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ON MULTIVALUED GENERAL MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and anlayze a new class of predictor-corrector algorithms for solving multivalued general mixed variational inequalities by using the auxiliary principle technique. The convergence of the proposed method only requires partially relaxed strongly monotonicity of the operator, which is weaker than co-coercivity. As special cases, we obtain a number of known and new results for solving various classes of variational inequalities. Our results represent a refinement of the previously known results.

1. Introduction

Variational inequalities theory has emerged an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, physical, regional, social, pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. Variational inequalities have been generalized and extended in different directions using the novel and innovative techniques. An important and useful generalization of variational inequalities is called the multivalued general mixed variational inequality. For applications and numerical methods, see [3-10] and the references therein. There are several numerical methods including resolvent equations, descent and decomposition for solving mixed variational inequalities. To implement the resolvent method and its variant forms, one has to calculate the resolvent of the operator, which is itself a difficult problem. To overcome this drawback, the auxiliary principle technique has been developed. Glowinski, Lions and Tremolieres [4] used this technique to study the existence of a solution of the mixed variational inequalities whereas Noor [5-7] has used the auxiliary principle technique to develop some iterative methods for solving various classes of variational inequalities and optimization problems. It has been shown that a substantial number of numerical methods can be obtained as special cases from this technique, see [5-7] and references therein. On the other hand, there are no such type of predictor-corrector methods for solving multivalued general mixed variational inequalities. The main reason is that the the technique of updating the solution cannot be extended for multivalued general mixed variational inequalities. In this paper, we use the auxiliary principle technique to suggest a class of two-step predictor-corrector iterative methods for multivalued variational inequalities. In particular, we show that one can obtain various forward-backward splitting, modified

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resolvent, and other methods as special cases form these methods. We also prove that the convergence of the suggested methods requires only the partially relaxed strongly monotonicity, which is a weaker condition than the co-coercivity. Consequently, our results represent an improvement and refinement of the previously known results. Our results can be considered as an extension of the results of Noor [9,10] and Tseng [15] for solving general mixed variational inequalities and complementarity problems.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let $C(H)$ be a family of all nonempty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator and $g : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty closed convex set in H and $\varphi : H \longrightarrow R \cup \{+\infty\}$ be a nonlinear nondifferentiable function.

We consider the problem of finding $u \in H, \nu \in T(u)$ such that

$$\langle \nu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \qquad \forall g(v) \in H. \tag{2.1}$$

The inequality of type (2.1) is called the *multivalued general mixed variational inequality*. It can be shown that a wide class of multivalued odd order and nonsymmetric free, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequalities (2.1), see, for example, [8,14].

We note that if $T : H \longrightarrow H$ is a single-valued operator, then problem(2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \quad \forall g(v) \in H, \tag{2.2}$$

which is known as the general mixed variational inequality, see [5,6,8-10]. It can be shown that a class of quasi variational inequalities and nonconvex programming problems can be studied by the general variational inequality approach.

We remark that if $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in H, \nu \in T(u)$ such that

$$\langle \nu, v - u \rangle + \varphi(v) - \varphi(u) \geqslant 0, \qquad \forall v \in H, \tag{2.3}$$

which are called the generalized mixed variational inequalities. For the applications, numerical methods and formulations, see [3-15].

If φ is the indicator function of a closed convex set K in H, then problem (2.4) is equivalent to finding $u \in H$, $g(u) \in K$, $\nu \in T(u)$ such that

$$\langle \nu, g(v) - g(u) \rangle \geqslant 0, \quad \forall g(v) \in K, \tag{2.4}$$

which is known as the multivalued variational inequality, introduced and studied by Noor [7] recently. In particular, for $g \equiv I$, the identity operator, then problem is called the generalized variational inequality problem introduced and studied by Fang and Peterson [2].

If $K^* = \{u \in H : \langle u, v \rangle \geqslant 0, \forall v \in K\}$ is a polar cone of a convex cone K in H, then problem (2.4) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad \nu \in T(u) \subseteq K^*, \text{and} \quad \langle \nu, g(u) \rangle = 0, \tag{2.5}$$

which is known as the multivalued complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem(2.5) is called the multivalued quasi(implicit) complementarity problem, see the references for the formulation and numerical methods.

If φ is a proper, convex and lower semicontinuous function, then problem (2.1) is equivalent to finding $u \in H, \nu \in T(u)$ such that

$$0 \in \nu + \partial\varphi(g(u)) \tag{2.6}$$

which is called the problem of finding a zero of a sum of two maximal monotone operators. Such type of problems have been studied extensively in recent years by many authors, see, for example, [3,11,14,15] and the references therein.

We also need the following well known result and concepts.

LEMMA 2.1. *$\forall u, v \in H$, we have*

$$\langle u, v \rangle = \frac{1}{2}\{||u+v||^2 - ||u||^2 - ||v||^2\}. \tag{2.7}$$

DEFINITION 2.1. $\forall u_1, u_2, z \in H, w_1 \in T(u_1), w_2 \in T(u_2)$, the multivalued operator $T : H \to C(H)$ is said to be:

(i) *g-partially relaxed strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle w_1 - w_2, g(z) - g(u_2) \rangle \geqslant -\alpha||g(u_1) - g(z)||^2$$

(ii) *g-co-coercive*, if there exists a constant $\mu > 0$ such that

$$\langle w_1 - w_2, g(u_1) - g(u_2) \rangle \geqslant \mu||w_1 - w_2||^2.$$

(iii) *M-Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$M(T(u_1), T(u_2)) \leqslant \delta||u_1 - u_2||,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$. We remark that if $z = u_1$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator T. For $g \equiv I$, the indentity operator, Definition 2.1 reduces to the definition of partially relaxed strongly monotonicity and co-coercivity of the operator. It has been shown in [7] that g-co-coercivity implies g-partially relaxed strongly monotonicity, but not conversely. Consequently, it follows that the concept of g-partially relaxed strongly monotonicity is weaker than co-coercivity.

3. Main Results

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [4].

For a given $u \in H$, $g(u) \in K$, consider the problem of finding a unique $w \in H$, $\eta \in T(w)$ satisfying the auxiliary variational inequality

$$\langle \rho\eta + g(w) - g(u), g(v) - g(w)\rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \quad \forall g(v) \in H, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the multivalued general mixed variational inequality (2.1). This observation enables us to suggest the following predictor-corrector method for solving the multivalued general mixed variational inequalities (2.1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle + \varphi(g(v)) - \varphi(g(u_{n+1})) \geqslant 0, \quad \forall g(v) \in H \tag{3.2}$$

$$\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(w_{n+1}), T(w_n)) \tag{3.3}$$

$$\langle \beta\nu_n + g(w_n) - g(u_n), g(v) - g(w_n)\rangle + \varphi(g(v)) - \varphi(g(w_n)) \geqslant 0, \quad \forall g(v) \in H \tag{3.4}$$

$$\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leqslant M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Note that if $g \equiv I$, the identity operator, then Algorithm 3.1 reduces to the following predictor-corrector method for solving the mixed variational inequalities (2.3), which appears to be a new one.

ALGORITHM 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned}
&\langle \rho\eta_n + u_{n+1} - w_n, v - u_{n+1}\rangle + \varphi(v) - \varphi(u_{n+1}) \geqslant 0, \quad \forall v \in H, \\
&\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(w_{n+1}), T(w_n)) \\
&\langle \beta\nu_n + w_n - u_n, v - w_n\rangle + \varphi(v) - \varphi(w_n) \geqslant 0, \quad \forall v \in H \\
&\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leqslant M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots
\end{aligned}$$

Using the technique of the resolvent operator, Algorithm 3.1 can be written as

ALGORITHM 3.3. For a given $u_0 \in H$, compute u_{n+1} such that $\eta_n \in T(w_n)$, $\nu_n \in T(u_n)$, by the iterative schemes

$$\begin{aligned}
g(u_{n+1}) &= J_\varphi[g(w_n) - \rho\eta_n], \\
g(w_n) &= J_\varphi[g(u_n) - \beta\nu_n], \quad n = 0, 1, 2, \ldots
\end{aligned}$$

where $J_\varphi = (I+\rho\partial\varphi)^{-1}$ is the resolvent operator associated with the maximal monotone operator $\partial\varphi$, the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \longrightarrow R \cup \{\infty\}$, see [5,9]. Algorithm 3.3 is a two-step forward-backward splitting method for solving multivalued general mixed variational inequalities (2.1), which appears to be a new one.

If T is a single-valued operator, then algorithm 3.1 collapses to the following predictor-corrector method for solving general mixed variational inequalities (2.2), which is due to Noor [5].

ALGORITHM 3.4. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\langle \rho T(w_n)+g(u_{n+1})-g(w_n), g(v)-g(u_{n+1})\rangle +\varphi(g(v))-\varphi(g(u_{n+1}))\geqslant 0, \quad \forall g(v)\in H$$
$$\langle \beta T(u_n)+g(w_n)-g(u_n), g(v)-g(w_n) > +\varphi(g(v))-\varphi(g(w_n))\geqslant 0, \quad \forall g(v)\in H.$$

We remark that Algorithm 3.4 can be written in the equivalent form as

ALGORITHM 3.5. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$g(w_n) = J_\varphi[g(u_n) - \beta T u_n]$$
$$g(u_{n+1}) = J_\varphi[g(w_n) - \rho T w_n], \quad n = 0,1,2\ldots$$

which can be written in the following form, if g is invertible,

$$g(u_{n+1}) = J_\varphi[I-\rho T g^{-1}]J_\varphi[I-\beta T g^{-1}]g(u_n), \quad n=0,1,2\ldots$$

which is a two-step forward-backward splitting algorithms. For $g \equiv I$, the identity operator, Algorithm 3.5 can be written as

ALGORITHM 3.6. For a given $u_0 \in H,$ compute the sequence $\{u_n\}$ by the iterative scheme

$$u_{n+1} = J_T\{[J_\varphi[I-\rho T]J_\varphi[I-\rho T]+\rho T]u_n\}, \quad n=0,1,2,\ldots$$

which is called the two-step forward-backward splitting method and $J_T = (I+\rho T)^{-1}$ is the resolvent operator associated with the maximal monotone operator $T.$ If $J_T \equiv P_K,$ the projection of H onto the closed convex set K, then Algorithm 3.6 coincides with the modified forward-backward splitting method of Tseng [15], which he suggested by using the updating technique of the solution. For the applications of the splitting methods in partial differential equations, see Ames [1] and the references therein. For appropriate and suitable choice of the operators and the space H, one can obtain various new and known methods for solving variational inequalities and complementarity problems.

Using the technique of Noor [5-7], one can study convergence analysis of Algorithm 3.1. However, we include its proof for the sake of completeness.

LEMMA 3.1. *Let $u \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $T : H \longrightarrow C(H)$ is a g-partially relaxed strongly monotone operator with constant $\alpha > 0$, then*

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(u_n) - g(u)||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2. \quad (3.6)$$

Proof. Let $u \in H$, $\quad \nu \in T(u)$ be solution of (2.1). Then

$$\langle \rho\nu, g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \quad \forall g(v) \in H \quad (3.7)$$
$$\langle \beta\nu, g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \quad \forall g(v) \in H, \quad (3.8)$$

where $\rho > 0$ and $\beta > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.7) and $v = u$ in (3.2), we have

$$\langle \rho\nu, g(u_{n+1}) - g(u)\rangle + \varphi(g(u_{n+1})) - \varphi(g(u)) \geqslant 0 \quad (3.9)$$

and

$$\langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle + \varphi(g(u)) - \varphi(g(u_{n+1})) \geqslant 0. \quad (3.10)$$

Adding (3.10) and (3.9), we have

$$\begin{aligned}\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &\geqslant \rho\langle \eta_n - \nu, g(u_{n+1}) - g(u)\rangle \\ &\geqslant -\alpha\rho||g(u_{n+1}) - g(w_n)||^2,\end{aligned} \quad (3.11)$$

where we have used the fact that T is g-partially relaxed strongly monotone with constant $\alpha > 0$.

Setting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(w_n)$ in (2.6), we obtain

$$\begin{aligned}\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &= \frac{1}{2}\{||g(u) - g(w_n)||^2 - ||g(u) - g(u_{n+1})||^2 \\ &\quad - ||g(u_{n+1}) - g(w_n)||^2\}.\end{aligned} \quad (3.12)$$

Combining (3.11) and (3.12), we have

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(w_n) - g(u)||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2. \quad (3.13)$$

Taking $v = u$ in (3.4) and $v = w_n$ in (3.8), we have

$$\langle \beta\nu, g(w_n) - g(u)\rangle + \varphi(g(w_n)) - \varphi(g(u)) \geqslant 0 \quad (3.14)$$

and

$$\langle \beta\nu_n + g(w_n) - g(u_n), g(u) - g(w_n)\rangle + \varphi(g(u)) - \varphi(g(w_n)) \geqslant 0. \quad (3.15)$$

Adding (3.14) and (3.15) and rearranging the terms, we have

$$\begin{aligned}\langle g(w_n) - g(u_n), g(u) - g(w_n)\rangle &\geqslant \beta\langle \nu_n - \nu, g(w_n) - g(u)\rangle \\ &\geqslant -\beta\alpha||g(y_n) - g(w_n)||^2,\end{aligned} \quad (3.16)$$

since T is a g-partially relaxed strongly monotone operator with constant $\alpha > 0$.

Now taking $v = g(w_n) - g(u_n)$ and $u = g(u) - g(w_n)$ in (2.6), (3.16) can be written as

$$||g(u) - g(w_n)||^2 \leqslant ||g(u) - g(u_n)||^2 - (1 - 2\beta\alpha)||g(u_n) - g(w_n)||^2 \\ \leqslant ||g(u) - g(u_n)||^2, \quad \text{for } 0 < \beta < 1/2\alpha. \tag{3.17}$$

Now

$$||g(u_{n+1}) - g(w_n)||^2 = ||g(u_{n+1}) - g(u_n) + g(u_n) - g(w_n)||^2 \\ = ||g(u_{n+1}) - g(u_n)||^2 + ||g(u_n) - g(w_n)||^2 \\ + 2 < g(u_{n+1}) - g(u_n), g(u_n) - g(w_n) > . \tag{3.18}$$

Combining (3.13), (3.17) and (3.18), we obtain

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(u_n) - g(u)||^2 - (1 - 2\beta\alpha)||g(u_{n+1}) - g(u_n)||^2,$$

the required result (3.6).

□

THEOREM 3.1. *Let H be a finite dimensional space. Let $g : H \longrightarrow H$ be invertible and $0 < \rho < \frac{1}{2\alpha}$. Let $T : H \longrightarrow C(H)$ be M-Lipschitz continuous operator. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in H$ is the exact solution of (2.1), then $\lim_{n\to\infty} u_n = u$.*

Proof. Let $u \in H$ be a solution of (1). Since $0 < \rho < \frac{1}{2\alpha}$. From (3.6) ,it follows that the sequence $\{||g(u) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)||g(u_{n+1}) - g(u_n)||^2 \leqslant ||g(u_0) - g(u)||^2,$$

which implies that

$$\lim_{n\to\infty} ||g(u_{n+1}) - g(u_n)|| = 0. \tag{3.19}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n and u_n by u_{n_j} in (3.2) and (3.4) taking the limit $n_j \longrightarrow \infty$ and using (3.19), we have

$$\langle \hat{v}, g(v) - g(\hat{u})\rangle + \varphi(g(v)) - \varphi(g(\hat{u})) \geqslant 0, \quad \forall g(v) \in H,$$

which implies that $\hat{u}$ solves the multivalued mixed variational inequality (2.1) and

$$\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is invertible, thus

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

It remains to show that $v \in T(u)$. From (3.5) and using the M-Lipschitz continuity of T, we have

$$||v_n - v|| \leqslant M(T(u_n), T(u)) \leqslant \delta||u_n - u||,$$

which implies that $v_n \longrightarrow v$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(v, T(u)) &\leqslant ||v - v_n|| + d(v, T(u)) \\ &\leqslant ||v - v_n|| + M(T(u_n), T(u)) \\ &\leqslant ||v - v_n|| + \delta||u_n - u|| \quad \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

where $d(v, T(u)) = inf\{||v - z|| : z \in T(u)\}$. and $\delta > 0$ is the M-Lipschitz continuity constant. From the above inequality, it follows that $d(v, T(u)) = 0$. This implies that $v \in T(u)$, since $T(u) \in C(H)$. This completes the proof.

□

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