$

$$\langle F_1(x^*, y^*), x - x^* \rangle \geqslant 0 \quad \text{for all} \quad x \in X,$$

and

$$\langle F_2(x^*, y^*), g(y) - g(y^*) \rangle \geqslant 0 \quad \text{for all} \quad g(y) \in Y,$$

where X and Y, respectively, are nonempty closed convex subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$ and related $F_1 : X \times Y \to \mathbf{R}^m$ and $F_2 : X \times Y \to \mathbf{R}^n$ are any mappings such that $F = (F_1, F_2)$ is g-γ-partially relaxed monotone, is presented. Here $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

1. Introduction

It was Cohen [2], who first introduced the auxiliary problem principle to describe and analyze iterative optimization algorithms such as gradient or subgradient as well as decomposition/coordination algorithms, and later applied this approach to the approximation-solvability of a class of nonlinear mixed variational inequalities involving strongly monotone mappings in a reflexive Banach space setting. In this particular case, the estimate for the approximate solutions turns out to be a standard contraction type and so the sequence of approximate solutions converges to a given solution of the original nonlinear variational inequality. This auxiliary problem principle may work for other mappings but the outcome for the estimates may be totally different, for instance in the works of Zhu and Marcotte [22] for cocoercive mappings and of Verma [18] for partially relaxed monotone mappings, where the estimates are forced to hold for a finite-dimensional normed linear spaces, because of the limitations to the convergence analysis. If we choose some cocoercive or a partially relaxed monotone mapping in a reflexive Banach or a Hilbert space setting, the estimates turn out in such a manner that one can obtain only a bounded sequence which can have only a weakly convergent subsequence, unlike in a finite-dimensional normed space such as $\mathbf{R}^n$. Motivated by these developments, Verma [18] applied a generalized version of Cohen's auxiliary problem

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principle to the approximation-solvability of a class of mixed variational inequalities involving the partially relaxed monotone mappings — a computationoriented class — introduced by Verma [16] — which is more general than cocoercive and strongly monotone mappings — in a finite dimensional Hilbert space / $\mathbf{R}^n$ space setting and the convergence analysis is similar to that of Zhu and Marcotte [22].

We intend in this paper to discuss, based on a generalized auxiliary problems principle, the approximation-solvability of a system of nonlinear variational and quasi-variational inequalities involving the g - γ -partially relaxed monotone mappings in $\mathbf{R}^n$. The obtained results complement similar investigations by Cohen [2], Zhu and Marcotte [22], and Verma [16, 18, 19]. For more select details on variational inequalities and related algorithms, we recommend [1- 22].

Let $\langle \cdot\ ,\ \cdot \rangle$ and $||\cdot||$ denote, respectively, the scalar product and the Euclidean norm on $\mathbf{R}^n$. Let K be a nonempty closed convex subset of $\mathbf{R}^n$ and $T : K \to \mathbf{R}^n$ any mapping on K. For a positive definite matrix M, the matrix norm is defined by

$$||x||_M = \langle Mx, x\rangle^{1/2}.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be α-cocoercive [19] if for all x, $y \in K$, we have

$$||x-y||^2 \geqslant \alpha^2||T(x)-T(y)||^2 + ||\alpha(T(x)-T(y))-(x-y)||^2,$$

where $\alpha > 0$ is a constant.

A mapping $T : K \to \mathbf{R}^n$ is called α-cocoercive [3, 9] if there exists a constant $\alpha > 0$ such that

$$\langle T(x)-T(y), x-y\rangle \geqslant \alpha||T(x)-T(y)||^2 \quad \text{for all} \quad x, y \in K.$$

A mapping $T : K \to \mathbf{R}^n$ is called g - α -cocoercive if there exists a constant $\alpha > 0$ such that

$$\langle T(x)-T(y), g(x)-g(y)\rangle \geqslant \alpha||T(x)-T(y)||^2 \quad \text{for all} \quad x, y \in K,$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

T is called r-strongly monotone if for each x, $y \in K$, we have

$$\langle T(x)-T(y), x-y\rangle \geqslant r||x-y||^2 \quad \text{for a constant} \quad r > 0.$$

This implies that

$$||T(x)-T(y)|| \geqslant r||x-y||,$$

that is, T is r-expanding, and when $r = 1$, it is expanding. This class of mappings satisfies the following implications:

the r-monotonicity

$\downarrow$

the r-expansiveness

$\downarrow$

the expansiveness

A mapping $T : K \to \mathbf{R}^n$ is called g-r-strongly monotone if for each x, $y \in K$, we have

$$\langle T(x) - T(y), g(x) - g(y) \rangle \geqslant r||g(x) - g(y)||^2 \quad \text{for a constant} \quad r > 0,$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

This implies that

$$||T(x) - T(y)|| \geqslant r||g(x) - g(y)||,$$

that is, T is g-r-expanding, and when $r = 1$, it is g-expanding. This class of mappings satisfies the following implications:

$$\begin{array}{c} \text{the} \quad g\text{-}r\text{-monotonicity} \\ \downarrow \\ \text{the} \quad g\text{-}r\text{-expansiveness} \\ \downarrow \\ \text{the} \quad g\text{-expansiveness} \end{array}$$

A mapping $T : K \to \mathbf{R}^n$ is called r-relaxed monotone if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geqslant -r||x - y||^2 \quad \text{for all} \quad x, y \in K.$$

The mapping T is called β-Lipschitz continuous (or β-Lipschitzian) if there exists a constant $\beta \geqslant 0$ such that

$$||T(x) - T(y)|| \leqslant \beta||x - y|| \quad \text{for all} \quad x, y \in K.$$

A mapping $T : K \to \mathbf{R}^n$ is called g-β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$||T(x) - T(y)|| \leqslant \beta||g(x) - g(y)|| \quad \text{for all} \quad x, y \in K,$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

We note that if T is α-cocoercive and expanding, then T is α-strongly monotone. On the top of that, if T is α-strongly monotone and β-Lipschitz continuous, then T is (α/β^2)-cocoercive for $\beta > 0$. Clearly, every α-cocoercive mapping T is $(1/\alpha)$-Lipschitz continuous.

LEMMA 1.1. *For all elements v, $w \in K$, we have*

$$||v||^2 + \langle v, w \rangle \geqslant -(1/4)||w||^2.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be γ-partially relaxed monotone [16] if for all x, y, $z \in K$, we have

$$\langle T(x) - T(y), z - y \rangle \geqslant -\gamma||z - x||^2 \quad \textit{for} \quad \gamma > 0.$$

PROPOSITION 1.1 [16]. *Let $T : K \to \mathbf{R}^n$ be an α-cocoercive mapping on K. Then T is $(1/4\alpha)$-partially relaxed monotone.*

PROPOSITION 1.2. *Let $T : K \to \mathbf{R}^n$ be an α-partially relaxed monotone mapping on K. Then T is not, in general, $(1/4\alpha)$-cocoercive, that is, the converse of Proposition 1.1 is not true in general.*

Proof. Since T is α-partially relaxed monotone, applying Lemma 1.2, we have for all $x,\ y,\ z \in K$ that

$$\begin{aligned}\langle T(x) - T(y), x - y\rangle &= \langle T(x) - T(y), z - y\rangle + \langle T(x) - T(y), x - z\rangle \\ &\geqslant -\alpha||z - x||^2 + \langle T(x) - T(y), x - z\rangle \\ &= -\alpha\{||z - x||^2 + (1/\alpha)\langle T(x) - T(y), z - x\rangle\} \\ &\leqslant (1/4\alpha)||T(x) - T(y)||^2,\end{aligned}$$

that means, T is not $(1/4\alpha)$-cocoercive.

A mapping $T : K \to \mathbf{R}^n$ is said to be r-partially strongly monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), z - y\rangle \geqslant r||x - y||^2 \quad \text{for} \quad r > 0.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be γ-r-partially relaxed-strongly monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), z - y\rangle \geqslant -\gamma||z - x||^2 + r||x - y||^2 \quad \text{for constants} \quad \gamma, r > 0.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be γ-r-partially relaxed-relaxed monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), z - y\rangle \geqslant -\gamma||z - x||^2 - r||x - y||^2 \quad \text{for constants} \quad \gamma, r > 0.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be g-γ-partially relaxed relaxed monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), g(z) - g(y)\rangle \geqslant -\gamma||g(z) - g(x)||^2 \quad \text{for} \quad \gamma > 0,$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

A mapping $T : K \to \mathbf{R}^n$ is said to be g-r-partially strongly monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), g(z) - g(y)\rangle \geqslant r||g(x) - g(y)||^2 \quad \text{for} \quad r > 0.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be g-γ-r-partially relaxed-strongly monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x)-T(y), g(z)-g(y)\rangle \geqslant -\gamma||g(z)-g(x)||^2+r||x-y||^2 \quad \text{for constants} \quad \gamma, r > 0.$$

A mapping $T : K \to \mathbf{R}^n$ is said to be g-γ-r-partially relaxed-relaxed monotone [16] if for all $x,\ y,\ z \in K$, we have

$$\langle T(x) - T(y), g(z) - g(y)\rangle \geqslant -\gamma||g(z) - g(x)||^2 - r||g(x) - g(y)||^2 \quad \text{for constants} \quad \gamma, r > 0.$$

PROPOSITION 1.3. *Let* $T : K \to \mathbf{R}^n$ *be an* g - α *-cocoercive mapping on* K*. Then* T *is* g - $(1/4\alpha)$ *-partially relaxed monotone.*

Proof. For all x, y, $z \in K$, we have

$$\begin{aligned}
&\langle T(x) - T(y), g(z) - g(y)\rangle \\
&\qquad = \langle T(x) - T(y), g(x) - g(y)\rangle + \langle T(x) - T(y), g(z) - g(x)\rangle \\
&\qquad \geqslant \alpha||T(x) - T(y)||^2 + \langle T(x) - T(y), g(z) - g(x)\rangle \\
&\qquad = \alpha\{||T(x) - T(y)||^2 + (1/\alpha)\langle T(x) - T(y), g(z) - g(x)\rangle\} \\
&\qquad \geqslant -(1/4\alpha)||g(z) - g(x)||^2,
\end{aligned}$$

that is, T is g - $(1/4\alpha)$ -partially relaxed monotone.

PROPOSITION 1.4. *Let* $T : K \to \mathbf{R}^n$ *be a* g - α *-partially relaxed monotone mapping on* K*. Then* T *is not, in general,* g - $(1/4\alpha)$ *-cocoercive, that is, the converse of Proposition 1.3 is not true in general.*

Proof. Since T is g - α -partially relaxed monotone, applying Lemma 1.2, we have for all x, y, $z \in K$ that

$$\begin{aligned}
&\langle T(x) - T(y), g(x) - g(y)\rangle \\
&\qquad = \langle T(x) - T(y), g(z) - g(y)\rangle + \langle T(x) - T(y), g(x) - g(z)\rangle \\
&\qquad \geqslant -\alpha||g(z) - g(x)||^2 + \langle T(x) - T(y), g(x) - g(z)\rangle \\
&\qquad = -\alpha\{||g(z) - g(x)||^2 + (1/\alpha)\langle T(x) - T(y), g(z) - g(x)\rangle\} \\
&\qquad \leqslant (1/4\alpha)||T(x) - T(y)||^2,
\end{aligned}$$

that means, T is not g - $(1/4\alpha)$ -cocoercive.

We remark that the class of g -partially relaxed monotone mappings satisfies the following implications:

$$\begin{gathered}
\text{the} \quad g\text{-}r\text{-partial relaxed-strong monotonicity} \\
\downarrow \\
\text{the} \quad g\text{-}\gamma\text{-}r\text{-partial relaxed monotonicity} \\
\downarrow \\
\text{the} \quad g\text{-}\gamma\text{-partial relaxed monotonicity}
\end{gathered}$$

Before we conclude the section, we need to extend the notion of the γ -partial relaxed monotonicity to the case of a pair of $\mathbf{R}^m$ and $\mathbf{R}^n$. Let X and Y, respectively, be nonempty closed convex subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$. Let $F_1 : X \times Y \to \mathbf{R}^m$ and $F_2 : X \times Y \to \mathbf{R}^n$ be any mappings such that the mapping $F : X \times Y \to \mathbf{R}^m \times \mathbf{R}^n$ is defined by

$$F(x,y) = (F_1(x,y), F_2(x,y)) \quad \text{for all} \quad (x,y) \in X \times Y.$$

A mapping $F : X \times Y \to \mathbf{R}^m \times \mathbf{R}^n$ is said to be g - γ -partially relaxed monotone if there exists a constants $\gamma > 0$ such that for all x_1, $x_2 \in X$ and for all y_1, y_2, $y_3 \in Y$, we have

$$\langle F_1(x_1,y_1) - F_1(x_2,y_2), x_1 - x_2\rangle + \langle F_2(x_1,y_1) - F_2(x_2,y_2), g(y_3) - g(y_2)\rangle \geqslant -\gamma||g(y_3) - g(y_1)||^2,$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is any mapping.

We consider a system of nonlinear variational and quasivariational inequalities $(SNVQVI)$

$$\langle F_1(x^*,y^*), x - x^*\rangle \geqslant 0 \quad \text{for all} \quad x \in X, \tag{1.1}$$

and

$$\langle F_2(x^*,y^*), g(y) - g(y^*)\rangle \geqslant 0 \quad \text{for all} \quad g(y) \in Y. \tag{1.2}$$

2. Algorithms and solvability

Before we discuss the approximation-solvability of $SNVQVI$ (1.1)–(1.2), we need to describe the system of algorithms [21] as follows: for given initial iterates $x^0 \in X$ and $y^0 \in Y$, we have

$$\langle F_1(x^{k+1},y^k), x - x^{k+1}\rangle \geqslant 0 \quad \text{for all} \quad x \in X, \tag{2.1}$$

and

$$\langle \rho(F_2(x^{k+1},y^k) + M(g(y^{k+1})) - M(g(y^k)), g(y) - g(y^{k+1})\rangle + \langle h'(y^{k+1}) - h'(y^k), g(y) - g(y^{k+1})\rangle \geqslant 0 \quad \text{for all} \quad g(y) \in Y, \tag{2.2}$$

where h' is g-b-strongly monotone and g-c-Lipschitz continuous on Y, and M is a symmetric matrix.

For $M = 0$ in (2.1)–(2.2), we have a system of algorithms as follows: for given initial iterates $x^0 \in X$ and $y^0 \in Y$, compute

$$\langle F_1(x^{k+1},y^k), x - x^{k+1}\rangle \geqslant 0 \quad \text{for all} \quad x \in X, \tag{2.3}$$

and

$$\langle \rho F_2(x^{k+1},y^k) + h'(y^{k+1}) - h'(y^k), g(y) - g(y^{k+1})\rangle \geqslant 0 \quad \text{for all} \quad g(y) \in Y, \tag{2.4}$$

where h' is g-b-strongly monotone and g-c-Lipschitz continuous on Y. Here h' denotes the derivative of h.

LEMMA 2.1 [18]. *Let $h : K \to \mathbf{R}$ be continuously differentiable on a convex subset K of $\mathbf{R}^n$. Then we have the following conclusions:*

(i) If the gradient h' is g-b-strongly monotone, then

$$h(x) - h(y) \geqslant \langle h'(y), g(x) - g(y)\rangle + (b/2)||g(x) - g(y)||^2$$

for all $x, y \in K$ and $g(x), g(y) \in K$.

(ii) *If the gradient* h' *is* g-c*-Lipschitz continuous, then*

$$h(x) - h(y)\langle h'(y), g(x) - g(y)\rangle + (c/2)||g(x) - g(y)||^2$$

for all $x, y \in K$ *and* $g(x), g(y) \in K$*, where* $g : \mathbf{R}^n \to \mathbf{R}^n$ *is any mapping such that for any* $t \geqslant 0$*,* $g(tx) = tg(x)$*.*

The proof is included for the sake of the completeness.

Proof. (i) For any x, $y \in K$, we define a function

$$\Phi(t) := h((1-t)x + ty) - h(x) - t\langle h'(x), g(y) - g(x)\rangle \quad \text{for} \quad t \in [0, 1].$$

Then we have

$$\Phi'(t) = \langle h'((1-t)x + ty) - h'(x), g(y) - g(x)\rangle$$

and

$$\begin{aligned}
\Phi(1) - \Phi(0) =& h(y) - h(x) - \langle h'(x), g(y) - g(x)\rangle \\
=& \int_0^1 \Phi'(t)dt \quad \text{(by the Fundamental theorem of calculus)} \\
\geqslant& \int_0^1 (1/t)b||t(g(y) - g(x))||^2 dt \\
=& (b/2)||g(y) - g(x)||^2.
\end{aligned}$$

(ii) For any x, $y \in K$, we define a function

$$\Phi(t) := h((1-t)x + ty) - h(x) - t\langle h'(x), g(y) - g(x)\rangle \quad \text{for} \quad t \in [0, 1].$$

Then we have

$$\Phi'(t) = \langle h'((1-t)x + ty) - h'(x), g(y) - g(x)\rangle$$

and

$$\begin{aligned}
\Phi(1) - \Phi(0) =& h(y) - h(x) - \langle h'(x), g(y) - g(x)\rangle \\
=& \int_0^1 \Phi'(t)dt \quad \text{(by the Fundamental theorem of calculus)} \\
\leqslant& \int_0^1 |\Phi'(t)|dt \\
\leqslant& \int_0^1 c||t(g(y) - g(x))|| \quad ||(g(y) - g(x))||dt \\
=& (c/2)||(g(y) - g(x))||^2.
\end{aligned}$$

For $g \equiv I$, we arrive at [22]:

LEMMA 2.2. *Let* $h : K \to \mathbf{R}$ *be continuously differentiable on a convex subset* K *of* $\mathbf{R}^n$. *Then we have the following conclusions:*

(i) *If the gradient* h' *is* b*-strongly monotone, then*

$$h(x) - h(y) \geqslant \langle h'(y), x - y\rangle + (b/2)||x - y||^2 \quad \textit{for all} \quad x, y \in K.$$

(ii) *If the gradient* h' *is* c*-Lipschitz continuous, then*

$$h(x) - h(y) \leqslant \langle h'(y), x - y\rangle + (c/2)||x - y||^2 \quad \textit{for all} \quad x, y \in K.$$

We are just about ready to present, based on (2.1)–(2.2), the approximation-solvability of the *SNVQVI* problem (1.1)–(1.2) involving g-γ-partially relaxed monotone mappings in Euclidean spaces.

THEOREM 2.1. *Let* $F_1 : X \times Y \to \mathbf{R}^m$ *be* $r(y)$*-expanding in its first variable and* g-s*-Lipschitzian in its second variable. Suppose that* $F = (F_1, F_2)$ *is* g-γ*-partially relaxed monotone and* F_2 *is* g-μ*-Lipschitz continuous in its second variable. Then the sequence* $\{(x^k, y^k)\}$ *generated by the algorithm (2.1)–(2.2) converges to a solution of the SNVQVI (1.1)–(1.2) for g expansive and*

$$0 < \rho < (b/2\gamma)/[1 - (\lambda \min(M)/2\gamma)] \qquad \textit{if} \quad b(\lambda \min(G)) - 2\rho\gamma > 0$$

and

$$0 < \rho < \infty \qquad \textit{if} \quad b(\lambda \min(G)) - 2\rho\gamma \leqslant 0,$$

where $G = I + (\rho/b)M$ *is positive definite.*

Proof. Proof. Assume (x^*, y^*) is a solution of the *SNVQVI* (1.1)–(1.2) and define a function Λ^* by

$$\begin{aligned}\Lambda^*(g(y)) :=& h(y^*) - h(y) - \langle h'(y), g(y^*) - g(y)\rangle \\ &+ \rho/2\langle M(g(y) - g(y^*)), g(y) - g(y^*)\rangle \\ \geqslant& (b/2)||g(y^*) - g(y)||^2 + \rho/2\langle M(g(y) - g(y^*)), g(y) - g(y^*)\rangle \\ =& (b/2)||g(y) - g(y*)||_G^2 \quad \text{for} \quad g(y) \in Y,\end{aligned}$$

where $G = I + (\rho/b)M$ is positive definite.

Now we can write

$$\begin{aligned}\Lambda^*(g(y^k)) - \Lambda^*(g(y^{k+1})) =& h(y^{k+1}) - h(y^k) - \langle h'(y^k), g(y^{k+1}) - g(y^k)\rangle \\ &+ \langle h'(y^{k+1}) - h'(y^k), g(y^*) - g(y^{k+1})\rangle \\ &+ \rho/2\langle M(g(y^k) - g(y^*)), g(y^k) - g(y^*)\rangle \\ &- \rho/2\langle M(g(y^{k+1}) - g(y^*)), g(y^{k+1}) - g(y^*)\rangle \\ \geqslant& (b/2)||g(y^k) - g(y^{k+1})||^2 \\ &+ \rho\langle F_2(x^{k+1}, y^k), g(y^{k+1}) - g(y^*)\rangle \\ &+ \rho\langle M(g(y^{k+1}) - g(y^k)), g(y^{k+1}) - g(y^*)\rangle \\ &+ \rho/2\langle M(g(y^k) - g(y^*)), g(y^k) - g(y^*)\rangle \\ &- \rho/2\langle M(g(y^{k+1}) - g(y^*)), g(y^{k+1}) - g(y^*)\rangle .\end{aligned} \tag{2.3}$$

for $y = y^*$ in (2.2).

If we replace y by y^{k+1} in (1.2) and combine with (2.3), we obtain

$$\begin{aligned}\Lambda^*(g(y^k)) - \Lambda^*(g(y^{k+1})) &\geqslant (b/2)\|g(y^{k+1}) - g(y^k)\|^2 + \rho\langle F_2(x^{k+1}, y^k) \\ &\quad - F_2(x^*, y^*), g(y^{k+1}) - g(y^*)\rangle \\ &\quad + \rho\langle M(g(y^{k+1}) - g(y^k)), g(y^{k+1}) - g(y^*)\rangle \\ &\quad + \rho/2\langle M(g(y^k) - g(y^*)), g(y^k) - g(y^*)\rangle \\ &\quad - \rho/2\langle M(g(y^{k+1}) - g(y^*)), g(y^{k+1}) - g(y^*)\rangle \\ = (b/2)\|g(y^{k+1}) - g(y^k)\|^2 &+ \rho\langle F_2(x^{k+1}, y^k) - F_2(x^*, y^*), g(y^{k+1}) - g(y^*)\rangle \\ &\quad + \rho/2\langle M(g(y^k) - g(y^{k+1})), g(y^k) - g(y^{k+1})\rangle .\end{aligned} \tag{2.4}$$

Setting $x = x^{k+1}$ in (1.1) and $x = x^*$ in (2.1) and combining with (2.4) yields

$$\begin{aligned}\Lambda^*(g(y^k)) - \Lambda^*(g(y^{k+1})) &\geqslant (b/2)\|g(y^{k+1}) - g(y^k)\|^2 \\ &\quad + \rho\{\langle F_2(x^{k+1}, y^k) - F_2(x^*, y^*), g(y^{k+1}) - g(y^*)\rangle \\ &\quad + \langle F_1(x^{k+1}, y^k) - F_1(x^*, y^*), x^{k+1} - x^*\rangle \} \\ &\quad + \rho/2\langle M(g(y^k) - g(y^{k+1})), g(y^k) - g(y^{k+1})\rangle .\end{aligned} \tag{2.5}$$

It follows from the g-γ-partial relaxed monotonicity of F that

$$\begin{aligned}\Lambda^*(g(y^k)) - \Lambda^*(g(y^{k+1})) &\geqslant (b/2)\|g(y^{k+1}) - g(y^k)\|^2 - \rho\gamma\|g(y^{k+1}) - g(y^k)\|^2 \\ &\quad + \rho/2\langle M(g(y^k) - g(y^{k+1})), g(y^k) - g(y^{k+1})\rangle \\ &= (b/2)\|g(y^{k+1}) - g(y^k)\|_G^2 - \rho\gamma\|g(y^{k+1}) - g(y^k)\|^2 \\ &\geqslant (b/2)\lambda \min(G)\|g(y^{k+1}) - g(y^k)\|^2 - \rho\gamma\|g(y^{k+1}) - g(y^k)\|^2 \\ &= (1/2)[b\lambda \min(G) - 2\rho\gamma]\|g(y^{k+1}) - g(y^k)\|^2.\end{aligned} \tag{2.6}$$

Under the conditions of theorem, $\Lambda^*(g(y^k)) - \Lambda^*(g(y^{k+1}))$ is positive and so the sequence $\{\Lambda^*(g(y^k))\}$ is strictly decreasing for $y^k \neq y^{k+1}$. As a result, $\{\Lambda^*(g(y^k))\}$ converges to some number. This yields that difference of two successive terms tends to zero, and so we have

$$\|g(y^{k+1}) - g(y^k)\| \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\Lambda^*(g(y^k)) \geqslant (b/2)\|g(y^k) - g(y^*)\|^2$ and the sequence $\{\Lambda^*(g(y^k))\}$ is strictly decreasing, we can claim that the sequence $\{g(y^k)\}$ is bounded. Thus, there exists a subsequence $\{g(y^{k'})\}$ which converges to a limit point $g(y')$ of the sequence $\{g(y^k)\}$ on Y. Since g is expansive, it implies

$$\{y^{k'}\} \to y' \quad \text{as} \quad k \to \infty.$$

Next, assume that $x(y)$ satisfies

$$\langle F_1(x(y), y), x - x(y)\rangle \geqslant 0 \quad \text{for all} \quad x \in X, \tag{2.7}$$

and $x' = x(y')$. Since F_1 is $r(y)$-expanding in its first variable and g-s-Lipschitz continuous in its second variable, it implies

$$r(y)||x(y^k) - x(y')|| \leqslant ||F_1(x(y^k), y^k) - F_1(x(y'), y') \leqslant s||g(y^k) - g(y')|| \to 0.$$

that is, $x(y)$ is continuous in y. As $\{yk'\} \to y'$, it implies $x^{k'+1} \to x'$, and so (x', y') satisfies (1.1) for $(x^*, y^*) = (x', y')$. Since $||g(y^{k+1}) - g(y^k)|| \to 0$ and, $\{g(y^k)\} \to g(y')$ and $\{y^k\} \to y'$ as $k \to \infty$, taking limit in (2.2) yields that (x', y') satisfies (1.2) as well. Thus, (x', y') is a solution of the *SNVQVI* (1.1)–(1.2).

To show the entire sequence $\{(x^k, y^k)\} \to (x', y')$, assume $(x^*, y^*) = (x', y')$. Then, since h' is g-b-strongly monotone and g-c-Lipschitz continuous, it yields

$$(b/2)||g(y') - g(y^k)||^2 \leqslant \Lambda^*(g(y^k)) \leqslant (c/2)||g(y') - g(y^k)||^2 \quad \text{(by Lemma 2.1)}.$$

This and the convergence of the subsequence $\{g(y^{k'})\}$ to $g(y')$ yields the convergence of the entire sequence $\{g(y^k)\}$ to $g(y')$. As a result, the expansiveness of g implies that the entire sequence $\{yk\}$ converges to y'. This along with the continuity of $x(y)$ ensures the convergence of the entire sequence $\{x^{k+1}\}$ to x'. This completes the proof.

Theorem 2.1 also holds for F_1 being $r(y)$-strongly monotone in the first variable.

For $M = 0$, Theorem 2.1 reduces to Verma [18, Theorem 2.1]:

THEOREM 2.2. *Let $F_1 : X \times Y \to \mathbf{R}^m$ be $r(y)$-expanding in its first variable and g-s-Lipschitzian in its second variable. Suppose that $F = (F_1, F_2)$ is g-γ-partially relaxed monotone and F_2 is g-μ-Lipschitz continuous in its second variable. Then the sequence $\{(xk, yk)\}$ generated by the algorithm (2.3)–(2.4) converges to a solution of the SNVQVI (1.1)–(1.2) for $0 < \rho < b/2\gamma$ and for g expanding.*

We remark that one can obtain the extension of the algorithmic system (2.1)–(2.2) as follows: for given initial iterates $x^0 \in X$ and $y^0 \in Y$, find x^{k+1} and x^{k+1} such that

$$\langle F_1(x^{k+1}, y^k), x - x^{k+1}\rangle \geqslant 0 \quad \textit{for all} \quad x \in X, \tag{2.8}$$

and

$$\langle \rho(F_2(x^{k+1}, y^k) + L(y^{k+1}) - L(y^k) + M(g(y^{k+1})) - M(g(y^k)), g(y) - g(y^{k+1})\rangle + \langle h'(y^{k+1}) - h'(y^k), g(y) - g(y^{k+1})\rangle \geqslant 0 \quad \textit{for all} \quad g(y) \in Y, \tag{2.9}$$

where h' is g-b-strongly monotone and g-c-Lipschitz continuous on Y, M is a symmetric matrix and $L : Y \to \mathbf{R}^n$ is any mapping.

REFERENCES

[1] C. BAIOCCHI AND A. CAPELO, *Variational and Quasivariational Inequalities*, Wiley & Sons, New York, 1984.

[2] G. COHEN, *Auxiliary problem principle extended to variational inequalities*, J. Optim. Theo. Appl. **59** (2) (1988), 325–333.t.

[3] J. C. DUNN, *Convexity, monotonicity and gradient processes in Hilbert spaces*, J. Math. Anal. Appl. **53** (1976), 145–158.

[4] J. S. GUO AND J. C. YAO, *Extension of strongly nonlinear quasivariational inequalities*, Appl. Math. Letters **5** (3) (1992), 35–38.
[5] B. S. HE, *A new method for a class of linear variational inequalities*, Math. Programming **66** (1994), 137–144.
[6] B. S. HE, *A class of projection and contraction methods for monotone variational inequalities*, Applied Math. Optim. **35** (1997), 69–76.
[7] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities*, Academic Press, New York, 1980.
[8] G. M. KORPELEVICH, *The extragradient method for finding saddle points and other problems*, Matecon **12** (1976), 747–756.
[9] P. MARCOTTE AND J. H. WU, *On the convergence of projection methods*, J. Optim. Theory Appl. **85** (1995), 347–362.
[10] J. S. PANG AND D. CHAN, *Iterative methods for variational and complementarity problems*, Math. Programming **24** (1982), 284–313.
[11] R. U. VERMA, *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, ZAMM **77** (5) (1997), 387–391.
[12] R. U. VERMA, *RKKM mapping theorems and variational inequalities*, Math. Proc. Royal Irish Acad. **98A** (2) (1998), 131–138.
[13] R. U. VERMA, *Generalized pseudocontractions and nonlinear variational inequalities*, Publ. Debrecen **33** (1-2) (1998), 23–28.
[14] R. U. VERMA, *An iterative algorithm for a class of nonlinear variational inequalities involving generalized pseudocontractions*, Math. Sci. Res. Hot–Line **2** (5) (1998), 17–21.
[15] R. U. VERMA, *Strongly nonlinear quasivariational inequalities*, Math. Sci. Res. Hot–Line **3** (2) (1999), 11–18.
[16] R. U. VERMA, *Approximation–solvability of nonlinear variational inequalities involving partially relaxed monotone (prm) mappings*, Adv. Nonlinear Var. Inequal. **2** (2) (1999), 137–148.
[17] R. U. VERMA, *An extension of a class of nonlinear quasivariational inequality problems based on a projection method*, Math. Sci. Res. Hot–Line **3** (5) (1999), 1–10.
[18] R. U. VERMA, *Approximation–solvability of a system of nonlinear quasivariational inequalities involving partially relaxed monotone mappings*, Math. Sci. Res. Hot–Line **4** (4) (2000), 47–61.
[19] R. U. VERMA, *On a class of nonlinear implicit variational inequalities involving partially relaxed monotone and partially strongly monotone mappings*, Math. Inequal. Appl. (to appear).
[20] R. U. VERMA, *A new class of iterative algorithms for approximation–solvability of nonlinear variational inequalities*, Computers Math. Appl. (to appear).
[21] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications I*, Springer–Verlag, New York, 1986.
[22] D. L. ZHU AND P. MARCOTTE, *Co–coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optim. **6** (3) (1996), 714–726.

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