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AN INEQUALITY ON WEIGHTED ORLICZ SPACES FOR A VECTOR–VALUED EXTENSION OF THE HARDY–LITTLEWOOD MAXIMAL OPERATOR ON S^n AND $P^n(\mathbb{R})$

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Abstract. Partitions of the sphere S^n and of the real projective space $P^n(\mathbb{R})$ are constructed and are applied to study inequalities on weighted Orlicz spaces for a vector-valued extension of the Hardy-Littlewood maximal operator for functions with values in an UMD Banach space.

Introduction

The boundedness of the Hardy-Littlewood maximal operator on weighted L^p spaces and for real-valued functions defined on the unit sphere S^n, on the real projective space $P^n(\mathbb{R})$ and on more general spaces, was studied in Calderón [4], Macias-Segovia [9] and Aimar-Macias [1]. The boundedness of this operator on weighted Orlicz spaces and for real-valued functions on S^n was studied in Kazaryan [7].

C. Fefferman and E. M. Stein introduced in [5] a technique to study the Hardy-Littlewood maximal operator. The dyadic decomposition of $\mathbb{R}^n$ is used as a fundamental tool in this technique. The idea is to obtain an integral estimate for the dyadic maximal operator and then, by a transference method, to obtain an integral estimate for the Hardy-Littlewood maximal operator. This technique was applied e. g. in Bourgain [2] and in Tozoni [12] to study integral estimates for vector-valued extensions of this operator and in Sawyer [11] and Ruiz-Torrea [10] to study weighted integral estimates for others maximal operators.

In Section 1 we construct partitions $\mathscr{A}_k^n$, $k \geqslant 0$, of S^n which induce partitions of $P^n(\mathbb{R})$, similar to the dyadic partitions of $\mathbb{R}^n$ and we prove several properties of these partitions. To construct our partitions we use induction on n and we consider S^1 with the usual dyadic partitions. The elements of the partitions are obtained making use of spherical coordinates.

We give below a geometric rule to build the partitions $\mathscr{A}_k^2$ of the sphere S^2.

The elements of $\mathscr{A}_2^2$ are eight geodesic triangles. All of them have their sides on the equator line and on meridian lines, and all of them have one of the poles

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$\mathbb{1} = (1,0,0)$ and $-\mathbb{1} = (-1,0,0)$ as a vertex. If $Q, Q' \in \mathscr{A}_2^2$, them there exists a rotation $u \in SO(3)$ such that $Q' = u(Q)$.

The elements of $\mathscr{A}_{k+1}^2$ are obtained dividing the elements of $\mathscr{A}_k^2$. The elements of $\mathscr{A}_k^2$, $k \geqslant 2$, are geodesic triangles and geodesic rectangles. In each $\mathscr{A}_k^2$, $k \geqslant 2$, there exist exactly eight geodesic triangles. Now we give a rule to obtain the elements of $\mathscr{A}_{k+1}^2$, dividing the elements of $\mathscr{A}_k^2$.

Let Q be a triangle in $\mathscr{A}_k^2$, $k \geqslant 2$, let B_1 and B_2 be the middle points of the sides of Q which are on meridian lines and let B_3 be the middle point of the side which is parallel to the equator plane. Let $\overline{B_1B_2}$ be the geodesic segment joining the points B_1 and B_2 and let B_4 be the middle point of this segment. Then Q is the union of a geodesic triangle Q_1 and two geodesic rectangles Q_2 and Q_3, all in $\mathscr{A}_{k+1}^2$, which are obtained when we divide Q using the geodesic segments $\overline{B_1B_2}$ and $\overline{B_3B_4}$.

Now, let Q be a geodesic rectangle of $\mathscr{A}_k^2$, $k \geqslant 3$, and let C_1 and C_2 be the middle points of the sides of Q which are on meridian lines and let C_3 and C_4 be the middle points of the sides of Q which are parallel to the equator plane. Then Q is the union of four geodesic rectangles Q_1, Q_2, Q_3, Q_4 in $\mathscr{A}_{k+1}^2$, which are obtained when we divide Q using the geodesic segments $\overline{C_1C_2}$ and $\overline{C_3C_4}$.

In Section 2 we show that the partitions constructed in Section 1 can be used to introduce the technique of Fefferman-Stein in the case of S^n and $P^n(\mathbb{R})$. Applying this technique we prove a weighted integral inequality for a vector-valued extension of the Hardy-Littlewood maximal operator for functions with values in an UMD Banach space. With this result we obtain the version for S^n and $P^n(\mathbb{R})$ of the estimates for the Hardy-Littlewood maximal operator proved in [5, 2, 12] and a vector-valued extension of a result in [7].

We point out that the natural dyadic partitions of S^n are not good enough to build the theory and to prove the results in Section 2. The problem occurs as consequence of the failure of the relation dyadic-nondyadic given in the condition (b) of Theorem 1.1.

1. Dyadic Partitions of S^n and $P^n(\mathbb{R})$

In this section we construct partitions $\mathscr{A}_k^n$ of the unit n-sphere S^n in $\mathbb{R}^{n+1}$ proceeding by induction on the dimension n. These partitions induce partitions of the real projective space $P^n(\mathbb{R})$. The main result of this section is Theorem 1.1. To prove Theorem 1.1 we need three lemmas where we obtain properties of the partitions $\mathscr{A}_k^n$.

If $x \in \mathbb{R}^n$, we write $|x|_n = (x \cdot x)^{1/2}$, where $x \cdot y$ is the usual scalar product of x and y in $\mathbb{R}^n$ and thus $S^n = \{x \in \mathbb{R}^{n+1} : |x|_{n+1} = 1\}$. We denote by $SO(n+1)$ the group of proper rotations in $\mathbb{R}^{n+1}$ and by du the normalized left Haar measure on $SO(n+1)$. If $x \in S^n$ and $\ell > 0$, we denote by $B_n(x,\ell)$ the intersection of S^n with the closed ball in $\mathbb{R}^{n+1}$ with center x and radius ℓ.

Let $\mathscr{D}_1 = [0,2\pi]$ and for $n \geqslant 2$ let $\mathscr{D}_n = [0,\pi]^{n-1} \times [0,2\pi]$. We define the application $\xi_n : \mathscr{D}_n \to S^n$ by $\xi_n(\theta_1,\dots,\theta_n) = (x_1,\dots,x_{n+1})$ where

$$x_1 = \cos\theta_1, \quad x_i = \cos\theta_i \prod_{j=1}^{i-1} \sin\theta_j, \quad 2 \leqslant i \leqslant n, \quad x_{n+1} = \prod_{j=1}^{n} \sin\theta_j.$$

We observe that, if $\theta = (\theta_1, \dots, \theta_n)$ and $\theta' = (\theta_2, \dots, \theta_n)$, then

$$\xi_n(\theta) = (\cos\theta_1, \sin\theta_1\ \xi_{n-1}(\theta')).$$

The Lebesgue measure of a measurable set $A \subset S^n$ will be denoted by $\sigma(A)$. If G is a measurable set in $\mathscr{D}_n$ we have that

$$\sigma(\xi_n(G)) = \int_G \sin^{n-1}\theta_1 \dots \sin\theta_{n-1} d\theta,$$

where $d\theta = d\theta_1 \dots d\theta_n$. We write $\omega_n = \sigma(S^n)$.

Now, for integers k and j with $k \geqslant 0$ and $1 \leqslant j \leqslant 2^k$, let I_j^k denote the dyadic interval $[(j-1)2^{-k+1}\pi, j2^{-k+1}\pi)$. We define

$$\mathscr{G}_k^1 = \{I_j^k : 1 \leqslant j \leqslant 2^k\};$$
$$\mathscr{G}_0^n = \{[0,\pi]^{n-1} \times [0,2\pi)\},\ n \geqslant 2;$$
$$\mathscr{G}_1^2 = \{[0,\pi/2) \times [0,2\pi),\ [\pi/2,\pi] \times [0,2\pi)\};$$
$$\mathscr{G}_1^n = \{[0,\pi/2) \times [0,\pi]^{n-2} \times [0,2\pi),\ [\pi/2,\pi] \times [0,\pi]^{n-2} \times [0,2\pi)\},\ n \geqslant 3.$$

For n, $k \geqslant 2$, we define the family $\mathscr{G}_k^n$, using induction on n, as the family formed by the sets

$$[0, 2^{-k+1}\pi) \times [0,\pi/2)^{n-1},\ [\pi - 2^{-k+1}\pi, \pi] \times [0,\pi/2)^{n-1},$$
$$(0, 2^{-k+1}\pi) \times G, \quad [\pi - 2^{-k+1}\pi, \pi) \times G,$$

where $G \in \mathscr{G}_2^{n-1}$, $G \neq [0,\pi/2)^{n-1}$, and if $k \geqslant 3$, also formed by the sets

$$I_j^k \times G,\ I_{2^{k-1}-j+1}^k \times G,$$

where $G \in \mathscr{G}_r^{n-1}$, $3 \leqslant r \leqslant k$ and $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$. For $n \geqslant 1$ and $k \geqslant 0$ we define

$$\mathscr{A}_k^n = \{\xi_n(G) : G \in \mathscr{G}_k^n\}, \quad \mathscr{A}^n = \bigcup_{k \geqslant 0} \mathscr{A}_k^n.$$

For each $k \geqslant 0$, $\mathscr{A}_k^n$ is a partition of S^n, and, for every $Q \in \mathscr{A}_{k+1}^n$, $k \geqslant 0$, there exists $Q' \in \mathscr{A}_k^n$, such that $Q \subset Q'$.

Let $Q_1 = \xi_n(I_j^k \times G)$ and $Q_2 = \xi_n(I_{2^{k-1}-j+1}^k \times G)$ two elements of $\mathscr{A}_k^n$, where $G \in \mathscr{G}_r^{n-1}$. Then there exists $u \in SO(n+1)$ such that $u(Q_1) = Q_2$. As consequence of this remark, it will be enough to consider in the proofs of the results of this section, only the elements of the type Q_1.

If $\#F$ denotes the number of elements of a finite set F, then we have that $\#\mathscr{A}_k^1 = 2^k$ for all $k \geqslant 0$, $\#\mathscr{A}_0^n = 1$, $\#\mathscr{A}_1^n = 2$, $\#\mathscr{A}_2^n = 2^{n+1}$ and for $k \geqslant 3$,

$$\#\mathscr{A}_k^n = 2(\#\mathscr{A}_2^{n-1} + \sum_{r=3}^{k} 2^{r-3}\#\mathscr{A}_r^{n-1}).$$

LEMMA 1.1. **(a)** *Let $n \geqslant 2$, $k \geqslant 1$ and $Q_1 = \xi_n(G_1) \in \mathscr{A}_k^n$, $Q_2 = \xi_n(G_2) \in \mathscr{A}_{k+1}^n$ such that $Q_2 \subset Q_1$. If $G_1 = I_1^1 \times G_1'$ with $G_1' \in \mathscr{G}_1^{n-1}$, then $G_2 = I_i^2 \times G_2'$*

with $i \in \{1,2\}$ *and* $G_2' \in \mathscr{G}_2^{n-1}$. *If* $k \geqslant 2$ *and* $G_1 = I_j^k \times G_1'$ *with* $G_1' \in \mathscr{G}_r^{n-1}$ *and* $2 \leqslant r \leqslant k$, *then* $G_2 = I_i^{k+1} \times G_2'$ *where* $i \in \{2j-1, 2j\}$ *and* $G_2' \in \mathscr{G}_2^{n-1} \cup \mathscr{G}_3^{n-1}$ *if* $j = 1$ *and* $G_2' \in \mathscr{G}_{r+1}^{n-1}$ *if* $2 \leqslant j \leqslant 2^{k-2}$.

(b) *Let* $n \geqslant 1$, $k \geqslant 0$ *and* $Q \in \mathscr{A}_k^n$. *If* $k = 0$, *then* Q *is the union of 2 elements of* $\mathscr{A}_1^n$, *and if* $k = 1$, *then* Q *is the union of* 2^n *elements of* $\mathscr{A}_2^n$. *For* $k \geqslant 2$, Q *is the union of, at least* $n+1$ *and at most* 2^n, *elements of* $\mathscr{A}_{k+1}^n$.

Proof. Let us prove (a). It is trivial for $k = 1$. Consider $k \geqslant 2$ and let i and s such that $G_2 = I_i^{k+1} \times G_2'$, $G_2' \in \mathscr{G}_s^{n-1}$. Since $I_j^k = I_{2j-1}^{k+1} \cup I_{2j}^{k+1}$ and $G_2 \subset G_1$, then $i \in \{2j-1, 2j\}$ and $G_2' \subset G_1'$. If $r = 2$, then $i \in \{1,2\}$ and hence $s \in \{2,3\}$. If $r \geqslant 3$, then $2^{r-2}+1 \leqslant i \leqslant 2^{r-1}$ and hence $s = r+1$.

Now, let us prove (b). It is obvious for $k = 0$. We have that $\#\mathscr{A}_2^n = 2^{n-1}(\#\mathscr{A}_2^1) = 2^{n+1}$ and $\#\mathscr{A}_1^n = 2$, and hence we obtain (b) for $k = 1$.

We will prove (b) for $k \geqslant 2$ and $n \geqslant 1$, using induction on n. It is obvious that (b) is true for $n = 1$. Now, suppose that any element of $\mathscr{A}_k^{n-1}$ is the union of at least n and at most 2^{n-1} elements of $\mathscr{A}_{k+1}^{n-1}$, for all $k \geqslant 2$. Let $Q_1 = \xi_n(G_1) \in \mathscr{A}_k^n$, $Q_2 = \xi_n(G_2) \in \mathscr{A}_{k+1}^n$ with $G_1 = I_j^k \times G_1'$, $G_1' \in \mathscr{G}_r^{n-1}$, $2 \leqslant r \leqslant k$ and $G_2 = I_i^{k+1} \times G_2'$, $G_2' \in \mathscr{G}_s^{n-1}$, $2 \leqslant s \leqslant k+1$ and suppose that $Q_2 \subset Q_1$. If $j = 1$, then it follows by (a) that $i \in \{1,2\}$ and $G_2' = G_1'$ for $i = 1$, $G_2' \in \mathscr{G}_3^{n-1}$ for $i = 2$. Therefore, by the induction hypothesis, we can conclude that Q_1 is the union of $q+1$ elements of the type Q_2, where q is an integer, $n \leqslant q \leqslant 2^{n-1}$. If $j \geqslant 2$, then it follows by (a) that $i \in \{2j-1, 2j\}$ and $G_2' \in \mathscr{G}_{r+1}^{n-1}$. Therefore, by the induction hypothesis, we can conclude that Q_1 is the union of $2q$ elements of the type Q_2, where q is an integer, as above. $\square$

It follows by the above proof that the elements of $\mathscr{A}_k^2$ are unions of 3 or 4 elements of $\mathscr{A}_{k+1}^2$ if $k \geqslant 2$, 4 if $k = 1$ and 2 if $k = 0$, and the elements of $\mathscr{A}_k^3$ are unions of 4, 5, 6 or 8 elements of $\mathscr{A}_{k+1}^3$ if $k \geqslant 2$, 8 if $k = 1$ and 2 if $k = 0$.

LEMMA 1.2. *Let* $n \geqslant 1$, $k \geqslant 0$, $\delta_k^1 = 2^{-k}\pi$, $\delta_k^2 = (1+2^{-2}\pi)2^{-k+1}\pi$ *and* $\delta_k^n = 2^{n-k}\pi$ *for* $n \geqslant 3$. *If* $Q \in \mathscr{A}_k^n$, *then there exists* $x \in Q$ *such that* $Q \subset B_n(x, \delta_k^n)$.

Proof. If $Q \in \mathscr{A}_0^n$, then $Q = S^n = B_n(x, \delta_0^n)$ for any $x \in S^n$. If $Q \in \mathscr{A}_1^n$, then Q is a half of the n-sphere S^n. Therefore, since $\delta_1^n > \sqrt{2}$, there exists $x \in Q$ such that $Q \subset B_n(x, \delta_1^n)$.

We will prove the lemma for $k \geqslant 2$ and $n \geqslant 1$, using induction on n. The proof is trivial for $n = 1$. Now, suppose that the lemma is true for $n-1$.

Let $Q = \xi_n(G) \in \mathscr{A}_k^n$, where $G = I_j^k \times G'$, with $G' \in \mathscr{G}_2^{n-1}$ if $j = 1$ and $G' \in \mathscr{G}_r^{n-1}$ if $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$, $3 \leqslant r \leqslant k$. We set $\phi = j2^{-k+1}\pi$, $t = 2^{-k+1}\pi$ and $\alpha = 2^{-r+1}\pi$. If $j = 2^{r-3}+i$ and $1 \leqslant i \leqslant 2^{r-3}$, then $\phi = \pi t/4\alpha + it$ and

$$Q = \{(\cos\theta_1, \sin\theta_1\ \xi_{n-1}(\theta')) : \phi - t \leqslant \theta_1 \leqslant \phi,\ \theta' \in G'\}.$$

It is enough to consider $i = 2^{r-3}$, that is, $\phi = \pi t/2\alpha$. Let $Q' = \xi_{n-1}(G')$. Since $Q' \in \mathscr{A}_r^{n-1}$, it follows by the induction hypothesis that there exists $x' \in Q'$ such that

$Q' \subset B_{n-1}(x', \delta_r^{n-1})$. If $\theta = (\theta_1, \theta') \in G$, then $x = (\cos\phi,\ \sin\phi\ x')$, $\xi_n(\phi, \theta') = (\cos\phi,\ \sin\phi\ \xi_{n-1}(\theta'))$ and $\xi_n(\theta) = (\cos\phi_1, \sin\theta_1\ \xi_{n-1}(\theta'))$ are in Q and

$$\begin{aligned}|\xi_n(\theta) - \xi_n(\phi, \theta')|_{n+1} =&((\cos\theta_1 - \cos\phi)^2 + (\sin\theta_1 - \sin\phi)^2|\xi_{n-1}(\theta')|_n^2)^{1/2}\\ =&|\xi_1(\theta_1) - \xi_1(\phi)|_2 \leqslant t\end{aligned}$$

and

$$|\xi_n(\phi, \theta') - x|_{n+1} = \sin\phi\ |\xi_{n-1}(\theta') - x'|_n \leqslant \sin\phi\ \delta_r^{n-1}.$$

Therefore

$$\begin{aligned}|\xi_n(\phi) - x|_{n+1} \leqslant&|\xi_n(\theta) - \xi_n(\phi, \theta')|_{n+1} + |\xi_n(\phi, \theta') - x|_{n+1}\\ \leqslant&t + \phi\delta_r^{n-1} \leqslant \delta_k^n.\end{aligned}$$

Then $\xi_n(\theta) \in B_n(x, \delta_k^n)$ for all $\theta \in G$, that is, $Q \subset B_n(x, \delta_k^n)$. □

LEMMA 1.3. *Let $n \geqslant 1$, $k \geqslant 3$, $G_k^n = I_{2k-2}^k \times \ldots \times I_{2k-2}^k \times I_1^k \in \mathscr{G}_k^n$, $Q_k^n = \xi_n(G_k^n)$, $\alpha^{n,k} = (\pi/2 - 2^{-k}\pi, \ldots, \pi/2 - 2^{-k}\pi,\ 2^{-k}\pi) \in \mathscr{D}_n$, $x^{n,k} = \xi_n(\alpha^{n,k})$ and $\rho_k^n = 2^{(1-n)/2}(2 - 2\cos 2^{-k}\pi)^{1/2}$. Then $B_n(x^{n,k}, \rho_k^n) \subset Q_k^n$.*

Proof. Suppose that $y = \xi_n(\theta_1, \theta') \notin Q_k^n$. If $\theta_1 \notin I_{2k-2}^k$, then

$$\begin{aligned}|y - x^{n,k}|_{n+1} \geqslant&((\cos\theta_1 - \cos\alpha_1^{n,k})^2 + (\sin\theta_1|\xi_{n-1}(\theta')|_n - \sin\alpha_1^{n,k}|x^{n-1,k}|_n)^2)^{1/2}\\ =&|\xi_1(\theta_1) - \xi_1(\alpha_1^{n,k})|_2\\ >&(2 - 2\cos 2^{-k}\pi)^{1/2}.\end{aligned}$$

Now consider $\theta_1 \in I_{2k-2}^k$. Since $y \notin Q_k^n$, we have that $\theta' \notin G_k^{n-1}$ and thus $\xi_{n-1}(\theta') \notin Q_k^{n-1}$. If $\alpha_1^{n,k} \leqslant \theta_1 \leqslant \pi/2$, let $v_1 = \sin\alpha_1^{n,k}\ \xi_{n-1}(\alpha^{n-1,k})$, $v_2 = \sin\alpha_1^{n,k}\xi_{n-1}(\theta')$ and $\beta = (\sin\theta_1 - \sin\alpha_1^{n,k})/\sin\alpha_1^{n,k}$. Since $\beta \geqslant 0$ and $|v_1|_n = |v_2|_n$, we obtain

$$\begin{aligned}|y - x^{n,k}|_{n+1} \geqslant&|\sin\theta_1\ \xi_{n-1}(\theta') - \sin\alpha_1^{n,k}\ \xi_{n-1}(\alpha^{n-1,k})|_n\\ =&|\beta v_2 + (v_2 - v_1)|_n\\ \geqslant&|v_2 - v_1|_n\\ =&\cos 2^{-k}\pi\ |\xi_{n-1}(\theta') - x^{n-1,k}|_n\ .\end{aligned}$$

If $\pi/2 - 2^{-k+1}\pi \leqslant \theta_1 < \alpha_1^{n,k}$, then choosing conveniently v_1, v_2 and β we obtain

$$|y - x^{n,k}|_{n+1} > \cos 2^{-k+1}\pi\ |\xi_{n-1}(\theta') - x^{n-1,k}|_n.$$

Therefore, from the above inequalities, we can conclude using induction on n, that for $n \geqslant 1$ and $k \geqslant 3$,

$$\begin{aligned}|y - x^{n,k}|_{n+1} >&(\cos 2^{-k+1}\pi)^{n-1}(2 - 2\cos 2^{-k}\pi)^{1/2}\\ \geqslant&2^{(1-n)/2}(2 - 2\cos 2^{-k}\pi)^{1/2}.\end{aligned}$$

Then, if $y \notin Q_k^n$, it follows by the above inequality that $y \notin B_n(x^{n,k}, \rho_k^n)$, that is, $B_n(x^{n,k}, \rho_k^n) \subset Q_k^n$. □

We are now in conditions of proving the main result of this section.

THEOREM 1.1. *For all $n \geqslant 1$, there exists a constant D_n, depending only on n, such that:*

(a) *If $k \geqslant 0$, $Q_1 \in \mathscr{A}_k^n$ and $Q_2 \in \mathscr{A}_{k+1}^n$ with $Q_2 \subset Q_1$, then*

$$\sigma(Q_1) \leqslant D_n \sigma(Q_2).$$

(b) *For all $Q \in \mathscr{A}_k^n$, $k \geqslant 0$, there exist $x \in Q$ and $0 \leqslant \ell \leqslant 2$, such that $Q \subset B_n(x,\ell)$ and*

$$\sigma(B_n(x,\ell)) \leqslant D_n \sigma(Q).$$

(c) *For all $x \in S^n$ and all $0 \leqslant \ell \leqslant 2$, there exist $k \geqslant 0$, $Q \in \mathscr{A}_k^n$ and $u \in SO(n+1)$, such that $B_n(x,\ell) \subset u(Q)$ and*

$$\sigma(Q) \leqslant D_n \sigma(B_n(x,\ell)).$$

Proof. We will prove (a) for the constant $D_n = 2^{n(n+1)/2}$, using induction on n. The proof is trivial in the case $n = 1$, $k \geqslant 0$ and in the case $n \geqslant 1, k = 0$. Suppose that (a) is true for $n-1$ and $k \geqslant 1$. Let $k \geqslant 1$, $Q_1 = \xi_n(G_1) \in \mathscr{A}_k^n$, $Q_2 = \xi_n(G_2) \in \mathscr{A}_{k+1}^n$ such that $Q_2 \subset Q_1$. If $G_1 = I_j^k \times G_1'$, $G_1' \in \mathscr{G}_r^{n-1}$, $1 \leqslant r \leqslant k$, then it follows from Lemma 1.1(a) that $G_2 = I_i^{k+1} \times G_2'$ where $i \in \{2j-1, 2j\}$ and $G_2' \in \mathscr{G}_r^{n-1} \cup \mathscr{G}_{r+1}^{n-1}$. By the induction hypothesis we obtain

$$\begin{aligned}\sigma(Q_1) =&\sigma(\xi_{n-1}(G_1'))\int_{I_j^k} \sin^{n-1}\theta_1 d\theta_1 \\ \leqslant& 2^{n(n-1)/2}\sigma(\xi_{n-1}(G_2'))\int_{I_j^k} \sin^{n-1}\theta_1 d\theta_1.\end{aligned} \tag{1.1}$$

It is enough to prove (a) for $i = 2j-1$. If $j \geqslant 2$, taking into account that $\sin(t+2^{-k}\pi) \leqslant 2\sin t$ for $2^{-k}\pi \leqslant t \leqslant \pi/2$, we obtain

$$\int_{I_j^k} \sin^{n-1}\theta_1 d\theta_1 \leqslant 2^n \int_{I_{2j-1}^{k+1}} \sin^{n-1}\theta_1 d\theta_1.$$

Now, since $\sin 2t \leqslant 2\sin t$, we obtain

$$\int_{I_1^k} \sin^{n-1}\theta_1 d\theta_1 \leqslant 2^n \int_{I_1^{k+1}} \sin^{n-1}\theta_1 d\theta_1.$$

Therefore, it follows from (1.1) that

$$\sigma(Q_1) \leqslant 2^{n(n+1)/2}\sigma(\xi_{n-1}(G_2'))\int_{I_{2j-1}^{k+1}} \sin^{n-1}\theta_1 d\theta_1 = 2^{n(n+1)/2}\sigma(Q_2).$$

Let us prove (b). The case $n = 1$ is true for $D_1 = 1$. Since the elements of $\mathscr{A}_0^n$ and $\mathscr{A}_1^n$ are balls in S^n, we can suppose $k \geqslant 2$. Let $n = 2$, $k \geqslant 2$ and $Q = \xi_2(G) \in \mathscr{A}_k^2$, where $G = I_j^k \times G'$, with $G' \in \mathscr{G}_2^1$ if $j = 1$ and $G' \in \mathscr{G}_r^{n-1}$ if $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$, $3 \leqslant r \leqslant k$. We set $\phi = (j-1)2^{-k+1}\pi$, $t = 2^{-k+1}\pi$ and $\alpha = 2^{-r+1}\pi$. If $j = 2^{r-3}+i$ and $1 \leqslant i \leqslant 2^{r-3}$, then $\phi = \pi t/4\alpha + (i-1)t$. Considering ϕ, t and α fixed, the element Q of smallest area is obtained when $i = 1$, and thus it is enough to consider $\phi = \pi t/4\alpha$. By Lemma 1.2 there exists $x \in Q$ such that $Q \subset B_2(x, \delta_k^2)$, where $\delta_k^2 = (1 + 2^{-2}\pi)2^{-k+1}\pi$. We have that $\sigma(B_2(x, \delta_k^2)) = \pi(\delta_k^2)^2 < 4\pi t^2$ and $\sigma(Q) = \alpha(\cos\phi - \cos(\phi + t))$. For $j = 1$ we have

$$\frac{\sigma(B_2(x, \delta_k^2))}{\sigma(Q)} \leqslant 2\pi^2,$$

and for $j \geqslant 2$,

$$\frac{\sigma(B_2(x, \delta_k^2))}{\sigma(Q)} < \frac{4\pi t^2}{\alpha(\cos\phi - \cos(\phi + t))} < 16\left(\frac{\phi}{\sin\phi}\right)^2 \leqslant 4\pi^2.$$

Now let $n \geqslant 3$ and $k \geqslant 2$. We will proceed by induction on n. Therefore let us suppose that the result is true for $n-1$. Let $Q = \xi_n(G) \in \mathscr{A}_k^n$, where $G = I_j^k \times G'$, with $G' \in \mathscr{G}_2^{n-1}$ if $j = 1$ and $G' \in \mathscr{G}_r^{n-1}$ if $2^{r-3} + 1 \leqslant j \leqslant 2^{r-2}$, $3 \leqslant r \leqslant k$, and let ϕ, t and α as before. Again, it is enough to consider $\phi = \pi t/4\alpha$. By Lemma 1.2, there exists $x \in Q$ such that $Q \subset B_n(x, \delta_k^n)$, where $\delta_k^n = 2^{n-1}t$. For $j = 1$ we have

$$\frac{\sigma(B_n(x, \delta_k^n))}{\sigma(Q)} \leqslant 2^{n^2} D_{n-1}.$$

Since $s/2 \leqslant \sin s \leqslant s$ for $0 \leqslant s \leqslant \pi/2$, then

$$\frac{2^{n-1}}{n\pi^{n-1}}[(\phi + t)^n - \phi^n] \leqslant \int_\phi^{\phi+t} \sin^{n-1} s ds \leqslant \frac{1}{n}[(\phi + t)^n - \phi^n]. \tag{1.2}$$

But for $0 \leqslant \ell \leqslant \pi/2$,

$$\sigma(B_n(x, \ell)) = \omega_{n-1} \int_0^{2\,\mathrm{arc}\,\sin(\ell/2)} \sin^{n-1} s ds$$

and hence

$$\frac{\omega_{n-1}2^{n-1}}{n\pi^{n-1}}\ell^n \leqslant \sigma(B_n(x, \ell)) \leqslant \frac{\omega_{n-1}2^n}{n}\ell^n. \tag{1.3}$$

Then it follows by (1.2), (1.3) and by the induction hypothesis that, for $j \geqslant 2$,

$$\begin{aligned}
\sigma(Q) =& \sigma(\xi_{n-1}(G')) \int_\phi^{\phi+t} \sin^{n-1}\theta_1 d\theta_1 \\
\geqslant& \frac{2^{n-1}}{nD_{n-1}\pi^{n-1}}\ \sigma(B_{n-1}(x', \delta_r^{n-1}))\ [(\phi + t)^n - \phi^n] \\
\geqslant& \frac{\omega_{n-2}2^{(n-1)(n-2)-1}}{n(n-1)D_{n-1}\pi^{n-2}}\ \frac{t^{n-1}[(\phi + t)^n - \phi^n]}{\phi^{n-1}}
\end{aligned}$$

and

$$\sigma(B_n(x, \delta_k^n)) \leqslant \frac{\omega_{n-1}2^{n^2}}{n}\ t^n.$$

Therefore
$$\frac{\sigma(B_n(x,\delta_k^n))}{\sigma(Q)} \leqslant C_n \frac{t\phi^{n-1}}{(\phi+t)^n-\phi^n} \leqslant \frac{C_n}{n}.$$

Let us prove (c). Consider Q_k^n, ρ_k^n and $x^{n,k}$ as in Lemma 1.3. We observe that it is enough to prove (c) for $0<\ell\leqslant\rho_3^n$. Let $u\in SO(n+1)$ such that $ux^{n,k}=x$. Thus, it follows by Lemma 1.3 that $B_n(x,\ell)\subset u(Q_k^n)$ if $\rho_{k+1}^n<\ell\leqslant\rho_k^n$ for $k\geqslant 3$. Therefore, it is enough to prove that there exists a constant D_n such that

$$\frac{\sigma(Q_k^n)}{\sigma(B_n(x^{n,k},\rho_{k+1}^n))} \leqslant D_n, \quad k\geqslant 3. \tag{1.4}$$

Consider $n=2$, $k\geqslant 3$ and let $t=2^{-k+1}\pi$. Then
$$\frac{\sigma(Q_k^2)}{\sigma(B_2(x^{2,k},\rho_{k+1}^2))} = \frac{1}{\pi}\,\frac{t\sin t}{1-\cos(t/4)} < 11.$$

Now consider $n\geqslant 3$, $k\geqslant 3$, $t=2^{-k+1}\pi$. Applying (1.2), (1.3) and the inequality $t^2/3\leqslant 1-\cos t$, $0\leqslant t\leqslant \pi/2$, we obtain

$$\sigma(Q_k^n) \leqslant \frac{1}{n!}\prod_{j=1}^{n}\left[\left(\frac{\pi}{2}\right)^j - \left(\frac{\pi}{2}-t\right)^j\right] \leqslant \left(\frac{\pi}{2}\right)^{n(n-1)/2} t^n;$$
$$\sigma(B_n(x^{n,k},\rho_{k+1}^n)) \geqslant \frac{\omega_{n-1}2^{n-1}}{n\pi^{n-1}}\,(\rho_{k+1}^n)^n \geqslant \frac{\omega_{n-1}2^{-(n^2+2)/2}}{n\pi^{n-1}3^{n/2}}\,t^n$$

and consequently we obtain (1.4) . □

REMARK 1.1. In the proof of Theorem 1.1(b), the radius ℓ of $B_n(x,\ell)$ is the same for all $Q\in\mathscr{A}_k^n$. Therefore, for any two elements Q_1, $Q_2\in\mathscr{A}_k^n$, we obtain
$$D_n^{-1}\sigma(Q_1)\leqslant\sigma(Q_2)\leqslant D_n\sigma(Q_1).$$
The above inequality shows that the measures of the elements of $\mathscr{A}_k^n$ are proportional and the constant of proportionality depends only on n.

REMARK 1.2. Consider the equivalence relation Δ in S^n defined by: $x\Delta y \Leftrightarrow y = x$ *or* $y=-x$. Then $P^n(\mathbb{R})$ is the quotient space $S^n/\Delta=\{\bar{x}:\ x\in S^n\}$ where $\bar{x}$ is the equivalence class $\{x,-x\}$ of $x\in S^n$. Let ψ be the projection map from S^n onto $P^n(\mathbb{R})$, $\psi(x)=\bar{x}$. If $A\subset S^n$ then we can identify the subset $\bar{A}=\psi(A)$ of $P^n(\mathbb{R})$ with the subset $A\cup -A=\psi^{-1}(\bar{A})$ of S^n. Let us denote by $\bar{\sigma}$ the image measure of σ by ψ, that is, $\bar{\sigma}(B)=\sigma(\psi^{-1}(B))$ for all Borel subsets B of $P^n(\mathbb{R})$. For $k\geqslant 1$ we define
$$\bar{\mathscr{A}}_k^n=\{\bar{Q}=\psi(Q):\ Q\in\mathscr{A}_k^n\}.$$
We point out that, if we show that $-Q\in\mathscr{A}_k^n$ for all $Q\in\mathscr{A}_k^n$, $k\geqslant 1$, then it will be easy to check that $\bar{\mathscr{A}}_k^n$ is a partition of $P^n(\mathbb{R})$ and that, all the properties of the partitions $\mathscr{A}_k^n$ of S^n also hold for the partitions $\bar{\mathscr{A}}_k^n$ of $P^n(\mathbb{R})$, in particular the conditions (a), (b) and (c) of Theorem 1.1 hold for the same constant D_n.

We will show that $-Q \in \mathscr{A}_k^n$ if $Q \in \mathscr{A}_k^n$ using induction on n. It is obvious that $-Q \in \mathscr{A}_1^n$ if $Q \in \mathscr{A}_1^n$. If $Q = \xi_2(I_j^k \times I_l^2) \in \mathscr{A}_k^2$, $j = 1, 2^{k-1}$ and $1 \leqslant l \leqslant 4$, then $-Q = \xi_2(I_i^k \times I_u^2) \in \mathscr{A}_k^2$ for $i = 2^{-k}$ if $j = 1$, $i = 1$ if $j = 2^{-k}$, $u = l+2$ if $1 \leqslant l \leqslant 2$ and $u = l-2$ if $3 \leqslant l \leqslant 4$. If $Q = \xi_2(I_j^k \times I_l^r) \in \mathscr{A}_k^n$ with $3 \leqslant r \leqslant k$, $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$ and $1 \leqslant l \leqslant 2^r$, then since $-\xi_2(\theta_1, \theta_2) = \xi_2(\pi - \theta_1, \theta_2 - \pi)$, we have that $-Q = \xi_2(I_i^k \times I_u^r) \in \mathscr{A}_k^2$ for $i = 2^{k-1} - j + 1$, $u = l + 2^{r-1}$ if $1 \leqslant l \leqslant 2^{r-1}$ and $u = l - 2^{r-1}$ if $2^{r-1}+1 \leqslant l \leqslant 2^r$. If $Q = \xi_2(I_{2^{k-1}-j+1}^k \times I_l^r)$ we have $-Q = \xi_2(I_i^k \times I_u^r) \in \mathscr{A}_k^2$ for $i = j$, $u = l + 2^{r-1}$ if $1 \leqslant l \leqslant 2^{r-1}$ and $u = l - 2^{r-1}$ if $2^{r-1}+1 \leqslant l \leqslant 2^r$.

Now suppose that $-Q \in \mathscr{A}_k^{n-1}$ if $Q \in \mathscr{A}_k^{n-1}$, $k \geqslant 1$. Let $Q = \xi_n(G) \in \mathscr{A}_k^n$ where $G = I_j^k \times G'$ with $G' \in \mathscr{G}_2^{n-1}$ if $j = 1$ and $G' \in \mathscr{G}_r^{n-1}$ if $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$, $3 \leqslant r \leqslant k$. If $Q' = \xi_{n-1}(G')$, then $-Q' = \xi_{n-1}(G^*)$ where $G^* \in \mathscr{G}_2^{n-1}$ if $j = 1$ and $G^* \in \mathscr{G}_r^{n-1}$ if $2^{r-3}+1 \leqslant j \leqslant 2^{r-2}$, $3 \leqslant r \leqslant k$. Therefore $-Q = \xi_n((\pi - I_j^k) \times G^*) = \xi_n(I_i^k \times G^*) \in \mathscr{A}_k^n$ for $i = 2^{k-1} - j + 1$.

2. Vector Valued Maximal Operators

In this section X will denote a Banach space or a Banach lattice with norm $\|\cdot\|$ and with the UMD property (see [3]) and W will denote a positive integrable function on S^n. Our reference for Banach space and Banach lattice is [8].

The definitions and results in this section will be enunciated and proved only for functions defined on the sphere S^n. By Remark 1.2 all the results of Section 1, including Theorem 1.1, also hold for partitions $\bar{\mathscr{A}}_k^n$ of the real projective space $P^n(\mathbb{R})$. As consequence all the results in this section will also hold for $P^n(\mathbb{R})$.

We denote by $L_X^p(W)$, $1 \leqslant p < \infty$, the space of all measurable X-valued functions f on S^n such that

$$\|f\|_{L_X^p(W)} = \left(\int_{S^n} \|f(x)\|^p W(x) d\sigma(x) \right)^{1/p} < \infty.$$

The space $L_X^p(W)$ is a Banach space with the norm $\|\cdot\|_{L_X^p(W)}$. For $W = 1$ we write $L_X^p(W) = L_X^p$.

Let $1 < p < \infty$. If there exists a constant C such that

$$\left(\frac{1}{\sigma(B_n(x,\ell))} \int_{B_n(x,\ell)} W d\sigma \right) \left(\frac{1}{\sigma(B_n(x,\ell))} \int_{B_n(x,\ell)} W^{-1/(p-1)} d\sigma \right)^{(p-1)} \leqslant C, \quad (2.1)$$

for all $\ell > 0$ and $x \in S^n$, we say that W is a weight in the Muckenhoupt's class $A_p(S^n)$. If $W \in A_p(S^n)$, we denote by $C(p, W)$ the smallest constant C that satisfies (2.1). Now, if there exists a constant C such that

$$\left(\frac{1}{\sigma(Q)} \int_Q W d\sigma \right) \left(\frac{1}{\sigma(Q)} \int_Q W^{-1/(p-1)} d\sigma \right)^{(p-1)} \leqslant C, \quad (2.2)$$

for all $Q \in \mathscr{A}^n$, we say that W is a weight in the class $A_p(\mathscr{A}^n)$. The class $A_\infty(S^n)$ (respectively $A_\infty(\mathscr{A}^n)$) is defined as the union of the classes $A_p(S^n)$ (respectively $A_p(\mathscr{A}^n)$) for $1 < p < \infty$.

Let f be a real-valued integrable function on S^n. The Hardy-Littlewood maximal operator and the dyadic maximal operator are defined at f respectively by

$$M(f)(x) = \sup_B \frac{1}{\sigma(B)} \int_B |f(y)| d\sigma(y)$$

and

$$M_d(f)(x) = \sup_{\substack{x \in Q \\ Q \in \mathscr{A}^n}} \frac{1}{\sigma(Q)} \int_Q |f(y)| d\sigma(y)$$

where the supremum is taken over all $B = B_n(t,\ell)$ with $\ell > 0$ and $t \in S^n$ such that $x \in B_n(t,\ell)$.

LEMMA 2.1. *Let D_n be the constant in Theorem 1.1. Then, for all $1 < p \leqslant \infty$, all real-valued integrable function f and $x \in S^n$,*

$$A_p(S^n) \subset A_p(\mathscr{A}^n), \tag{2.3}$$

$$M_d(f)(x) \leqslant D_n M(f)(x). \tag{2.4}$$

Proof. Let $1 < p < \infty$ and $W \in \mathscr{A}_p(S^n)$. If $Q \in \mathscr{A}^n$, then by Theorem 1.1(b), there exist $t \in Q$ and $\ell > 0$ such that $Q \subset B_n(t,\ell)$ and $\sigma(B_n(t,\ell)) \leqslant D_n \sigma(Q)$. Therefore it follows by (2.1) that

$$\left(\frac{1}{\sigma(Q)} \int_Q W d\sigma\right) \left(\frac{1}{\sigma(Q)} \int_Q W^{-1/(p-1)} d\sigma\right) \leqslant D_n^p C(p,W)$$

and for a real-valued integrable function f we have that

$$\frac{1}{\sigma(Q)} \int_Q |f(y)| d\sigma(y) \leqslant \frac{D_n}{\sigma(B_n(t,\ell))} \int_{B_n(t,\ell)} |f(y)| d\sigma(y).$$

Consequently $W \in A_p(\mathscr{A}^n)$ and $M_d(f)(x) \leqslant D_n M(f)(x)$ for all $x \in S^n$. □

The next two results follow from martingale theory when we consider the increasing sequence of σ-fields $(\mathscr{F}_k)_{k \geqslant 0}$, where $\mathscr{F}_k$ is the σ-field generated by $\mathscr{A}_k^n$.

THEOREM 2.1. ([6]). *Let $1 < p < \infty$. Then $W \in A_p(\mathscr{A}^n)$ if and only if the maximal operator M_d is bounded on $L^p_{\mathbb{R}}(W)$.*

Throughout this section Φ is a non-decreasing continuous function on $[0,\infty)$ with $\Phi(0) = 0$ and that satisfies the Δ_2-condition, that is, there exists a constant $c > 0$ such that,

$$\Phi(2\lambda) \leqslant c\Phi(\lambda), \qquad \lambda > 0. \tag{2.5}$$

The following result is an immediate consequence of Theorem 1.1(a) and of Remark 3.1 in [12].

THEOREM 2.2. *Let $W \in A_\infty(\mathscr{A}^n)$ and let X be a UMD Banach space with a normalized unconditional basis $(e_j)_{j\geqslant 1}$. Then there exists a constant K, depending only on X, Φ and W, such that for all $f = \sum_{j=1}^\infty f_j e_j \in L_X^1$,*

$$\int_{S^n} \Phi\left(\sup_{k\geqslant 1} \|\sum_{j=1}^k M_d(f_j)e_j\|\right) W d\sigma \leqslant K \int_{S^n} \Phi(M_d(\|f\|)) W d\sigma. \tag{2.6}$$

REMARK 2.1. Consider $\mathscr{D}_n$, $n \geqslant 2$ with the usual dyadic partition, that is, for $k \geqslant 1$ let

$$\mathscr{G}_k^{n,0} = \{I_{j_1}^k \times \ldots \times I_{j_n}^k : 1 \leqslant j_1, \ldots, j_{n-1} \leqslant 2^{k-1}, 1 \leqslant j_n \leqslant 2^k\}$$

and

$$\mathscr{A}_k^{n,0} = \{\xi_n(G) : G \in \mathscr{G}_k^{n,0}\}.$$

Then $\mathscr{A}_k^{n,0}$ is a natural partition of S^n for each $k \geqslant 1$. An easy calculation can show that the condition (a) of Theorem 1.1 is satisfied for these partitions. Since $Q_k^n = \xi_n(I_{2^{k-2}}^k \times \cdots \times I_{2^{k-2}}^k) \in \mathscr{A}_k^{n,0}$, $k \geqslant 3$, then it follows by Lemma 1.3 that the condition (c) of Theorem 1.1 is also satisfied.

We point out that the condition (b) of Theorem 1.1 is not satisfied for the elements of $\mathscr{A}_k^{n,0}$ around $\mathbb{1} = \xi_n(0,0,\ldots,0) = (1,0,\ldots,0)$. In fact, consider $Q_k = \xi_n(I_1^k \times \cdots \times I_1^k)$ and $a_k = \xi_n(2^{-k+1}\pi, 0, \ldots, 0)$. Suppose that $Q_k \subset B_n(x_k, r_k)$. Then $\mathbb{1}$, $a_k \in B_n(x_k, r_k)$ and hence

$$r_k \geqslant \frac{1}{2}|\mathbb{1} - a_k|_{n+1} = \left[\frac{1}{2}(1 - \cos 2^{-k+1}\pi)\right]^{1/2} = \ell_k.$$

Applying (1.2), (1.3) and the inequality $t^2/3 \leqslant 1 - \cos t$, $0 \leqslant t \leqslant \pi/2$, we obtain

$$\sigma(B_n(x_k, r_k)) \geqslant \sigma(B_n(x_k, \ell_k)) \geqslant \frac{\omega_{n-1} 2^{(n-1)/2}}{n\pi^{n-1} 3^{n/2}} (2^{-k+1}\pi)^n$$

and

$$\sigma(Q_k) \leqslant \frac{1}{n!} (2^{-k+1}\pi)^{n(n+1)/2}.$$

Therefore we have that

$$\lim_{k\to\infty} \frac{\sigma(B_n(x_k, r_k))}{\sigma(Q_k)} = \infty$$

and thus the condition (b) of Theorem 1.1 is not satisfied in this case.

Since the condition (a) of Theorem 1.1 hold, then the results of Theorem 2.1 and Theorem 2.2 also hold in this case, but the relations (2.3) and (2.4) in Lemma 2.1 don't hold as consequence of the failure of the condition (b).

THEOREM 2.3. *Let $W \in A_\infty(S^n)$, let X be a UMD Banach space with a normalized unconditional basis $(e_j)_{j\geqslant 1}$, and suppose that Φ is a convex function. Then there exists a constant K, depending only on X, Φ and W, such that*

$$\int_{S^n} \Phi\Big(\sup_{k\geqslant 1} \|\sum_{j=1}^{k} M(f_j)e_j\|\Big) W d\sigma \leqslant K \int_{S^n} \Phi(M(\|f\|))W d\sigma \tag{2.7}$$

for all $f = \sum_{j=1}^{\infty} f_j e_j \in L^1_X$. Moreover, if $1 < p < \infty$, $W \in A_p(S^n)$ and $f \in L^p_X(W)$, then $\sum_{j=1}^{\infty} M(f_j)e_j$ converges in $L^p_X(W)$ to a function $\widetilde{M}(f)$ and the operator $\widetilde{M}$ is bounded on $L^p_X(W)$.

Proof. We observe that X is a Banach lattice with absolute value $|\sum_j x_j e_j| = \sum_j |x_j| e_j$. Let $f = \sum_{j=1}^{\infty} f_j e_j \in L^1_X$ and let us denote

$$\widetilde{M}(f)(y) = \sum_{j=1}^{k} M(f_j)(y)e_j, \quad \widetilde{M}_d(f)(y) = \sum_{j=1}^{k} M_d(f_j)(y)e_j.$$

Let $x \in S^n$ and $B = B_n(t,\ell)$ such that $x \in B$. It follows by Theorem 1.1(c) that, there exist $Q \in \mathscr{A}^n$ and $u \in SO(n+1)$, such that $B \subset u(Q)$ and $\sigma(Q) \leqslant D_n \sigma(B)$. Therefore

$$\begin{aligned}\frac{1}{\sigma(B)} \int_B |f(y)| d\sigma(y) &\leqslant \frac{D_n}{\sigma(Q)} \int_{u(Q)} |f(y)| d\sigma(y)\\ &= \frac{D_n}{\sigma(Q)} \int_Q |f(uz)| d\sigma(z)\\ &\leqslant D_n \widetilde{M}_d(f\circ u)(u^{-1}x).\end{aligned}$$

Integrating both sides of the above inequality on $SO(n+1)$ and with respect to the Haar measure du, we obtain

$$\frac{1}{\sigma(B)} \int_B |f(y)| d\sigma(y) \leqslant D_n \int_{SO(n+1)} \widetilde{M}_d(f\circ u)(u^{-1}x) du$$

and thus

$$\widetilde{M}(f)(x) \leqslant D_n \int_{SO(n+1)} \widetilde{M}_d(f\circ u)(u^{-1}x) du. \tag{2.8}$$

Since $W \in A_\infty(S^n)$, we can choose $1 < p < \infty$ such that $W \in A_p(S^n)$. Then, it follows by (2.1) that $W\circ u \in A_p(S^n)$ and $C(p, W\circ u) = C(p,W)$ for all $u \in SO(n+1)$. Therefore it follows by (2.4), (2.5), (2.6), (2.8), by Jensen's inequality and Fubini's theorem that,

$$\begin{aligned}
&\int_{S^n} \Phi(\|\widetilde{M}(f)(x)\|)W(x)d\sigma(x) \leqslant \\
&\leqslant \int_{S^n} \Phi\left(D_n \int_{SO(n+1)} \|\widetilde{M}_d(f \circ u)(u^{-1}x)\|du\right) W(x)d\sigma(x) \\
&\leqslant c^b \int_{S^n} \Phi\left(\int_{SO(n+1)} \|\widetilde{M}_d(f \circ u)(u^{-1}x)\|du\right) W(x)d\sigma(x) \\
&\leqslant c^b \int_{S^n} \int_{SO(n+1)} \Phi(\|\widetilde{M}_d(f \circ u)(u^{-1}x)\|)W(x)dud\sigma(x) \\
&= c^b \int_{SO(n+1)} \int_{S^n} \Phi(\|\widetilde{M}_d(f \circ u)(y)\|)W \circ u(y)d\sigma(y)du \\
&\leqslant Kc^b \int_{SO(n+1)} \int_{S^n} \Phi(M_d(\|f \circ u\|)(y))W \circ u(y)d\sigma(y)du \\
&\leqslant Kc^b \int_{SO(n+1)} \int_{S^n} \Phi(D_n M(\|f \circ u\|)(y))W \circ u(y)d\sigma(y)du \\
&\leqslant Kc^{2b} \int_{SO(n+1)} \int_{S^n} \Phi(M(\|f \circ u\|)(u^{-1}x))W(x)d\sigma(x)du
\end{aligned}$$

where b is the unique integer satisfying $2^{b-1} < D_n \leqslant 2^b$. But $M(\|f \circ u\|)(u^{-1}x) = M(\|f\|)(x)$ and hence we obtain

$$\int_{S^n} \Phi(\|\widetilde{M}(f)(x)\|)W(x)d\sigma(x) \leqslant Kc^{2b} \int_{S^n} \Phi(M(\|f\|)(x))W(x)d\sigma(x). \tag{2.9}$$

Now, let $f = \sum_{j=1}^{\infty} f_j e_j$ and $f^k = \sum_{j=1}^{k} f_j e_j$, $k \geqslant 1$. Since (2.9) is true for all $f^k, k \geqslant 1$, it follows by the Monotone Convergence Theorem that (2.7) is true.

Finally, let $1 < p < \infty$, $\Phi(t) = t^p$, $W \in A_p(S^n)$ and $f = \sum_{j=1}^{\infty} f_j e_j \in L_X^p(W)$. Then by (2.7) we obtain

$$\|\sum_{j=\ell}^{\ell+m} M(f_j)e_j\|_{L_X^p(W)} \leqslant K^{1/p}\|M(\|\sum_{j=\ell}^{\ell+m} f_j e_j\|)\|_{L_{\mathbb{R}}^p(W)}$$

for all positive integers ℓ and m. But the operator M is bounded on $L_{\mathbb{R}}^p(W)$ (see [1, 4, 9]) and hence

$$\|\sum_{j=\ell}^{\ell+m} M(f_j)e_j\|_{L_X^p(W)} \leqslant K'\|\sum_{j=\ell}^{\ell+m} f_j e_j\|_{L_X^p(W)}.$$

From the above inequality we can conclude that $\sum_{j=1}^{\infty} M(f_j)e_j$ converges in $L_X^p(W)$ to a function $\widetilde{M}(f)$ and that the operator $f \mapsto \widetilde{M}(f)$ is bounded on $L_X^p(W)$. □

REMARK 2.2. Let X be a Banach lattice of real-valued measurable functions on a σ-finite measure space $(\Omega, \mathscr{B}, \mu)$ with the UMD property. Let us denote by $\mathscr{H}(X)$ the set of all functions of the type $f = \sum_{j=1}^{k} a_j f_j$ where $a_j \in X$, $f_j \in L_{\mathbb{R}}^1$, $k \geqslant 1$.

We have that $\mathscr{H}(X)$ is dense in L^p_X for $1 \leqslant p < \infty$. For $f \in \mathscr{H}(X)$ we define

$$\overline{M}_d(f)(x) = \sup_{\substack{x\in Q \\ Q\in\mathscr{A}^n}} \frac{1}{\sigma(Q)} \int_Q |f(y)|d\sigma(y)$$

and

$$\overline{M}(f)(x) = \sup_B \frac{1}{\sigma(B)} \int_B |f(y)|d\sigma(y),$$

where the supremum is taken over all $B = B_n(t,\ell)$ with $\ell > 0$ and $t \in S^n$ such that $x \in B_n(t,\ell)$.

It was proved in [12] that, if $W \in A_\infty(\mathscr{A}^n)$, there exists a constant K such that

$$\int_{S^n} \Phi(\|\overline{M}_d(f)\|)Wd\sigma \leqslant K \int_{S^n} \Phi(M_d(\|f\|))Wd\sigma$$

for all $f \in \mathscr{H}(X)$. Then, if $W \in A_\infty(S^n)$, we can obtain the inequality (2.9) for the operator $\overline{M}$ instead of $\widetilde{M}$ and for $f \in \mathscr{H}(X)$.

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