Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 465–479

ANALYSIS OF AN ELASTIC CONTACT PROBLEM WITH SLIP DEPENDENT COEFFICIENT OF FRICTION

C. CIULCU, D. MOTREANU AND M. SOFONEA

(*communicated by R. U. Verma*)

Abstract. We consider the problem of frictional contact between an elastic body and an obstacle, say a foundation. The elastic constitutive law is assumed to be nonlinear and the contact is modeled with a simplified version of Coulomb's law of dry friction. The novelty consists in the fact that the coefficient of friction depends on the slip. We present two alternative yet equivalent weak formulations of the problem and establish existence, uniqueness and continuous dependence results. The proofs are based on a new result obtained in [10] in the study of elliptic quasivariational inequalities. Moreover, we study the behavior of the solution with respect to the coefficient of friction and obtain a convergence result.

1. Introduction

We investigate a model for the process of frictional contact between an elastic body, which is acted upon by volume forces and surface tractions, and an obstacle, say a foundation. Situations which involve such type of problems abound in industry and everyday life. The contact of the braking pads with the wheel, the tire with the road and the piston with skirt are just three simple examples. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations and the engineering literature concerning this topic is extensive.

An early attempt to study frictional contact problems for elastic and viscoelastic materials within the framework of variational inequalities was made in [4]. Steady–state as well as time–dependent frictional contact problems for linearly and nonlinearly elastic materials may be found in [14]. In both these two books existence and uniqueness results for static contact problems using a simplified version of Coulomb's law of dry friction may be found. Important results in the study of the Signorini contact problem with non–local versions of Coulomb's law were obtained in [1, 3, 12], among others.

An important step in the understanding of the *stick–slip phenomenon* was done in [15] where it is pointed out that the coefficient of friction μ varies with the tangential displacement, i.e.

$$\mu = \mu(|\boldsymbol{u}_\tau|).$$

In this way a part of the elasto–plastic deformation of the interface is captured in the model. Stick–slip is then a result of the *slip weakening*, i.e. the fall of the friction force

Mathematics subject classification (2000): 74M10, 35J85, 49J40.

Key words and phrases: nonlinear elastic material, Coulomb's law, slip–dependent coefficient of friction, quasivariational inequality, weak solution, directional derivative.

Paper MIA-04-41

with slip. Generally speaking, the dependence of the friction forces upon the surface displacements is usually accepted when the slip is very small on laboratory scales (see for instance [13] and [16]).

The study of an elastostatic contact problem with slip dependent friction was presented in [6]. There, a linear elastic constitutive law has been considered and the existence of a weak solution to the model was proved by studying the local minima of the energy function. The uniqueness of the solution as well as its continuous dependence upon the loads have also been discussed.

The aim of this paper is to complete the study of the elastic contact problem presented in [6]. Thus, we consider here the case of nonlinear elastic constitutive laws, we relax the assumptions on the coefficient of friction and we use new arguments to study two variational formulations of the problem, in terms of the displacement and the stress field, respectively. For the variational problem in displacements we prove the existence of the solution using a new result obtained recently in [10]. Then, we present an equivalence result which allows us to deduce the existence of the solution for the variational problem in terms of stress. We also investigate the uniqueness of the solution as well as its dependence with respect to the external data and with respect to the coefficient of friction.

The paper is structured as follows. In Section 2 we state the mechanical problem and discuss the frictional contact conditions. In Section 3 we present the notation and preliminary material, list the assumption on the problem data, derive the variational formulations of the model and state our main results, Theorems 3.2–3.4. The proofs are established in Section 5 and are based on an abstract result in the study of elliptic quasivariational inequalities that we recall in Section 4. Finally, in Section 6 we study the dependence of the weak solution with respect to the coefficient of friction and prove a convergence result.

2. Problem statement

In this section we describe a model for the process and we discuss the boundary conditions. The physical setting is as follows. An elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. A volume force of density $\boldsymbol{f}_0$ acts in Ω and a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. The body is clamped on Γ_1 and thus the displacement field vanish there. On Γ_3 the body is in contact with an obstacle, the so–called foundation. We model the contact with a simplified version of Coulomb's law of dry friction, already used in [6], in which the coefficient of friction depends on the slip. Finally, we denote by $\mathbb{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of the symmetric matrices of order d.

The classical formulation of the contact problem is the following.

PROBLEM P. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}_d$

such that

$$\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u})) \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{2.4}$$

$$\sigma_\nu = S \qquad \text{on } \Gamma_3, \tag{2.5}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leqslant \mu(|\boldsymbol{u}_\tau|)\,|S|, \\ |\boldsymbol{\sigma}_\tau| \leqslant \mu(0)\,|S| & \text{if} \quad \boldsymbol{u}_\tau = \boldsymbol{0}, \\ \boldsymbol{\sigma}_\tau = -\mu(|\boldsymbol{u}_\tau|)|S|\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} & \text{if} \quad \boldsymbol{u}_\tau \neq \boldsymbol{0} \end{cases} \qquad \text{on} \quad \Gamma_3. \tag{2.6}$$

In (2.1)–(2.6) and below, in order to simplify the notation, we usually do not indicate explicitely the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equality (2.1) represents the elastic constitutive law of the material in which F is a given nonlinear function and $\varepsilon(\boldsymbol{u})$ denotes the small strain tensor; (2.2) represents the equilibrium equation, (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which $\boldsymbol{\nu}$ denotes the unit outward normal on Γ and $\boldsymbol{\sigma\nu}$ represents the Cauchy stress tensor. Finally, (2.5) and (2.6) represent the contact boundary conditions.

Condition (2.5) states that the normal stress σ_ν is prescribed on Γ_3 since here S denotes a given function. Condition (2.6) represents the associate friction law in which $\boldsymbol{\sigma}_\tau$ is the tangential stress, $\boldsymbol{u}_\tau$ denotes the tangential displacement and μ is the coefficient of friction. This law should be seen either as a mechanical model suitable for the proportional loadings or as a first approximation of a more realistic model, based on a friction law involving the time derivative of $\boldsymbol{u}_\tau$ (see for instance [2], [9]). Note that in (2.6) the coefficient of friction depends on the slip $|\boldsymbol{u}_\tau|$, which leads to a new and nonstandard mathematical problem.

3. Variational formulations and main results

In this section we derive two variational formulations for the mechanical problem (2.1)–(2.6), list the assumptions on the data and state our main existence and uniqueness results. To this end we need to introduce notation and preliminary material. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}_d$ by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}, \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}_d\,.$$

Here and below, $i,j = 1, 2, \ldots, d$, and the summation convention over repeated indices is adopted. Moreover, in the sequel, the index that follows a comma indicates a partial derivative, e.g., $u_{i,j} = \partial u_i / \partial x_j$.

We now introduce function spaces for the variables. Let

$$H = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \qquad H_1 = \{\boldsymbol{u} = (u_i) \mid u_i \in H^1(\Omega)\},$$

$$\mathscr{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathscr{H}_1 = \{\boldsymbol{\sigma} \in \mathscr{H} \mid \sigma_{ij,j} \in H\}.$$

The spaces H, $\mathscr{H}$, H_1 and $\mathscr{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i\,dx, \qquad (\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}},$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathscr{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H,$$

respectively. Here $\boldsymbol{\varepsilon} : H_1 \to \mathscr{H}$ and $\mathrm{Div} : \mathscr{H}_1 \to H$ are the *deformation* and the *divergence* operators, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms on the spaces H, $\mathscr{H}$, H_1 and $\mathscr{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathscr{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathscr{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\boldsymbol{v} \in H_1$, we also use the notation $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and *tangential* components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}. \tag{3.1}$$

Let H'_Γ be the dual of H_Γ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\boldsymbol{\sigma} \in \mathscr{H}_1$, $\boldsymbol{\sigma\nu}$ can be defined as the element in H'_Γ which satisfies

$$\langle \boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}\rangle = (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\boldsymbol{v})_H \qquad \forall\,\boldsymbol{v}\in H_1. \tag{3.2}$$

Denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the *normal* and *tangential* traces of $\boldsymbol{\sigma}$, respectively. If $\boldsymbol{\sigma}$ is continuously differentiable on $\Omega\cup\Gamma$, then

$$\sigma_\nu = (\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}, \tag{3.3}$$

$$\langle \boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}\rangle = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \tag{3.4}$$

for all $\boldsymbol{v}\in H_1$, where da is the surface measure element.

In the sequel we use V to denote the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v}\in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}.$$

Since *meas* $(\Gamma_1) > 0$, the following Korn's inequality holds:

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathscr{H}} \geqslant c_K\,|\boldsymbol{v}|_{H_1} \quad \forall\,\boldsymbol{v}\in V, \tag{3.5}$$

see, e.g., [11]. Here c_K denotes a positive constant which depends only on Ω and Γ_1.

On V we consider the inner product given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} \tag{3.6}$$

and let $|\cdot|_V$ be the associated norm. It follows from Korn's inequality (3.5) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V,|\cdot|_V)$ is a real Hilbert space.

Moreover, by the Sobolev's trace theorem and (3.5) we have a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)} \leqslant c_0 |\boldsymbol{v}|_V \qquad \forall \boldsymbol{v} \in V. \tag{3.7}$$

In the study of the mechanical problem (2.1)–(2.6) we assume that

$$\begin{cases} \text{(a) } F : \Omega \times \mathbb{S}_d \to \mathbb{S}_d . \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad |F(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - F(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leqslant M |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \qquad \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d ,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m > 0 \text{ such that} \\ \qquad (F(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - F(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m\, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \qquad \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d ,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) For any } \boldsymbol{\varepsilon} \in \mathbb{S}_d ,\ \boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{3.8}$$

$$\boldsymbol{f}_0 \in H , \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{3.9}$$

$$S \in L^\infty(\Gamma_3) \quad \text{and} \quad |S|_{L^\infty(\Gamma_3)} > 0. \tag{3.10}$$

$$\begin{cases} \text{(a) } \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ . \\ \text{(b) There exists } c_1^\mu \geqslant 0 \text{ and } c_2^\mu \geqslant 0 \text{ such that} \\ \qquad |\mu(\boldsymbol{x}, r)| \leqslant c_1^\mu |r| + c_2^\mu \quad \text{for all } r \in \mathbb{R}_+,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 . \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mu(\boldsymbol{x}, r) \text{ is Lebesgue measurable} \\ \quad \text{on } \Gamma_3,\ \text{for all } r \in \mathbb{R}_+ . \\ \text{(d) The mapping } r \mapsto \mu(\boldsymbol{x}, r) \text{ is continuous on } \mathbb{R}_+,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 . \end{cases} \tag{3.11}$$

$$\begin{cases} \text{There exists } L_\mu \geqslant 0 \text{ such that} \\ (\mu(\boldsymbol{x}, r_1) - \mu(\boldsymbol{x}, r_2))\, (r_2 - r_1) \leqslant L_\mu |r_1 - r_2|^2 \\ \forall r_1, r_2 \in \mathbb{R}_+,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 . \end{cases} \tag{3.12}$$

We make in the sequel some comments on the assumptions (3.8)–(3.12).

First, using the condition (3.8), we see that for all $\boldsymbol{\tau} \in \mathcal{H}$ the function $\boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{\tau}(\boldsymbol{x}))$ belongs to $\mathcal{H}$ and hence we may consider F as an operator defined on $\mathcal{H}$ with the range on $\mathcal{H}$. Moreover, $F : \mathcal{H} \to \mathcal{H}$ is a strongly monotone Lipschitz continuous operator and therefore F is invertible and its inverse $F^{-1} : \mathcal{H} \to \mathcal{H}$ is also a strongly monotone Lipschitz continuous operator.

We note that condition (3.8) is satisfied in the case of the linear elastic constitutive law

$$\sigma_{ij} = a_{ijkh} \varepsilon_{kh}(\boldsymbol{u}), \tag{3.13}$$

provided that $a_{ijkh} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$a_{ijkh}(\boldsymbol{x})\, \xi_h \xi_l \geqslant \alpha |\boldsymbol{\xi}|^2 \qquad \forall \boldsymbol{\xi} \in \mathbb{S}_d ,\ \ \text{a.e. } \boldsymbol{x} \in \Omega.$$

To provide examples of nonlinear constitutive laws which satisfy (3.8), for every tensor $\boldsymbol{\xi} \in \mathbb{S}_d$ we denote by $tr\,\boldsymbol{\xi}$ the trace of $\boldsymbol{\xi}$ and by $\boldsymbol{\xi}^D$ the deviatoric part of $\boldsymbol{\xi}$ given by

$$tr\,\boldsymbol{\xi} = \xi_{ii}, \qquad \boldsymbol{\xi}^D = \boldsymbol{\xi} - \frac{1}{d}(tr\,\boldsymbol{\xi})\boldsymbol{I}_d,$$

where $\boldsymbol{I}_d$ represents the identity tensor of the second order. Let K denote a nonempty closed convex set in $\mathbb{S}_d$ and let P_K represent the projection mapping. We also consider a fourth order symmetric and positively defined tensor $\mathscr{E} : \mathbb{S}_d \to \mathbb{S}_d$ and we take

$$F(\boldsymbol{\xi}) = \mathscr{E}\boldsymbol{\xi} + \frac{1}{\lambda}(\boldsymbol{\xi} - P_K\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \mathbb{S}_d\,, \tag{3.14}$$

where $\lambda > 0$. Using the properties of the projection mapping, it is straightforward to see that the elasticity operator F given by (3.14) satisfies condition (3.8). Constitutive laws of the form (3.14) have been considered by many authors, see e.g. [8], [14] and [17].

A second example of nonlinear elastic equations of the form (2.1) is provided by nonlinear Hencky materials (for detail, cf. e.g., [18]). For a Hencky material, the stress–strain relation is given by

$$\boldsymbol{\sigma} = K_0 \mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})\,\boldsymbol{I}_d + \psi(|\boldsymbol{\varepsilon}^D(\boldsymbol{u})|^2)\,\boldsymbol{\varepsilon}^D(\boldsymbol{u}),$$

so that the elasticity operator is

$$F(\boldsymbol{\xi}) = K_0 \mathrm{tr}\,\boldsymbol{\xi}\,\boldsymbol{I}_d + \psi(|\boldsymbol{\xi}^D|^2)\,\boldsymbol{\xi}^D \qquad \forall \boldsymbol{\xi} \in \mathbb{S}_d\,. \tag{3.15}$$

Here, $K_0 > 0$ is a material coefficient, the function ψ is assumed to be piecewise continuously differentiable, and there exist positive constants c_1, c_2, d_1 and d_2, such that for $s \geqslant 0$,

$$\psi(s) \leqslant d_1, \quad -c_1 \leqslant \psi'(s) \leqslant 0, \quad c_2 \leqslant \psi(s) + 2\,\psi'(s)\,s \leqslant d_2. \tag{3.16}$$

The elasticity operator F defined in (3.15) satisfies condition (3.8) provided that conditions (3.16) hold (for detail, c.f. e.g. [5]). We conclude that our results below apply for the constitutive laws (3.13), (3.14) and (3.15).

Next, we remark that (3.9) and (3.10) are standard regularity assumptions on the given forces and tractions. Condition $|S|_{L^\infty(\Gamma_3)} > 0$ is imposed here in order to obtain a genuine frictional problem. Indeed, if $S = 0$ a.e. on Γ_3 then by (2.5) and (2.6) it follows that the Cauchy stress vector vanish on Γ_3 and therefore problem (2.1)–(2.6) becomes a classical displacement–traction problem which can be studied using standard arguments.

We observe that the assumption (3.11) on the coefficient of friction μ are pretily general. Clearly, these assumptions are satisfied if μ is a bounded function which is continuously differentiable with respect to the second variable, as it was considered in [6].

We also remark that assumtions (3.11) and (3.12) are satisfied if $\mu : \Gamma_3 \to \mathbb{R}_+$ and $\mu \in L^\infty(\Gamma_3)$. This case corresponds to the case when the coefficient of friction does not depend on the slip. Problem (2.1)–(2.6) with this assumption was studied in [4], [14], in the case of linear elastic materials.

To obtain variational formulations of problem P we need to introduce further notations. Thus, using Riesz's representation theorem, we define $\boldsymbol{f} \in V$ by

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0\, \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2\, \boldsymbol{v}\, da + \int_{\Gamma_3} S\, v_\nu\, da \qquad \forall \boldsymbol{v} \in V, \tag{3.17}$$

and let $j : V \times V \to \mathbb{R}$ be the functional

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu(|\boldsymbol{u}_\tau|)|S||\boldsymbol{v}_\tau|\, da \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.18}$$

Keeping in mind assumptions (3.9)–(3.11) it follows that the integrals in (3.17) and (3.18) are well defined.

For all $\boldsymbol{\eta} \in V$, set

$$\Sigma(\boldsymbol{\eta}) = \{\boldsymbol{\tau} \in \mathscr{H} \mid (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} + j(\boldsymbol{\eta}, \boldsymbol{v}) \geqslant (\boldsymbol{f}, \boldsymbol{v})_V\ \forall\ \boldsymbol{v} \in V\} \tag{3.19}$$

and let $D(T)$ be the subset of $\mathscr{H}$ defined by

$$D(T) = \{\boldsymbol{\tau} \in \mathscr{H} \mid \exists \boldsymbol{v} \in V \text{ such that } F(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{\tau}\}. \tag{3.20}$$

Note that, taking $\boldsymbol{v} = \pm \boldsymbol{\vartheta}$ with $\boldsymbol{\vartheta} \in \mathscr{D}(\Omega)^d$ in (3.19), it follows that

$$\boldsymbol{\tau} \in \Sigma(\boldsymbol{\eta}) \quad \Longrightarrow \quad \mathrm{Div}\, \boldsymbol{\tau} + \boldsymbol{f}_0 = 0 \quad \text{in } \Omega. \tag{3.21}$$

Moreover, from (3.8) and Korn's inequality (3.5) it follows that the operator $F \circ \boldsymbol{\varepsilon} : V \to D(T)$ is invertible. Let $T : D(T) \to V$ denote its inverse. We obtain:

$$\boldsymbol{v} = T(\boldsymbol{\tau}) \iff F(\boldsymbol{\varepsilon}(\boldsymbol{v})) = \boldsymbol{\tau}. \tag{3.22}$$

We have the following result.

LEMMA 3.1. *If* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *are sufficiently smooth functions satisfying* (2.1)–(2.6), *then*

$$\begin{aligned} \boldsymbol{u} \in V, \quad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathscr{H}} &+ j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \\ &\geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{3.23}$$

$$\boldsymbol{\sigma} \in D(T) \cap \Sigma(\boldsymbol{u}), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathscr{H}} \geqslant 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\boldsymbol{u}). \tag{3.24}$$

Proof. The regularity $\boldsymbol{u} \in V$ follows from (2.3). Let $\boldsymbol{v} \in V$. Using (3.1)–(3.4), (2.2)–(2.5) we have

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} = (\boldsymbol{f}_0, \boldsymbol{v})_H + (\boldsymbol{f}_2, \gamma \boldsymbol{v})_{L^2(\Gamma_2)^d} + (S, v_\nu)_{L^2(\Gamma_3)} + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, da \tag{3.25}$$

and, keeping in mind (2.6) yields

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, da \geqslant - \int_{\Gamma_3} \mu(|\boldsymbol{u}_\tau|)|S||\boldsymbol{v}_\tau|\, da. \tag{3.26}$$

Using (3.25), (3.26), (3.17) and (3.18) we deduce that

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{v}) \geqslant (\boldsymbol{f}, \boldsymbol{v})_V. \tag{3.27}$$

The regularity $\boldsymbol{\sigma} \in D(T) \cap \Sigma(\boldsymbol{u})$ is now a consequence of (2.1), (3.19), (3.20) and (3.27). Moreover, from (2.6) and (3.18) we obtain

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau \, da = -j(\boldsymbol{u}, \boldsymbol{u}). \tag{3.28}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ in (3.25) and using again (3.17) and (3.28) it follows that

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{3.29}$$

The inequalities in (3.23) and (3.24) are now a consequence of (3.27), (3.29) and (3.19). □

Lemma 3.1, (2.1) and (3.22) lead us to consider the following two variational problems.

PROBLEM. P_1 *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *such that*

$$\boldsymbol{u} \in V, \quad (F(\varepsilon(\boldsymbol{u})), \varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V. \tag{3.30}$$

PROBLEM. P_2 *Find a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}_d$ *such that*

$$\boldsymbol{\sigma} \in D(T) \cap \Sigma(T(\boldsymbol{\sigma})), \quad (F^{-1}(\boldsymbol{\sigma}), \boldsymbol{\tau} - \boldsymbol{\sigma})_{\mathscr{H}} \geqslant 0 \quad \forall \boldsymbol{\tau} \in \Sigma(T(\boldsymbol{\sigma})). \tag{3.31}$$

We note that problems P_1 and P_2 are formally equivalent to the mechanical problem P. Indeed, if $\boldsymbol{u}$ represents a sufficiently regular solution of the variational problem P_1 and $\boldsymbol{\sigma}$ is defined by $\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u}))$ then, using the arguments of [4], it follows that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a solution of problem P. Similarly, if $\boldsymbol{\sigma}$ represents a regular solution of the variational problem P_2 and $\boldsymbol{u} \in V$ is given by $\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u}))$ then, using the same arguments, it follows that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a solution of the mechanical problem P. For this reason we may consider problems P_1 and P_2 as *variational formulations* of the mechanical problem P.

Our main results, which we establish in the next section are the following.

THEOREM 3.2. *Assume that conditions* (3.8)–(3.10) *hold. Then:*

1) *Under the assumption* (3.11), *problem* P_1 *has at least a solution.*

2) *Under the assumptions* (3.11) *and* (3.12), *there exists* $L_0 > 0$ *depending only on* Ω, Γ_1, Γ_3, F *and* S *such that if* $L_\mu < L_0$ *then problem* P_1 *has a unique solution which depends Lipschitz continuously on* $\boldsymbol{f} \in V$.

THEOREM 3.3. *Assume that conditions* (3.8)–(3.11) *hold. Then:*

1) *If* $\boldsymbol{u}$ *is a solution of problem* P_1 *then the function* $\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u}))$ *is a solution of problem* P_2.

2) *Conversely, if* $\boldsymbol{\sigma}$ *is a solution of problem* P_2 *then the element* $\boldsymbol{u} = T(\boldsymbol{\sigma})$ *is a solution of problem* P_1.

THEOREM 3.4. *Assume that conditions* (3.8) – (3.10) *hold. Then:*

1) *Under the assumption* (3.11), *problem* P_2 *has at least one solution.*

2) *Under the assumptions* (3.11) *and* (3.12), *if* $L_\mu < L_0$ *where* L_0 *is defined as in Theorem* 3.1, *then problem* P_2 *has a unique solution which depends Lipschitz continuously on* $\boldsymbol{f} \in V$.

We conclude that, under assumptions (3.8)–(3.11), the mechanical problem (2.1)–(2.6) has a unique weak solution $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$. This solution is unique and depends Lipschitz continuous on the data $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ if (3.12) holds with a sufficiently small constant L_μ or if the coefficient of friction is a given positive bounded function which does not depend on the slip. Finally, keeping in mind (3.21) and (3.9), we remark that if $\boldsymbol{\sigma}$ solves problem P_2 then $\boldsymbol{\sigma} \in \mathscr{H}_1$.

4. An abstract existence and uniqueness result

To prove Theorem 3.2 we need an abstract result on elliptic quasivariational inequalities that we recall in this section, for the convenience of the reader.

Everywhere in this section V will represent a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $|\cdot|_V$. We denote by "$\rightharpoonup$" the weak convergence on V. Let $A : V \longrightarrow V$ be a nonlinear operator, $j : V \times V \longrightarrow \mathbb{R}$ and $\boldsymbol{f} \in V$. With these data we consider the following quasivariational inequality: find $\boldsymbol{u} \in V$ such that

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \qquad \forall \boldsymbol{v} \in V. \tag{4.1}$$

In order to solve (4.1) we assume that A is strongly monotone and Lipschitz contiunous, i.e.

$$\begin{cases} \text{(a) there exists } m > 0 \text{ such that} \\ \qquad (A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geqslant m|\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V; \\ \text{(b) there exists } M > 0 \text{ such that} \\ \qquad |A\boldsymbol{u} - A\boldsymbol{v}|_V \leqslant M|\boldsymbol{u} - \boldsymbol{v}|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \end{cases} \tag{4.2}$$

The function j satisfies

$$j(\boldsymbol{\eta}, \cdot) : V \longrightarrow \mathbb{R} \text{ is a convex functional on } V, \quad \text{for all } \boldsymbol{\xi} \in V. \tag{4.3}$$

Keeping in mind (4.3), it is well known that there exists the directional derivative j_2' given by

$$j_2'(\boldsymbol{\eta}, \boldsymbol{u}; \boldsymbol{v}) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \Big[j(\boldsymbol{\eta}, \boldsymbol{u} + \lambda \boldsymbol{v}) - j(\boldsymbol{\eta}, \boldsymbol{u}) \Big] \qquad \forall \boldsymbol{\xi},\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{4.4}$$

We now formulate additional conditions on the function j.

$$\begin{cases} \text{For every sequence } \{\boldsymbol{u}_n\} \subset V \text{ with } |\boldsymbol{u}_n|_V \to \infty \\ \text{and every sequence } \{t_n\} \subset [0,1] \text{ one has} \\ \liminf\limits_{n \to \infty} \left[\dfrac{1}{|\boldsymbol{u}_n|_V^2} j_2'(t_n \boldsymbol{u}_n, \boldsymbol{u}_n; -\boldsymbol{u}_n) \right] < m. \end{cases} \tag{4.5}$$

$$\left\{\begin{array}{l}\text{For every sequence } \{\boldsymbol{u}_n\} \subset V \text{ with } |\boldsymbol{u}_n|_V \to \infty \\ \text{and every bounded sequence } \{\boldsymbol{\eta}_n\} \subset V \text{ one has} \\ \liminf\limits_{n\to\infty} \left[\dfrac{1}{|\boldsymbol{u}_n|_V^2} j_2'(\boldsymbol{\eta}_n, \boldsymbol{u}_n; -\boldsymbol{u}_n)\right] < m.\end{array}\right. \tag{4.6}$$

$$\left\{\begin{array}{l}\text{For all sequences } \{\boldsymbol{u}_n\} \subset V \text{ and } \{\boldsymbol{\eta}_n\} \subset V \text{ such that } \boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \in V, \\ \boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in V \text{ and for every } \boldsymbol{v} \in V, \text{ the inequality below holds} \\ \limsup\limits_{n\to\infty} [j(\boldsymbol{\eta}_n, \boldsymbol{v}) - j(\boldsymbol{\eta}_n, \boldsymbol{u}_n)] \leqslant j(\boldsymbol{\eta}, \boldsymbol{v}) - j(\boldsymbol{\eta}, \boldsymbol{u}).\end{array}\right. \tag{4.7}$$

$$\left\{\begin{array}{l} j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{u}) - j(\boldsymbol{v}, \boldsymbol{v}) \leqslant \beta |\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \\ \text{for some } \beta \in \mathbb{R} \text{ with } \beta < m.\end{array}\right. \tag{4.8}$$

In the study of the quasivariational inequality (4.1) we have the following result.

THEOREM 4.1. *Let* (4.2) – (4.3) *hold. Then:*

1) *Under the assumptions* (4.5) – (4.7) *there exists at least one element* $\boldsymbol{u} \in V$ *which solves* (4.1).

2) *Under the assumptions* (4.5) – (4.8), *problem* (4.1) *has unique solution* $\boldsymbol{u} = \boldsymbol{u}_f$ *which depends Lipschitz continuously on* $\boldsymbol{f}$ *with the Lipschitz constant* $(m-\beta)^{-1}$.

Theorem 4.1 has been obtained recently in [10] and therefore we do not provide here the details of the proof. We just specify that the proof is obtained in several steps and it is based on standard arguments of elliptic variational inequalities and topological degree theory. A trait of novelty of Theorem 4.1 consists, in our knowledge, in the consideration of conditions (4.5) and (4.6), formulated in terms of the directional derivative of the functional j.

5. Proof of the main results

In this section we provide the proofs of Theorem 3.2–3.4. We start with the proof of Theorem 3.2 which will be carried out in several steps. We assume in the sequel that (3.8)–(3.11) hold. We remark that j satisfies the condition (4.3). Moreover, we have the following results.

LEMMA 5.1. *The functional j satisfies the assumptions* (4.5), (4.6) *and* (4.7).

Proof. Let $\boldsymbol{\eta}, \boldsymbol{u} \in V$ and let $\lambda \in]0,1]$. Using (3.18) it results that

$$j(\boldsymbol{\eta}, \boldsymbol{u} - \lambda\boldsymbol{u}) - j(\boldsymbol{\eta}, \boldsymbol{u}) = -\lambda \int_{\Gamma_3} \mu(|\boldsymbol{\eta}_\tau|)|S||\boldsymbol{u}_\tau| da$$

and, keeping in mind (4.4) we deduce

$$j_2'(\boldsymbol{\eta}, \boldsymbol{u}; -\boldsymbol{u}) \leqslant 0 \qquad \forall \boldsymbol{\eta}, \boldsymbol{u} \in V. \tag{5.1}$$

We conclude by (5.1) that j satisfies conditions (4.5) and (4.6).

Let now consider the sequences $\{\boldsymbol{u}_n\} \subset V$, $\{\boldsymbol{\eta}_n\} \subset V$ such that $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \in V$ and $\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \in V$. Using the compactness property of the trace map, (3.11) and Krasnoselski's theorem (see for instance [7]) it follows that

$$\mu(|\boldsymbol{\eta}_{n\tau}|) \to \mu(|\boldsymbol{\eta}_\tau|) \qquad \text{in} \quad L^2(\Gamma_3),$$

$$|\boldsymbol{u}_{n\tau}| \to |\boldsymbol{u}_\tau| \qquad \text{in} \quad L^2(\Gamma_3).$$

Therefore, we deduce that

$$j(\boldsymbol{\eta}_n, \boldsymbol{v}) \to j(\boldsymbol{\eta}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \qquad j(\boldsymbol{\eta}_n, \boldsymbol{u}_n) \to j(\boldsymbol{\eta}, \boldsymbol{u}),$$

which shows that the functional j satisfies the condition (4.7). □

LEMMA 5.2. *If* (3.12) *holds, then the functional* j *satisfies the inequality*

$$j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{u}) - j(\boldsymbol{v}, \boldsymbol{v}) \leqslant c_0^2 L_\mu |S|_{L^\infty(\Gamma_3)} |\boldsymbol{u} - \boldsymbol{v}|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.2}$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in V$. Using (3.12) and (3.18) it follows that

$$j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{u}) - j(\boldsymbol{v}, \boldsymbol{v}) =$$

$$\int_{\Gamma_3} |S| \Big(\mu(|\boldsymbol{u}_\tau|) - \mu(|\boldsymbol{v}_\tau|) \Big) \Big(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau| \Big) da \leqslant L_\mu |S|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |\boldsymbol{u}_\tau - \boldsymbol{v}_\tau|^2 da.$$

Using now (3.7) in the previous inequality we deduce (5.2). □

We have now all the ingredients to prove Theorems 3.2–3.4.

Proof of Theorem 3.2. 1) Using condition (3.8), we see that for all $\boldsymbol{\tau} \in \mathscr{H}$ the function $\boldsymbol{x} \mapsto F(\boldsymbol{x}, \boldsymbol{\tau}(\boldsymbol{x}))$ belongs to $\mathscr{H}$ and hence we may consider F as an operator defined on $\mathscr{H}$ with the range on $\mathscr{H}$. Moreover, using Riesz's representation theorem we may define the operator $A : V \to V$ by the relation

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (F(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.3}$$

Keeping in mind (3.8) (b),(c) and (3.6) we deduce that A satisfies the conditions (4.2). We also recall that the functional j given by (3.18) satisfies the condition (4.3). Thus, using Lemma 5.1 and Theorem 4.1.1) we deduce that problem P_1 has at least a solution $u \in V$.

2) Let (3.12) hold and let

$$L_0 = \frac{m}{c_0^2 |S|_{L^\infty(\Gamma_3)}}. \tag{5.4}$$

Clearly L_0 depends only on Ω, Γ_1, Γ_3, F and S. Let now assume that $L_\mu < L_0$. Then, there exists $\beta \in \mathbb{R}$ such that $c_0^2 L_\mu |S|_{L^\infty(\Gamma_3)} < \beta < m$. Using (5.2) we conclude that the functional j satisfies condition (4.8). Therefore, by Theorem 4.1.2) we obtain that problem P_1 has a unique solution which depends Lipschitz continuously on f. □

Proof of Theorem 3.3. 1) Let u be a solution of problem P_1 and let

$$\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u})). \tag{5.5}$$

It follows from (3.20) and (3.22) that

$$\boldsymbol{\sigma} \in D(T) \quad \text{and} \quad T(\boldsymbol{\sigma}) = \boldsymbol{u}. \tag{5.6}$$

Taking now $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \boldsymbol{0}$ in (3.30) and using (5.5) we obtain

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{5.7}$$

Using now (3.30), (5.6) and (5.7) we deduce that $\boldsymbol{\sigma} \in \Sigma(T(\boldsymbol{\sigma}))$. Moreover, from (3.19), (5.5) and (5.7) we deduce the inequality in (3.31) which proves that $\boldsymbol{\sigma}$ is a solution of problem P_2.

2) Conversely, let $\boldsymbol{\sigma}$ be the solution of problem P_2 and let $\boldsymbol{u} = T(\boldsymbol{\sigma}) \in V$. Using (3.22) we deduce that (5.5) holds and, moreover, by (3.31) we find

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} \geqslant 0 \qquad \forall\, \boldsymbol{\tau} \in \Sigma(\boldsymbol{u}). \tag{5.8}$$

Using now the subdifferentiability of the seminorm $j(\boldsymbol{u}, \cdot)$ on V and (3.6) we deduce that there exists $\tilde{\boldsymbol{\tau}} \in \mathscr{H}$ such that

$$(\tilde{\boldsymbol{\tau}}, \varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.9}$$

Taking $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \boldsymbol{0}$ in (5.9) we obtain

$$(\tilde{\boldsymbol{\tau}}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{5.10}$$

From (3.19), (5.9) and (5.10), it follows that $\tilde{\boldsymbol{\tau}} \in \Sigma(\boldsymbol{u})$. Therefore, taking $\boldsymbol{\tau} = \tilde{\boldsymbol{\tau}}$ in (5.8) and using again (5.10) we deduce

$$(\boldsymbol{f}, \boldsymbol{u})_V \geqslant (\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{u}).$$

The converse inequality follows from (3.19) since $\boldsymbol{\sigma} \in \Sigma(T(\boldsymbol{\sigma}))$ and $T(\boldsymbol{\sigma}) = \boldsymbol{u}$. Therefore, we conclude that

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \tag{5.11}$$

Using again (3.19) we have

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathscr{H}} + j(\boldsymbol{u}, \boldsymbol{v}) \geqslant (\boldsymbol{f}, \boldsymbol{v})_V \tag{5.12}$$

and from (5.5), (5.11) and (5.12) it results that $\boldsymbol{u}$ is a solution of the problem P_1. □

Proof of Theorem 3.4. Theorem 3.4 is now a straightforward consequence of Theorems 3.2 and 3.3. □

6. A continuous dependence result

Next, we investigate the behavior of the weak solution to the problem (2.1)–(2.6) with respect to perturbations of the coefficient of friction μ. To this end, let us suppose in the sequel that conditions (3.8)–(3.12) hold with $L_\mu < L_0$, where L_0 is given by (5.4). Therefore, using the results in Section 3 we deduce that problem P_1 has a unique solution $\boldsymbol{u} \in V$ and problem P_2 has a unique solution $\boldsymbol{\sigma} \in \mathscr{H}_1$. Moreover, $\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u}))$. For every $\alpha \geqslant 0$, let μ^α be a perturbation of μ which satisfies (3.11) and (3.12) with the constant $L_\mu^\alpha < L_0$. Let us also introduce the functionals j^α, which are obtained by replacing μ by μ^α in j, and let $\Sigma^\alpha(\boldsymbol{\eta})$ be given by (3.19), replacing j by j^α. We consider now the following problems.

PROBLEM. P_1^α *For* $\alpha \geqslant 0$, *find a displacement field* $\boldsymbol{u}^\alpha : \Omega \to \mathbb{R}^d$ *such that*

$$\boldsymbol{u}^\alpha \in V, \quad (F(\varepsilon(\boldsymbol{u}^\alpha)), \varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}^\alpha))_{\mathcal{H}} + j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v}) - j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^\alpha)_V \quad \forall \boldsymbol{v} \in V. \tag{6.1}$$

PROBLEM. P_2^α *For* $\alpha \geqslant 0$, *find a stress field* $\boldsymbol{\sigma}^\alpha : \Omega \to \mathbb{S}_d$ *such that*

$$\boldsymbol{\sigma}^\alpha \in D(T) \cap \Sigma^\alpha(T(\boldsymbol{\sigma}^\alpha)), \quad (F^{-1}(\boldsymbol{\sigma}^\alpha), \boldsymbol{\tau} - \boldsymbol{\sigma}^\alpha)_{\mathcal{H}} \geqslant 0 \quad \forall \boldsymbol{\tau} \in \Sigma^\alpha(T(\boldsymbol{\sigma}^\alpha)). \tag{6.2}$$

Using Theorems 3.2 and 3.4 we deduce that for each $\alpha \geqslant 0$ problems P_1^α and P_2^α have a unique solution $\boldsymbol{u}^\alpha \in V$ and $\boldsymbol{\sigma}^\alpha \in \mathcal{H}_1$, respectively. Moreover, by Theorem 3.3 it follows that $\boldsymbol{\sigma}^\alpha = F(\varepsilon(\boldsymbol{u}^\alpha))$. Suppose now that the coefficient of friction satisfy the following assumption:

$$\begin{cases} \text{There exists } \theta : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\ \text{(a) } |\mu^\alpha(x,t) - \mu(x,t)| \leqslant \theta(\alpha)\,|t| \quad \forall\, t \in \mathbb{R}_+, \text{ a.e. on } \Gamma_3; \\ \text{(b) } \lim_{\alpha \to 0} \theta(\alpha) = 0. \end{cases} \tag{6.3}$$

We have the following result.

THEOREM 6.1. *Let* (6.3) *hold. Then*

$$\boldsymbol{u}^\alpha \to \boldsymbol{u} \text{ in } V, \quad \boldsymbol{\sigma}^\alpha \to \boldsymbol{\sigma} \text{ in } \mathcal{H}_1 \quad \text{as } \alpha \to 0. \tag{6.4}$$

Proof. Let $\alpha \geqslant 0$. Using (3.30) and (6.1) we obtain

$$\begin{aligned} &(F(\varepsilon(\boldsymbol{u}^\alpha)) - F(\varepsilon(\boldsymbol{u})), \varepsilon(\boldsymbol{u}^\alpha) - \varepsilon(\boldsymbol{u}))_{\mathcal{H}} \\ &\leqslant j(\boldsymbol{u}, \boldsymbol{u}^\alpha) - j(\boldsymbol{u}, \boldsymbol{u}) + j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \\ &= \int_{\Gamma_3} |S| \big(\mu(|\boldsymbol{u}_\tau|) - \mu^\alpha(|\boldsymbol{u}_\tau^\alpha|)\big)\,(|\boldsymbol{u}_\tau^\alpha| - |\boldsymbol{u}_\tau|)\,da. \end{aligned}$$

Thus, using (3.6) and (3.8), we deduce

$$\begin{aligned} &m\,|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V^2 \\ &\leqslant \int_{\Gamma_3} |S| \{ \big(\mu(|\boldsymbol{u}_\tau^\alpha|) - \mu^\alpha(|\boldsymbol{u}_\tau^\alpha|)\big) + \big(\mu(|\boldsymbol{u}_\tau|) - \mu(|\boldsymbol{u}_\tau^\alpha|)\big) \}\,(|\boldsymbol{u}_\tau^\alpha| - |\boldsymbol{u}_\tau|)\,da. \end{aligned} \tag{6.5}$$

Taking into account (6.3)(a) and (3.12), yields

$$\begin{aligned} &m\,|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V^2 \\ &\leqslant \theta(\alpha)|S|_{L^\infty(\Gamma_3)}\,|\boldsymbol{u}^\alpha|_{L^2(\Gamma_3)^d}|\boldsymbol{u}^\alpha - \boldsymbol{u}|_{L^2(\Gamma_3)^d} + |S|_{L^\infty(\Gamma_3)} L_\mu |\boldsymbol{u}^\alpha - \boldsymbol{u}|^2_{L^2(\Gamma_3)^d} \end{aligned}$$

and, using (3.7), it follows that

$$(m - c_0^2 L_\mu |S|_{L^\infty(\Gamma_3)})|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V \leqslant c_0^2 \theta(\alpha) |S|_{L^\infty(\Gamma_3)}\,|\boldsymbol{u}^\alpha|_V. \tag{6.6}$$

On the other hand, taking $\boldsymbol{v} = \mathbf{0}$ in (6.1) we obtain

$$(F(\varepsilon(\boldsymbol{u}^\alpha)), \varepsilon(\boldsymbol{u}^\alpha))_{\mathscr{H}} \leqslant -j^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) + (\boldsymbol{f}, \boldsymbol{u}^\alpha)_V \leqslant (\boldsymbol{f}, \boldsymbol{u}^\alpha)_V \tag{6.7}$$

and, using (3.8), we have

$$m|\boldsymbol{u}^\alpha|_V^2 \leqslant (F(\varepsilon(\boldsymbol{u}^\alpha)) - F(\mathbf{0}), \varepsilon(\boldsymbol{u}^\alpha))_{\mathscr{H}}. \tag{6.8}$$

Combining now (6.7) and (6.8), by (3.6) we deduce

$$|\boldsymbol{u}^\alpha|_V \leqslant \frac{1}{m}(|\boldsymbol{f}|_V + |F(\mathbf{0})|_{\mathscr{H}}) \quad \forall \alpha \geqslant 0. \tag{6.9}$$

Keeping in mind that $L_\mu < L_0$, using (6.9), (6.6) and (5.4), we find

$$|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V \leqslant K\theta(\alpha), \tag{6.10}$$

where $K > 0$ is a generic constant. Moreover, since $\boldsymbol{\sigma} = F(\varepsilon(\boldsymbol{u}))$ and $\boldsymbol{\sigma}^\alpha = F(\varepsilon(\boldsymbol{u}^\alpha))$, it follows that

$$|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}|_{\mathscr{H}} \leqslant M|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V.$$

Now, by (3.21) we have $\operatorname{Div} \boldsymbol{\sigma}^\alpha = \operatorname{Div} \boldsymbol{\sigma} = -\boldsymbol{f}_0$ and, using the previous inequality, we obtain

$$|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}|_{\mathscr{H}_1} = |\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}|_{\mathscr{H}} \leqslant M|\boldsymbol{u}^\alpha - \boldsymbol{u}|_V. \tag{6.11}$$

The convergence result (6.4) is now a consequence of (6.10), (6.11) and (6.3)(b). □

In addition to the mathematical interest in the result presented in Theorem 6.1, it is of importance in applications, as it indicates that small inaccuracies in the contact conditions lead to small inaccuracies in the solution.

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(Received July 8, 2000)

Claudiu Ciulcu
Departement of Mathematics
University of Craiova
Bd. A. I. Cuza, 1100 Craiova
Romania
e-mail: cciulcu@yahoo.com

Dumitru Motreanu
Laboratoire de Théorie des Systèmes
Université de Perpignan
52 Avenue de Villeneuve
66860 Perpignan
France
e-mail: motreanu@univ-perp.fr

Mircea Sofonea
Laboratoire de Théorie des Systèmes
Université de Perpignan
52 Avenue de Villeneuve
66860 Perpignan
e-mail: sofonea@univ-perp.fr