Mathematical
Inequalities
& Applications
Volume 4, Number 4 (2001), 491–497

INTEGRAL MEAN ESTIMATES FOR POLYNOMIALIS WITH RESTRICTED ZEROS

ABDUL AZIZ AND WALI MOHAMMAD SHAH

(*communicated by N. Govil*)

Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in the disk $|z| < K$. For $K = 1$, it is known that for $0 < q < \infty$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant B_q\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q},$$

where

$$B_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+R^n e^{in\alpha}\right|^q d\alpha\right\}^{1/q} \Big/ \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{in\alpha}\right|^q d\alpha\right\}^{1/q}.$$

In this paper we present a generalization of this result by considering the case $K \geqslant 1$. We shall also prove a similar result for polynomialis having all their zeros in $|z| \leqslant K$, where $K \geqslant 1$.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree at most n, then for each $R \geqslant 1$ and $q > 0$,

$$\max_{|z|=R}|P(z)| \leqslant R^n \max_{|z|=1}|P(z)| \tag{1}$$

and

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant R^n\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}. \tag{2}$$

Inequality (1) is a simple deduction from the maximum modulus principle (see [11, p. 346], or [8, Vol. 1, p. 137, prob. III 269]) and inequality (2) is a simple consequence of a classical result of Hardy [6] (see for example [9, Theorem 5.5]).

In both (1) and (2) equality holds only for $P(z) = cz^n$, $c \neq 0$, i. e., when all the zeros of $P(z)$ lie at the origin. Inequality (1) can be obtained by letting $q \to \infty$ in inequality (2). The inequalities (1) and (2) can be sharpened if we restict ourselves to

Mathematics subject classification (2000): 26D05, 30D15, 41A17.
Key words and phrases: Integral mean estimates, admissible operator, complex domain inequalities.

a class of polynomials having no zeros in $|z| < 1$. In fact, if $P(z) \neq 0$ for $|z| < 1$, then it was shown by Ankeny and Rivlin [1] that (1) can be replaces by

$$\max_{|z|=R>1} |P(z)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \tag{3}$$

where the corresponding refinement of (2) namely

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|P(Re^{i\theta})\right|^q d\theta \right\}^{1/q} \leqslant c_q \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q}, \quad R > 1, \tag{4}$$

where

$$c_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + R^n e^{i\alpha}\right|^q d\alpha \right\}^{1/q} \Big/ \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^q d\alpha \right\}^{1/q},$$

was proved by Boas and Rahman [5] for $1 \leqslant q < \infty$. Recently Rahman and Schmeisser [10] have shown that (4) remains true for $0 < q < 1$ as well. It can be easily seen that if we let $q \to \infty$ in (4), we get inequality (3).

Here we consider a class of polynomials having no zeros in $|z| < K$, where $K \geqslant 1$ and prove the following generalization of (4).

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geqslant K \geqslant 1$, then for every $R > 1$ and $q > 0$,*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|P(Re^{i\theta})\right|^q d\theta \right\}^{1/q} \leqslant B_q \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q}, \tag{5}$$

where

$$B_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + R^n e^{i\alpha}\right|^q d\alpha \right\}^{1/q} \Big/ \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|1 + t_K e^{i\alpha}\right|^q d\alpha \right\}^{1/q}$$

with $t_K = \left(\dfrac{1 + RK}{R + K} \right)^n$.

Inequality (5) reduces to (4) for $0 < q < \infty$ when $K = 1$.

REMARK 1. Letting $q \to \infty$ in (5), it follows that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geqslant K \geqslant 1$, then for $R > 1$,

$$\max_{|z|=R} |P(z)| \leqslant \frac{(R + K)^n (R^n + 1)}{(1 + RK)^n + (R + K)^n} \max_{|z|=1} |P(z)|. \tag{6}$$

Inequality (6) is a generalilzation of a result of Ankeny and Rivlin [1], proved by Aziz [4].

If $P(z)$ has all its zeros in $|z| \leqslant 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q} \leqslant \left\{ \int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{7}$$

Inequality (7) is due to Malik [7]. As an extension of (7) Aziz [3] proved that if $P(z)$ has all zeros in $|z| \leqslant K$, where $K \geqslant 1$, then for each $q > 1$,

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi}\left|1+K^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1}|P'(z)|. \tag{8}$$

Since in the proof of the inequality (8), the inequality (4) proved by Boas and Rahman for $1 \leqslant q < \infty$ was used, it was not clear, whether the restriction on q was indeed essential. Here we use Theorem 1 to show that the restriction on q is not needed. In fact we establish the following generalization of (7) which shows that (8) remains true for $0 < q < 1$ also. We prove

THEOREM 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant K$, where $K \geqslant 1$, then for each $q > 0$,*

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi}\left|1+K^n e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1}|P'(z)|. \tag{9}$$

The result is best possible and equality holds for the polynomial $P(z) = \alpha z^n + \beta K^n$, where $|\alpha| = |\beta|$.

2. Lemmas

The proof of Theorem 1 is based on a result of Arestov which we shall describe first.

For $\delta = (\delta_0, \delta_1, \ldots, \delta_n) \in c^{n+1}$ and

$$P(z) = \sum_{j=0}^{n} a_j z^j,$$

we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible, if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbf{C} : |z| \leqslant 1\}$

(ii) $P(z)$ has all its zeros in $\{z \in \mathbf{C} : |z| \geqslant 1\}$.

The result of Arestov may now be stated as follows:

LEMMA 1. [2. Theorem 4]. *Let $\phi(x) = \psi(\log x)$, where ψ is a convex nondecreasing function on $\mathbf{R}$. Then for all polynomials $P(z)$ of degree at most n and each admissible operator Λ_δ*

$$\int_0^{2\pi} \phi\left(\left|\Lambda_\delta P(e^{i\theta})\right|\right) d\theta \leqslant \int_0^{2\pi} \phi(c(\delta,n)|P(e^{i\theta})|)d\theta, \tag{10}$$

where

$$c(\delta,n)=\max(|\delta_0|,|\delta_n|).$$

In particular, Lemma 1 applies whith $\phi : x \to x^q$ for every $q \in (0,\infty)$ and $\phi : x \to \log x$ as well. Therefore, we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\Lambda_\delta P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leqslant c(\delta,n)\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q},0<q<\infty. \tag{11}$$

We also need

LEMMA 2. *If $P(z)$ is a polynomial of degree n, which does not vanish for $|z|\leqslant K$, $K\geqslant 1$. Then for all $R\geqslant 1$, $r\leqslant 1$ and for every θ, $0\leqslant\theta<2\pi$,*

$$|P(Rre^{i\theta})| \leqslant \left(\frac{Rr+K}{r+RK}\right)^n \left|R^nP\left(\frac{re^{i\theta}}{R}\right)\right|. \tag{12}$$

Proof. Since all the zeros of $P(z)$ lie in $|z|\geqslant K$, $K\geqslant 1$, we write

$$P(z)=c\prod_{j=1}^{n}(z-R_je^{i\theta_j}) \quad \text{where} \quad R_j\geqslant K, \quad j=1,2,\dots,n.$$

Therefore, for all $r\leqslant 1$, $R\geqslant 1$ and for every θ with $0\leqslant\theta<2\pi$, we have

$$\begin{aligned}\left|\frac{P(Rre^{i\theta})}{R^nP(\frac{re^{i\theta}}{R})}\right| &= \prod_{j=1}^{n}\left|\frac{Rre^{i\theta}-R_je^{i\theta_j}}{re^{i\theta}-RR_je^{i\theta_j}}\right| \\ &= \prod_{j=1}^{n}\left|\frac{Rre^{i(\theta-\theta_j)}-R_j}{re^{i(\theta-\theta_j)}-RR_j}\right| \\ &= \prod_{j=1}^{n}\left(\frac{R^2r^2+R_j^2-2RrR_j\cos(\theta-\theta_j)}{r^2+R^2R_j^2-2RrR_j\cos(\theta-\theta_j)}\right)^{1/2}. \end{aligned} \tag{13}$$

Now, after a short calculation one can easily verify that for every $r\leqslant 1$ and $R\geqslant 1$,

$$\frac{R^2r^2+R_j^2-2RrR_j\cos(\theta-\theta_j)}{r^2+R^2R_j^2-2RrR_j\cos(\theta-\theta_j)} \leqslant \left(\frac{Rr+R_j}{r+RR_j}\right)^2. \tag{14}$$

Since $R_j\geqslant K$, we see that

$$\frac{Rr+R_j}{r+RR_j}\leqslant\frac{Rr+K}{r+KR}. \tag{15}$$

From (13), (14) and (15), it follows that

$$\left|\frac{P(Re^{i\theta})}{R^nP(\frac{re^{i\theta}}{R})}\right| \leqslant \left(\frac{Rr+K}{r+RK}\right)^n,$$

for all $r\leqslant 1\leqslant R$ and for every θ, $0\leqslant\theta<2\pi$, from which the desired result follows immediately.

3. Proofs of the theorems

Proof of Theorem 1. Since the polynomial $P(z)$ has all its zeros in $|z| \geqslant K \geqslant 1$, it follows from Lemma 2 that for every $R \geqslant 1$ and for $|z| = r < 1$

$$|P(Rz)| \leqslant \left(\frac{R|z|+K}{|z|+RK}\right)^n |R^n P(z/R)|. \tag{16}$$

If $R = 1$, then Theorem 1 is trivial, so we assume that $R > 1$. Now, it can be easily verified that

$$\frac{R|z|+K}{|z|+RK} < 1, \qquad \text{for} \quad |z| = r < 1 \quad \text{and} \quad R > 1.$$

Using this in (16), we get

$$|P(Rz)| < |R^n P(z/R)| \qquad \text{for} \quad |z| < 1 \quad \text{and} \quad R > 1. \tag{17}$$

Let $F(z) = P(Rz) + e^{i\alpha} R^n P(z/R)$. We show for every α, $0 \leqslant \alpha < 2\pi$ and $R > 1$, that polynomial $F(z)$ does not vanish in $|z| < 1$. If this is not true, then there is a point $z = z_0$ with $|z_0| < 1$, such that $F(z_0) = 0$. This gives

$$0 = F(z_0) = P(Rz_0) + e^{i\alpha} R^n P(z_0/R), \quad \text{where} \quad |z_0| < 1.$$

This implies

$$|P(Rz_0)| = |R^n P(z_0/R)| \quad \text{where} \quad |z_0| < 1,$$

which clearly contradicts (17). Hence all zeros of $F(z) = P(Rz) + e^{i\alpha} R^n P(z/R)$ lie in $|z| \geqslant 1$, for every α, $0 \leqslant \alpha < 2\pi$ and $R > 1$. This shows that the operator Λ_δ defined by

$$\begin{aligned} \Lambda_\delta P(z) &= (1 + e^{i\alpha} R^n) a_0 + (R + e^{i\alpha} R^{n-1}) a_1 z + \cdots + (R^n + e^{i\alpha}) a_n z^n \\ &= P(Rz) + e^{i\alpha} R^n P(z/R) \end{aligned} \tag{18}$$

is an admissible operator. Applying (11), we obtain for $0 < q < \infty$

$$\int_0^{2\pi} |P(Re^{i\theta}) + e^{i\alpha} R^n P(e^{i\theta}/R)|^q d\theta \leqslant |R^n e^{i\alpha} + 1|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{19}$$

Integrating both sides of (19) with respect to α from 0 to 2π, we get for $0 < q < \infty$,

$$\int_0^{2\pi}\int_0^{2\pi} \left|P(Re^{i\theta}) + e^{i\alpha} R^n P\left(\frac{e^{i\theta}}{R}\right)\right|^q d\alpha\, d\theta \leqslant \int_0^{2\pi} |R^n e^{i\alpha} + 1|^q d\alpha \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{20}$$

Now for every real α and $t \geqslant s \geqslant 1$, it can be easily verified that $|t + e^{i\alpha}| \geqslant |S + e^{i\alpha}|$, which implies for every $q > 0$,

$$\int_0^{2\pi} |t + e^{i\alpha}|^q d\alpha \geqslant \int_0^{2\pi} |S + e^{i\alpha}|^q d\alpha. \tag{21}$$

Taking $r=1$ in Lemma 2, it follows from (12) that

$$\left|\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right| \geqslant \left(\frac{1+RK}{R+K}\right)^n = t_K \geqslant 1, \tag{22}$$

for every θ, $0 \leqslant \theta < 2\pi$ and $R>1$.

We take $t = \left|\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|$ and $S = t_K$, then from (22), $t \geqslant t_K \geqslant 1$ and we get with the help of (21),

$$\begin{aligned}\int_0^{2\pi} |P(Re^{i\theta}) + e^{i\alpha}R^nP(e^{i\theta}/R)|^q d\alpha &= |P(Re^{i\theta})|^q \int_0^{2\pi}\left|1+\frac{e^{i\alpha}R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|^q d\alpha \\ &= |P(Re^{i\theta})|^q \int_0^{2\pi}\left|e^{i\alpha}\left|\frac{R^nP(e^{i\theta}/R)}{P(Re^{i\theta})}\right|+1\right|^q d\alpha \\ &\geqslant |P(Re^{i\theta})|^q \int_0^{2\pi} |t_Ke^{i\alpha}+1|^q d\alpha.\end{aligned}$$

Using this in (20), we conclude that for $0<q<\infty$,

$$\int_0^{2\pi} |t_Ke^{i\alpha}+1|^q d\alpha \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \leqslant \int_0^{2\pi} |R^ne^{i\alpha}+1|^q d\alpha \int_0^{2\pi} |P(e^{i\theta})|^q d\theta,$$

which immediately leads to (5) and this completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 is identical with the proof of Theorem 1 of [3], except instead of using result of Boas and Rahman (inequality (4)) for $1 \leqslant q < \infty$, we use Theorem 1 with $K=1$ for $0<q<\infty$. We omit the details.

REFERENCES

[1] A. C. ANKENY AND T. J. RIVLIN, *On a theorem of S. Bernstein*, Pacific J. Math **5** (1955), 849–852.

[2] V. V. ARESTOV, *On inequalities for trigonometric polynomials and their derivatives*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 3–22.

[3] A. AZIZ, *Integral mean estimates for polynomials with restricted zeros*, J. Approx. Theory **55** (1988), 232–239.

[4] A. AZIZ AND Q. G. MOHAMMAD, *Growth of polynomials with zeros outside a circle*, Proc. Amer. Math. Soc. **81** (1981), 549–553.

[5] R. P. BOAS, JR. AND Q. I. RAHMAN, *L^p inequalities for polynomials and entire functions*, Arch. Rational Mech. Anal. **11** (1962), 34–39.

[6] G. H. HARDY, *The mean value of the modulus of na analytic function*, Proc. London Math. Soc. **14** (1915), 269–277.

[7] M. A. MALIK, *An integral mean estimate for polynomials*, Proc. Amer. Math. Soc. **91** (1984), 281–284.

[8] G. POLYA AND G. SZEGÖ, *Aufgaben and Lehrsätze aus der Analysis*, Springer–Verlag, Berlin, 1925.

[9] Q. I. RAHMAN AND G. SCHMEISSER, *Les inégalités de Markoff et de Bernstein*, Presses Univ. Montréal, Montréal, Quebec, 1983.

[10] Q. I. RAHMAN AND G. SCHMEISSER, *L^p inequalities for polynomials*, J. Approx. Theory **53** (1998), 26–32.

[11] M. RIESZ, *Über einen satz des Herrn Serge Bernstein*, Acta Math. **40** (1916), 337–347.

(Received July 12, 2000)

Abdul Aziz and Wali Moghammad Shah
Post Graduate Department of Mathematics
Universitiy of Kashmir
Hazratbal
Srinagar - 190006
India