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MATRIX INEQUALITIES BY MEANS OF BLOCK MATRICES

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Abstract. We first show a weak log-majorization inequality of singular values for partitioned positive semidefinite matrices which will imply some existing results of a number of authors, then present some basic matrix inequalities and apply them to obtain a number of matrix inequalities involving sum, ordinary product and Hadamard product.

1. Introduction

One of the most useful tools for deriving matrix inequalities is to utilize block matrices; usually they are 2×2 in most applications. In this paper, we shall show a weak log-majorization inequality of singular values for partitioned positive semidefinite matrices, from which some classical and recent results of Bhatia and Kittaneh [4], Wang, Xi and Zhang [12], and Zhan [13] will follow. We shall also develop a new technique that is complementary to the Schur complement; while by making use of Schur complements, a number of determinantal, trace, and other inequalities are exhibited in [16]. With the new technique we add more inequalities to these in [16].

We denote the eigenvalues of an $n \times n$ complex matrix X by $l_i(X)$, $i = 1, 2, \ldots, n$, and arrange them in modulus decreasing order $|l_1(X)| \geqslant |l_2(X)| \geqslant \cdots \geqslant |l_n(X)|$. The singular values of an $m \times n$ matrix X are denoted by $\sigma_1(X), \ldots, \sigma_n(X)$ and are also arranged in decreasing order. Note that $\sigma_i(X) = l_i(|X|)$ for each i, where $|X| = (X^*X)^{\frac{1}{2}}$. We further write

$$l(X) = (l_1(X), l_2(X), \ldots, l_n(X)), \quad \sigma(X) = (\sigma_1(X), \sigma_2(X), \ldots, \sigma_n(X)).$$

The leading principal $k \times k$ submatrix of a matrix X is denoted by $[X]_k$.

Write $X \geqslant 0$ if X is a positive semidefinite matrix and $X \geqslant Y$ if X and Y are Hermitian matrices such that $X - Y \geqslant 0$. The strict inequality $X > 0$ denotes the positive definiteness of X. Let $X \circ Y = (x_{ij}y_{ij})$ be the Hadamard (Schur) product of matrices X and Y of the same size and X^* the conjugate transpose of X.

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For complex vector $x=(x_1,x_2,\ldots,x_n)$, we denote $|x|=(|x_1|,|x_2|,\ldots,|x_n|)$, and for vectors $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ with nonnegative components in decreasing order, we write $\log x \prec_w \log y$ to mean

$$\prod_{i=1}^{k} x_i \leqslant \prod_{i=1}^{k} y_i, \quad k=1,2,\ldots,n.$$

As is well known, $\log x \prec_w \log y$ yields $x \prec_w y$ (see, e.g., [3, p. 42]). The latter means

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i, \quad k=1,2,\ldots,n.$$

The subscript "w" is dropped off in either $\prec_w$ if equality holds when $k=n$.

2. A Weak Log-Majorization Inequality

THEOREM 1. *Let A, B, and C be complex matrices such that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0,$$

where A is $m\times m$, C is $n\times n$, and B is $m\times n$. Let $\operatorname{rank}(B)=r$. *Then*

$$\log \sigma(B) \prec_w \log \mu, \tag{1}$$

where $\mu=(\mu_1,\mu_2,\ldots,\mu_n)$ with $\mu_i=\max\{l_i(A),l_i(C)\}$ if $i\leqslant r$, 0 otherwise. Thus

$$\sigma(B) \prec_w \mu. \tag{2}$$

And if A,B, and C are all square of the same size, then

$$\log |l(B)| \prec_w \log \mu. \tag{3}$$

Proof. We may assume that $B\neq 0$. Let $B=UDV^*$ be a singular value decomposition of the matrix B, where $D=\operatorname{diag}(\sigma_1(B),\ldots,\sigma_r(B))$, and U and V are $m\times r$ and $n\times r$ partial unitary matrices, respectively, i.e., $U^*U=V^*V=I_r$. Then

$$\begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}=\begin{pmatrix} U^*AU & D \\ D & V^*CV \end{pmatrix}\geqslant 0.$$

Taking the leading principal $k\times k$ submatrix of each block, $1\leqslant k\leqslant r$, we have

$$\begin{pmatrix} [U^*AU]_k & [D]_k \\ [D]_k & [V^*CV]_k \end{pmatrix}\geqslant 0.$$

It follows that, by taking determinant for each block,

$$\det[D]_k^2 \leqslant \det([U^*AU]_k)\det([V^*CV]_k).$$

Or equivalently, for each $1 \leqslant k \leqslant r$,

$$\prod_{i=1}^{k} \sigma_i(B)^2 \leqslant \prod_{i=1}^{k} l_i([U^*AU]_k) l_i([V^*CV]_k).$$

By the eigenvalue interlacing theorem (see, e.g., [17, p. 222–224]), we arrive at

$$\prod_{i=1}^{k} \sigma_i(B)^2 \leqslant \prod_{i=1}^{k} l_i(A) l_i(C) \leqslant \prod_{i=1}^{k} \mu_i^2.$$

The desired inequality (1), thus (2), follows immediately by taking square roots. (3) is similarly obtained by letting $B = WTW^*$, where T is an upper triangular matrix with diagonal entries $l_1(B), l_2(B), \ldots, l_n(B)$ and W is unitary. □

COROLLARY 1. *Let $A \geqslant 0$, $B \geqslant 0$ be of size $n \times n$. Then for any $z \in \mathbb{C}$*

$$\log \sigma(A - |z|B) \prec_w \log \sigma(A + zB) \prec_w \log \sigma(A + |z|B). \tag{4}$$

Proof. For the second part, by (1), it is sufficient to notice that

$$\begin{pmatrix} A + |z|B & A + zB \\ A + z^*B & A + |z|B \end{pmatrix} \geqslant 0;$$

whereas the first part is proven by using the elementary inequality (see [12] or [13])

$$|1 - |z|| \leqslant |1 - z| \leqslant 1 + |z|. \quad \square$$

(4) has just appeared in [13]. It refines the majorization inequality [12]

$$\log \sigma(A - B) \prec_w \log \sigma(A + B)$$

and implies the weaker inequality for unitarily invariant norms $\|\cdot\|_{\mathrm{ui}}$ [4]

$$\|A - |z|B\|_{\mathrm{ui}} \leqslant \|A + zB\|_{\mathrm{ui}} \leqslant \|A + |z|B\|_{\mathrm{ui}}.$$

We note that the following matrix inequalities do not hold in general:

$$|A - |z|B| \leqslant |A - zB| \leqslant A + |z|B.$$

For a counterexample, take $z = i$,

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Then $l(|A-B|) = (3,3)$, $l(|A+iB|) = (6.951\cdots, 1.294\cdots)$, and $l(A+B) = (9,1)$.

COROLLARY 2. *Let A be any $n \times n$ complex matrix. Then*

$$\log|l(A)| \prec \log \sigma(A). \tag{5}$$

Proof. By (3), it is sufficient to notice that

$$\begin{pmatrix} |A^*| & A \\ A^* & |A| \end{pmatrix} \geqslant 0. \quad \square$$

Inequality (5) is the well known Weyl's inequality (see, e.g., [7, p. 171]).

REMARK 2.1. Note that μ in (1) and (3) cannot be replaced by $\sigma(A)$ or $\sigma(C)$.

REMARK 2.2. In the proof of the theorem, we used the result [9, p. 142] that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0 \quad \Rightarrow \quad \det(B^*B) \leqslant \det A \det C,$$

where A, B, and C square matrices of the same size. (This does not hold in general if B is rectangular.) Using this result, one can also give a very simple proof to the determinantal inequality [9, p. 144]: Let l_i be complex numbers and $A_i \geqslant 0$. Then

$$|\det(l_1A_1 + \cdots + l_kA_k)| \leqslant \det(|l_1|A_1 + \cdots + |l_k|A_k).$$

This is because

$$\sum_{i=1}^{k} \begin{pmatrix} |l_i|A_i & l_iA_i \\ \bar{l}_iA_i & |l_i|A_i \end{pmatrix} \geqslant 0.$$

3. Some Basic Inequalities

Block matrices in the form $\begin{pmatrix} H & K \\ K & H \end{pmatrix}$ have played a pivotal role in proving some matrix inequalities. We shall give some elementary matrix inequalities by applying a result on the block matrix to some partitioned positive semidefinite matrices and then to further derive inequalities on sum, ordinary and Hadamard products.

Let H and K be (complex) Hermitian matrices of the same size. Then

$$\begin{pmatrix} H & K \\ K & H \end{pmatrix} \geqslant 0 \quad \Leftrightarrow \quad H \geqslant \pm K. \tag{6}$$

This is seen by noticing the matrix identity via nonsingular congruence (similarity)

$$\left[\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\right]\begin{pmatrix} H & K \\ K & H \end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\right] = \begin{pmatrix} H-K & 0 \\ 0 & H+K \end{pmatrix}.$$

Obviously the eigenvalues of the block matrix in (6) are those of $H \pm K$. A proof of (6) for the real case is given in [5] via quadratic forms, and a characterization of the matrices K, which comprise a convex set, for the given H by trace inequalities is presented in [2]. A majorization inequality of the eigenvalues of the matrices H and K in (6) is seen in [14]:

$$|l(K)| \prec_w l(H), \tag{7}$$

which is strengthened as, by Theorem 1,

$$\log|l(K)| \prec_w \log l(H), \tag{8}$$

while the (stronger) eigenvalue pairwise dominant inequalities

$$|l_i(K)| \leqslant l_i(H)$$

do not hold for all i, though $|l_1(K)| \leqslant l_1(H)$. (Thus $H \geqslant \pm K \not\Rightarrow H \geqslant |K|$.) Moreover, by using the block matrix in (6) and the Albert theorem [1], one has

$$H \geqslant \pm K \quad \Rightarrow \quad K = HH^+K = KH^+H,$$

where H^+ is the Moore-Penrose generalized inverse of H.

We now give our basic inequalities that easily follow from (6).

THEOREM 2. *Let A, B, and C be n-square complex matrices such that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0.$$

Then, with $\star$ for $+$ or $\circ$,

$$A \star C \geqslant \pm(B^* \star B) \tag{9}$$

and, if $AB = BA$,

$$A^{\frac{1}{2}} C A^{\frac{1}{2}} \geqslant B^* B. \tag{10}$$

Proof. Since the block matrix via a permutation congruence is also positive semi-definite, we have (by the Schur Hadamard product theorem; see, e.g., [17, p. 192])

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \star \begin{pmatrix} C & B^* \\ B & A \end{pmatrix} = \begin{pmatrix} A \star C & B^* \star B \\ B^* \star B & A \star C \end{pmatrix} \geqslant 0.$$

(9) thus follows from (6). For (10), notice that if B commutes with A, then B commutes with $A^{\frac{1}{2}}$ (see, e.g., [6, p. 322] or [17, p. 165]). Let A be nonsingular. Then

$$C \geqslant B^* A^{-1} B = B^* A^{-\frac{1}{2}} A^{-\frac{1}{2}} B = A^{-\frac{1}{2}} B^* B A^{-\frac{1}{2}},$$

from which, by pre- and post-multiplying both sides by $A^{\frac{1}{2}}$, we arrive at the desired inequality. The singular case of A follows from a continuity argument. □

With the assumption of the theorem, we note the following.

REMARK 3.1. For the sum in (9), the inequality $A + C \geqslant \pm(B + B^*)$ is also proven by observing

$$(I, \pm I) \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} I \\ \pm I \end{pmatrix} = A \pm (B + B^*) + C \geqslant 0.$$

REMARK 3.2. Applying (8) to (9), we have explicitly

$$\log |l(B+B^*)| \prec_w \log l(A+C)$$

and

$$\log |l(B\circ B^*)| \prec_w \log l(A\circ C).$$

In particular,

$$|\det(B+B^*)| \leqslant \det(A+C)$$

and

$$|\det(B\circ B^*)| \leqslant \det(A\circ C).$$

REMARK 3.3. The condition $AB = BA$ for (10) is not removable in general, and B and B^* on the right hand side cannot be switched. It is easy to find an example that

$$\begin{pmatrix} I & B \\ B^* & C \end{pmatrix} \geqslant 0 \Rightarrow C \geqslant B^*B, \quad \text{but } C \ngeqslant BB^*.$$

REMARK 3.4. If the matrices B and C in the theorem commute, then

$$C^{\frac{1}{2}}AC^{\frac{1}{2}} \geqslant BB^*.$$

4. Applications

Applications of Theorem 2 to some frequently used 2×2 block positive semidefinite matrices result in some interesting inequalities. We present some as examples.

Assume in the following that matrices A, B, and C are all n-square (some results also hold for the rectangular case). We itemize with the block positive semidefinite matrices followed by immediate inequalities and comments.

Inequalities of one matrix:

1. $\begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix} \geqslant 0$, for any $A > 0$, gives

   1i). $2I \leqslant A + A^{-1}$;

   1ii). $I \leqslant A\circ A^{-1}$.

   Comments: These are existing inequalities. Alternative proof of the first one is by a unitary diagonalization of A, while the second one's proof does not come that easy; it usually needs to prove $A\circ A^{-1} \geqslant (A\circ A^{-1})^{-1}$ first (see, e.g, [8]). If A and B are both positive definite and n-square, by noticing that

$$\begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix} \circ \begin{pmatrix} B^{-1} & I \\ I & B \end{pmatrix} = \begin{pmatrix} A\circ B^{-1} & I \\ I & A^{-1}\circ B \end{pmatrix} \geqslant 0,$$

   we obtain a result of Visick ([10, Theorem 5 ii]):

$$A\circ B^{-1} + B\circ A^{-1} \geqslant 2I.$$

2. $\begin{pmatrix} \sigma_1 I & A^* \\ A & \sigma_1 I \end{pmatrix} \geqslant 0$, where σ_1 is the largest singular value of (any) A, implies

2i). $|A+A^*| \leqslant 2\sigma_1 I$;
2ii). $|A \circ A^*| \leqslant \sigma_1^2 I$.
Comments: A direct proof for 2i) and 2ii) without using the theorem may not work out as smoothly, though they are weaker than the following inequality 3i).

3. $\begin{pmatrix} |A|^{2\alpha} & A^* \\ A & |A^*|^{2(1-\alpha)} \end{pmatrix} \geqslant 0$, for any A and $\alpha \in [0,1]$, gives

3i). $|A \star A^*| \leqslant |A|^{2\alpha} \star |A^*|^{2(1-\alpha)}$.
Comments: Taking $\alpha = 1$, we have the comparison of sum and ordinary product

$$|A+A^*| \leqslant A^*A + I$$

and the comparison of the Hadamard product and ordinary product

$$|A \circ A^*| \leqslant A^*A \circ I.$$

In particular, if A is positive semidefinite, with the above A replaced by $A^{\frac{1}{2}}$,

$$2A^{\frac{1}{2}} \leqslant A + I, \quad A^{\frac{1}{2}} \circ A^{\frac{1}{2}} \leqslant A \circ I = \operatorname{diag}(A).$$

And taking $\alpha = \frac{1}{2}$, we have the inequalities involving sum and the two products

$$|A+A^*| \leqslant |A| + |A^*|, \quad |A \circ A^*| \leqslant |A| \circ |A^*|.$$

(Note: Neither $|A+B| \leqslant |A| + |B|$ nor $|A \circ B| \leqslant |A| \circ |B|$ holds in general.)
Inequalities of two or three matrices:

4. $\begin{pmatrix} A & B \\ B^* & B^*A^{-1}B \end{pmatrix} \geqslant 0$, for any $A > 0$ and any B, gives

4i). $\pm(B \star B^*) \leqslant A \star (B^*A^{-1}B)$.
Comments: If $B = I$, then the block matrix is the same as the one in 1). Taking $B = J$, the all one matrix, and switching A and A^{-1}, one obtains a lower bound for the inverse of A:

$$\frac{1}{\Sigma(A)} J \leqslant A^{-1},$$

where $\Sigma(A) = \sum a_{ij}$ is the sum of all entries of A. Note also that for any $A > 0$

$$\begin{pmatrix} A & J \\ J & \Sigma(A^{-1})J \end{pmatrix} \geqslant 0.$$

With a similar block matrix for $B > 0$, one obtains a lower bound for $A \circ B$:

$$A \circ B \geqslant \frac{1}{\Sigma(A^{-1})\Sigma(B^{-1})} J.$$

5. $\begin{pmatrix} A & A^{\frac{1}{2}}CB^{\frac{1}{2}} \\ B^{\frac{1}{2}}C^*A^{\frac{1}{2}} & B \end{pmatrix} \geqslant 0$, for $A, B \geqslant 0$ and any contraction matrix C, gives

5i). $(A^{\frac{1}{2}}CB^{\frac{1}{2}}) \star (B^{\frac{1}{2}}C^*A^{\frac{1}{2}}) \leqslant A \star B$.
Comments: Taking $B = C = I$ for the Hadamard product yields $A^{\frac{1}{2}} \circ A^{\frac{1}{2}} \leqslant A \circ I$ as seen in 3). 5i) is equivalent to $(ACB) \star (BCA) \leqslant A^2 \star B^2$. Setting $C = I$ gives $AB + BA \leqslant A^2 + B^2$ and its Hadamard companion $AB \circ BA \leqslant A^2 \circ B^2$.

6. $\begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} \geqslant 0$, for any A and B, gives

6i). $\pm(A^*B \star B^*A) \leqslant A^*A \star B^*B$.
Comments: The Hadamard product case of 6i) is seen in [10, Corollary 12]. In particular, if we take $B = I$ for the Hadamard product, then $\pm(A^* \circ A) \leqslant A^*A \circ I$. Letting $A > 0$ and setting $B = A^{-1}$ results in 1ii).

7. $\begin{pmatrix} I + A^*A & A^* + B^* \\ A + B & I + BB^* \end{pmatrix} \geqslant 0$, for any A and B, gives

7i). $(A + B) \circ (A + B)^* \leqslant (I + A^*A) \circ (I + BB^*)$.
Comments: This Hadamard product matrix inequality is compared to the conventional product matrix inequality (by taking Schur complement)

$$(A + B)(I + A^*A)^{-1}(A + B)^* \leqslant I + BB^*.$$

8. $\begin{pmatrix} AA^* \circ I & A \circ B \\ A^* \circ B^* & B^*B \circ I \end{pmatrix} \geqslant 0$, for any A and B, gives

8i). $A \circ B + A^* \circ B^* \leqslant AA^* \circ I + B^*B \circ I$;
8ii). $A \circ A^* \circ B \circ B^* \leqslant AA^* \circ B^*B \circ I$.
Comments: For $A \geqslant 0$ and $B \geqslant 0$, 8i) gives the inequality of means for Hadamard product

$$A \circ B \leqslant \frac{A^2 + B^2}{2} \circ I.$$

It follows that for any correlation matrices A and B (with diagonal entries 1)

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leqslant I.$$

Notice that $AA^* \leqslant \sigma_1 I$. We put $B = A^t$, the transpose of A in 8ii). Then

$$A \circ A^* \circ A^t \circ \overline{A} \leqslant \sigma_1^4 I.$$

9. $\begin{pmatrix} AA^* \circ BB^* & A \circ B \\ A^* \circ B^* & I \end{pmatrix} \geqslant 0$, for any A and B, gives

9i). $(A \circ B)(A^* \circ B^*) \leqslant AA^* \circ BB^*$.

Comments: This has appeared in [15] and in a recent paper [10, Theorem 4].

10. $\begin{pmatrix} |A|\circ|B| & A^*\circ B^* \\ A\circ B & |A^*|\circ|B^*| \end{pmatrix} \geqslant 0$, for any A and B, gives

10i). $|A\circ B + A^*\circ B^*| \leqslant |A|\circ|B| + |A^*|\circ|B^*|$;

10ii). $|A\circ A^*\circ B\circ B^*| \leqslant |A|\circ|A^*|\circ|B|\circ|B^*|$.

Comments: By taking B to be a matrix of 0 and 1, one can get the inequalities for the specified entries of A. For example, if B is a permutation matrix, then $|B| = |B^*| = I$ and one gets the inequalities that compare any diagonal (entries) of A to the diagonals of $|A|$ and $|A^*|$. And one may also obtain inequalities for submatrices of A by setting $B = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}$.

11. $\begin{pmatrix} A & AB \\ B^*A & B^*AB \end{pmatrix} \geqslant 0$, for any $A \geqslant 0$ and any $n\times m$ matrix B, implies

11i). $B^*A + AB \leqslant A + B^*AB$;

11ii). $B^*A\circ AB \leqslant A\circ(B^*AB)$.

Comments: Setting $B = A^k$ yields the inequalities of shifting A

$$2A^{k+1} \leqslant A + A^{2k+1}, \quad k = 1,2,\ldots,$$

and

$$A^{k+1}\circ A^{k+1} \leqslant A\circ A^{2k+1}, \quad k = 1,2,\ldots.$$

Inequalities of generalized inverses:

12. $\begin{pmatrix} A & AA^+ \\ A^+A & A^+ \end{pmatrix} \geqslant 0$, for any $A \geqslant 0$, gives

12i). $A\circ A^+ \geqslant A^+A\circ AA^+$;

12ii). $A + A^+ \geqslant A^+A + AA^+$.

Comments: These are compared to the inequality of Visick in [11, p. 282]:

$$A\circ A^+ \geqslant (AA^+\circ AA^+)(A\circ A^+)^+(AA^+\circ AA^+).$$

Combining the above block matrices via sum or Hadamard product, one may get more block positive semidefinite matrices and thus more inequalities. For instance, if $A \geqslant 0$ and $B > 0$, both n-square, then

$$\begin{pmatrix} I & A \\ A & A^2 \end{pmatrix} \circ \begin{pmatrix} B & I \\ I & B^{-1} \end{pmatrix} = \begin{pmatrix} I\circ B & I\circ A \\ I\circ A & A^2\circ B^{-1} \end{pmatrix} \geqslant 0.$$

Thus

$$A^2\circ B^{-1} \geqslant (I\circ A)(I\circ B)^{-1}(I\circ A) = (\mathrm{diag}A)^2(\mathrm{diag}B)^{-1}.$$

Note that the right hand side involves only the diagonal entries of A and B. In addition, for any correlation matrix A and nonsingular correlation matrix B

$$A^2\circ B^{-1} \geqslant I.$$

More inequalities are available by substituting the above matrices with matrices involving Kronecker product and by using the fact that the Hadamard product is a principal submatrix of the Kronecker product when the matrices are square. One also gets majorization inequalities by applying Theorem 1 to the above block matrices.

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