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SPECTRAL ORDER $A \succ B$ IF AND ONLY IF $A^{2p-r} \geqslant (A^{\frac{-r}{2}} B^p A^{\frac{-r}{2}})^{\frac{2p-r}{p-r}}$ FOR ALL $p > r \geqslant 0$ AND ITS APPLICATION

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Dedicated to the memory of
Professor Chinami Watari
in deep sorrow

(*communicated by J. Pečarić*)

Abstract. New characterization of spectral order $A \succ B$ is given as follows:

$$A \succ B \text{ if and only if } A^{2p-r} \geqslant (A^{\frac{-r}{2}} B^p A^{\frac{-r}{2}})^{\frac{2p-r}{p-r}} \text{ for all } p > r \geqslant 0$$

and also its application is given.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by: $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Also an operator T is strictly positive (denoted by: $T > 0$) if T is positive and invertible. Olson [19] defined a new order $A \succ B$ among the selfadjoint operators as follows;

$$A \succ B \textit{ holds if and only if } E_t \leqslant F_t \textit{ holds for all } t$$

where $A = \int t dE_t$ and $B = \int t dF_t$, cf, also [4]. Moreover Olson [19] characterized the spectral order for positive operators as follows. For positive operators A and B

$$A \succ B \textit{ if and only if } A^n \geqslant B^n \textit{ for all natural numbers } n.$$

Appling Löwner-Heinz inequality, we recall that for positive operators A and B

$$A \succ B \textit{ if and only if } A^q \geqslant B^q \textit{ for all positive real numbers } q. \qquad (*)$$

A useful characterization of the spectral order is given in [23].

On the other hand, we recall the following order preserving operator inequalities.

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Theorem FI (Furuta inequality)**.**

If $A \geqslant B \geqslant 0$*, then for each* $r \geqslant 0$*,*

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$*.*

Figure 1

The original proof of Theorem FI is in [8], afterward in [2][17] and one page proof in [9] and the domain drawn for p, q and r in the Figure is the best possible one for (i) and (ii) of Theorem FI in [20]. The following Theorem GFI interpolates Furuta inequality itself and a useful inequality equivalent to the main theorem of log majorization in [1].

THEOREM GFI. (Generalized Furuta inequality) *If* $A \geqslant B > 0$*, then for each* $t \in [0,1]$ *and* $p \geqslant 1$*,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{GFI}$$

holds for any $r \geqslant t$ *and* $s \geqslant 1$*.*

The original proof of Theorem GFI is in [10], afterward in [5][14] and one page proof in [11]. The following result means the best possibility of the value of the power $\dfrac{1-t+r}{(p-t)s+r}$ in (GFI) shown in [22][24] and [6].

THEOREM BGFI. *For* $p \geqslant 1$*,* $t \in [0,1]$*,* $r \geqslant t$*,* $s \geqslant 1$ *and* $\alpha > 1$*, there exist* $A,B > 0$ *such that* $A \geqslant B > 0$ *and*

$$A^{(1-t+r)\alpha} \not\geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{(1-t+r)\alpha}{(p-t)s+r}}.$$

Very recently we posed the following question and gave a concrete negative answer to this question in [13] associated with Theorem BGFI.

QUESTION. *For* $A,B > 0$*,* $\log A \geqslant \log B$ *if and only if*

$$A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{Q}$$

holds for all $p \geqslant 1$*,* $r \geqslant t$*,* $s \geqslant 1$ *and* $t \in [0,1]$ *?*

Fujii and Nakamoto [7] have given the following exact answer in Theorem FN-1 to this question by using excellent idea and also they have shown Theorem FN-2 as an application of Theorem FN-1.

THEOREM FN-1. ([7]) *For $A, B > 0$, $A \geqslant B$ if and only if*

$$A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{Q}$$

holds for all $p \geqslant 1$, $r \geqslant t$, $s \geqslant 1$ and $t \in [0,1]$.

THEOREM FN-2. ([7]) *For $A, B > 0$, $A \succ B$ if and only if*

$$A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{Q}$$

holds for all $p, r \geqslant t \geqslant 1$ and $s \geqslant 1$.

In this paper, firstly we show a new characterization of the spectral order:

$$A \succ B \text{ if and only if } A^{2p-r} \geqslant (A^{\frac{-r}{2}} B^p A^{\frac{-r}{2}})^{\frac{2p-r}{p-r}} \text{ for all } p > r \geqslant 0.$$

Secondly we show two extensions of Theorem FN-2 and Theorem FN-1 as an application of this characterization of the spectral order.

2. Statement of results

THEOREM 1. *Let $A > 0$ and $B \geqslant 0$. Then $A \succ B$ if and only if*

$$A^{2p-r} \geqslant (A^{\frac{-r}{2}} B^p A^{\frac{-r}{2}})^{\frac{2p-r}{p-r}}$$

holds for all $p > r \geqslant 0$.

The following result is a simple corollary of Theorem 1.

COROLLARY 2. *Let $A > 0$ and $B \geqslant 0$. Then the following assertions are equivalent;*

(i) $A \succ B$.

(ii) $A^{q-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$ *holds for all $p \geqslant t \geqslant 0$, $2p \geqslant q \geqslant t$, $r \geqslant 0$ and s such that $(p-t)s \geqslant q-t$.*

The following result is an extension of Theorem FN-2 as an application of Theorem 1.

THEOREM 3. *Let $A > 0$ and $B \geqslant 0$. Then the following assertions are equivalent;*

(i) $A \succ B$.

(ii) *For each $\varepsilon \in (0,1]$*

$$A^{(r-t)\varepsilon} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{(r-t)\varepsilon}{(p-t)s+r}} \tag{Q-ε}$$

holds for all $p, r \geqslant t \geqslant 0$ and $s \geqslant 1$.

(iii) *For some $\alpha \geqslant 0$ and each $\varepsilon \in (0,1]$*

$$A^{(r-t)\varepsilon} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{(r-t)\varepsilon}{(p-t)s+r}} \tag{Q-ε}$$

holds for all $p, r \geqslant t \geqslant \alpha$ and $s \geqslant 1$.

The following result is an extension of Theorem FN-1.

COROLLARY 4. *Let* $A > 0$ *and* $B \geqslant 0$. *Then the following assertions are equivalent;*

(i) $A \geqslant B$.

(ii) *For each* $\varepsilon \in (0,1]$

$$A^{(r-t)\varepsilon} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{(r-t)\varepsilon}{(p-t)s+r}} \qquad \text{(Q-}\varepsilon\text{)}$$

holds for all $p \geqslant 1$, $t \in [0,1]$, $r \geqslant t$ *and* $s \geqslant 1$.

3. Proofs of results

We cite a simple proof of Theorem 1 by using the following known Theorem B.

THEOREM B. ([3][18][15][16][21]) *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then*

$$A^{1-t} \geqslant (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{1-t}{p-t}} \quad \textit{for} \quad 2p \geqslant 1 \geqslant p > t \geqslant 0.$$

Proof of Theorem 1. We have only to prove "only if" part since the reverse part "if" is obvious. The hypothesis $A \succ B$ is equivalent to $A^q \geqslant B^q$ for all $q > 0$ by (*). Put $A_1 = A^q$ and $B_1 = B^q$. Applying Theorem B for $A_1 \geqslant B_1$, then

$$A^{q-qt} \geqslant (A^{\frac{-qt}{2}}B^{qp}A^{\frac{-qt}{2}})^{\frac{q-qt}{qp-qt}}$$

for $2qp \geqslant q \geqslant qp > qt \geqslant 0$. Then replacing qp by p and qt by r, so we have

$$A^{q-r} \geqslant (A^{\frac{-r}{2}}B^pA^{\frac{-r}{2}})^{\frac{q-r}{p-r}} \quad \text{for} \quad 2p \geqslant q \geqslant p > r \geqslant 0. \qquad (1)$$

As we can choose $q = 2p$ in (1), we obtain

$$A^{2p-r} \geqslant (A^{\frac{-r}{2}}B^pA^{\frac{-r}{2}})^{\frac{2p-r}{p-r}} \quad \text{for} \quad p > r \geqslant 0,$$

so the proof of Theorem 1 is complete.

Proof of Corollary 2.

(i) $\Longrightarrow$ (ii). Put $A_1 = A^{2p-t}$ and $B_1 = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$. Then $A_1 \geqslant B_1 \geqslant 0$ holds for $p > t \geqslant 0$ by Theorem 1, so that Theorem FI ensures the following (2)

$$A_1^{q_1+r_1} \geqslant (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{q_1+r_1}{p_1+r_1}} \qquad (2)$$

holds for any $p_1 \geqslant q_1$, $q_1 \in [0,1]$ and $r_1 \geqslant 0$ because $(1+r_1)\frac{p_1+r_1}{q_1+r_1} \geqslant p_1 + r_1$ holds. Put $p_1 = \frac{(p-t)s}{2p-t}$, $q_1 = \frac{q-t}{2p-t} \in [0,1]$ and $r_1 = \frac{r}{2p-t} \geqslant 0$ in (2), and refining, we have the following desired result

$$A^{q-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for all $p \geqslant t \geqslant 0$, $2p \geqslant q \geqslant t$, $r \geqslant 0$ and s such that $(p-t)s \geqslant q-t$, so the proof of (ii) is complete.

(ii) $\Longrightarrow$ (i). Put $r = t = 0$ in (ii). Then we have $A^q \geqslant B^q$ for all $q \geqslant 0$, that is, (i) holds.

Proof of Theorem 3. We shall show a proof along an excellent idea of [7]. We need the following result;

THEOREM C. ([12, Theorem 2.1]) *Let* $M \geqslant A \geqslant m > 0$ *and* $h = \frac{M}{m}$. *If* $A \geqslant B \geqslant 0$, *then* $h^{p-1}A^p \geqslant B^p$ *for all* $p \geqslant 1$.

(i) $\Longrightarrow$ (ii). Put $q = p$ in (ii) of Corollary 2. Raise each side of (ii) of Corollary 2 to the power $\dfrac{(r-t)\varepsilon}{p-t+r} \in [0,1]$ by Löwner-Heinz theorem, we have (ii).

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (i). Let $M \geqslant A \geqslant m > 0$ and $h = \frac{M}{m}$. Applying Theorem C to (iii), we have

$$h_1^{p_1-1} A^{(r-t)\varepsilon p_1} \geqslant \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{(r-t)\varepsilon p_1}{(p-t)s+r}}$$

holds for all $p, r \geqslant t \geqslant \alpha$, $s \geqslant 1$ and also $h_1 = h^{(r-t)\varepsilon}$. Put $p_1 = \dfrac{(p-t)s+r}{(r-t)\varepsilon} \geqslant 1$ for $r > t$, then we have

$$h_1^{p_1-1} A^{(r-t)\varepsilon p_1} \geqslant A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}$$

and refining

$$h^{(p-t)s+r(1-\varepsilon)+t\varepsilon} A^{(p-t)s} \geqslant (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s \tag{3}$$

for all $p \geqslant t \geqslant \alpha$ and $s \geqslant 1$. Raise each side of (3) to the power $\frac{1}{s} \in [0,1]$ by Löwner-Heinz theorem and then $s \to \infty$, we obtain

$$h^{p-t} A^p \geqslant B^p \quad \text{for all} \quad p \geqslant t \geqslant \alpha, \tag{4}$$

so that we have $A^p \geqslant B^p$ for all $p \geqslant \alpha$ by putting $t = p$ in (4) and finally we obtain $A^p \geqslant B^p$ for all $p \geqslant 0$ by Löwner-Heinz theorem, that is, (i) holds.

Proof of Corollary 4. (i) $\Longrightarrow$ (ii). Raise each side of (GFI) of Theorem GFI to the power $\dfrac{(r-t)\varepsilon}{1-t+r} \in [0,1]$ by Löwner-Heinz theorem. Then we have (ii).

(ii) $\Longrightarrow$ (i). Proof is essentially contained in the proof of Theorem 3. In fact, by the same way as the proof of (iii) $\Longrightarrow$ (i) in Theorem 3 via $s \to \infty$,

$$h^{p-t} A^p \geqslant B^p \quad \text{for all} \quad p \geqslant 1 \geqslant t \geqslant 0 \tag{5}$$

so that we obtain $A \geqslant B$ because we can put $p = t = 1$ in (5).

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