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MAXIMAL SPECTRALLY SUBMULTIPLICATIVE MATRIX FAMILIES

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Abstract. Using radial sets of induced matrix norms, some maximal families $\mathscr{S} \subseteq \mathscr{M}_n$ such that $\rho(\mathbf{AB}) \leqslant \rho(\mathbf{A})\rho(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathscr{S}$ are identified. In the process new notions about matrix norms (such as adjustability of the spectral radius) are defined and used. As a result, certain further questions are raised.

1. Introduction and preliminaries

Let $\mathscr{M}_n$ denote $n \times n$ complex matrices. For $\mathbf{A} \in \mathscr{M}_n$, $\rho(\mathbf{A})$ is the **spectral radius** or maximum absolute value of an eigenvalue of $\mathbf{A}$. It is well known that for $\mathbf{A}, \mathbf{B} \in \mathscr{M}_n$, there is no general relationship between $\rho(\mathbf{AB})$ and $\rho(\mathbf{A})\rho(\mathbf{B})$, though $\rho(\mathbf{AB}) \leqslant \rho(\mathbf{A})\rho(\mathbf{B})$ would sometimes be convenient for applications.

We say that $\mathscr{F} \subseteq \mathscr{M}_n$ is a *spectrally submultiplicative (matrix) family*, $\mathscr{S}$-family for short, if

$$\rho(\mathbf{AB}) \leqslant \rho(\mathbf{A})\rho(\mathbf{B}),$$

whenever $\mathbf{A}, \mathbf{B} \in \mathscr{F}$, and we say that $\mathscr{F}$ is *maximal* if it is not a proper subset of another $\mathscr{S}$-family. It is relatively straightforward to construct large $\mathscr{S}$-families. For example, all polynomials in a given matrix, or, more generally, a commuting family is an $\mathscr{S}$-family. However, identification of maximal $\mathscr{S}$-families is less straightforward. It is our purpose here to give a technique for constructing maximal $\mathscr{S}$-families using matrix norms. In the process some interesting new notions about norms are encountered. Recall [3] that, in addition to vector norms properties, a *matrix norm* ($|||.||| \to R^+$) is *submultiplicative* ($|||\mathbf{AB}||| \leqslant |||\mathbf{A}|||\ |||\mathbf{B}|||$ for all $\mathbf{A}, \mathbf{B} \in \mathscr{M}_n$). A matrix norm $|||\cdot|||$ is *induced* (by the vector norm $\|\cdot\|$ on C^n) if $|||\mathbf{A}||| = \max_{\|x\|=1} \|\mathbf{Ax}\|$ for each $\mathbf{A} \in \mathscr{M}_n$. Here we study the property of the $\mathscr{S}$-family consisting of matrices having norms equal to their spectral radii. Examples and properties of such matrices are given in [1,2,5,6].

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2. $\mathscr{S}$ -families, radial matrices, and adjustability

As is known [3], if $|||\cdot|||$ is a matrix norm on $\mathscr{M}_n$, then $\rho(\mathbf{A}) \leqslant |||\mathbf{A}|||$ for all $\mathbf{A} \in \mathscr{M}_n$. A matrix $\mathbf{A}$ is called *radial* w.r.t. the matrix norm $|||\cdot|||$, if $\rho(\mathbf{A}) = |||\mathbf{A}|||$, and we say $\mathbf{A}$ is $|||\cdot|||$-radial. Let $\mathscr{R}_{|||\cdot|||} = \{\mathbf{A} \in \mathscr{M}_n : \rho(\mathbf{A}) = |||\mathbf{A}|||\}$ for the matrix norm $|||\cdot|||$. Of course, $\mathscr{R}_{|||\cdot|||}$ may consist of zero matrix only, but if $|||\cdot|||$ is induced, then $\mathscr{R}_{|||\cdot|||}$ consits of other elements, too.

THEOREM 1. *If $|||\cdot|||$ is a matrix norm on $\mathscr{M}_n$, then $\mathscr{R}_{|||\cdot|||}$ is an $\mathscr{S}$-family.*

Proof. In fact, for $\mathbf{A}, \mathbf{B} \in \mathscr{R}_{|||\cdot|||}$, $\rho(\mathbf{AB}) \leqslant |||\mathbf{AB}||| \leqslant |||\mathbf{A}|||\ |||\mathbf{B}||| = \rho(\mathbf{A})\rho(\mathbf{B})$. □

We will look for matrix norms such that $\mathscr{R}_{|||\cdot|||}$ is a *maximal* $\mathscr{S}$-family. It should be noted that not every maximal $\mathscr{S}$-family is an $\mathscr{R}_{|||\cdot|||}$ for some matrix norm $|||\cdot|||$. An example of such a family is the set of all upper triangular matrices $\mathscr{T}_n \subset \mathscr{M}_n$. Suppose that $\mathbf{A} \notin \mathscr{T}_n$, i.e. $a_{ij} \neq 0$ for some $i > j$, and $\{\mathbf{A}\} \cup \mathscr{T}_n$ is an $\mathscr{S}$-family. Then for matrix $\mathbf{B} \in \mathscr{T}_n$ with b_{ji} as the only non-zero entry $\rho(\mathbf{B}) = 0$ and $\rho(\mathbf{AB}) = a_{ij}b_{ji} > \rho(\mathbf{A})\rho(\mathbf{B}) = 0$. Therefore $\mathscr{T}_n$ is a maximal $\mathscr{S}$-family but $\mathscr{T}_n \neq \mathscr{R}_{|||\cdot|||}$ for any matrix norm $|||\cdot|||$ as $\rho(\mathbf{B}) = 0$ and $\mathbf{B} \neq \mathbf{0}$.

For each matrix norm $|||\cdot|||$ there is a minimal matrix norm $N(\cdot)$, smaller than $|||\cdot|||$. Here the order relation is pointwise, for instance, $N(\mathbf{A}) \leqslant |||\mathbf{A}|||$ for all $\mathbf{A} \in M_n$ and therefore $\mathscr{R}_{|||\cdot|||} \subseteq \mathscr{R}_{N(\cdot)}$. It is known that a matrix norm is minimal if and only if it is induced ([3], Theorem 5.6.32). We will focus on the class of induced matrix norms. In fact, for some induced matrix norms, $\mathscr{R}_{|||\cdot|||}$ is a maximal $\mathscr{S}$-family. In order to see conditions for this, and some specific examples, we need to define a new property that a matrix norm may have. We call the matrix norm $|||\cdot|||$ (spectrally) *adjustable* if for any $\mathbf{A} \in \mathscr{M}_n$, there is a $\mathbf{P} \in \mathscr{M}_n$ such that (i) $\rho(\mathbf{P}) = |||\mathbf{P}||| = 1$ and (ii) $\rho(\mathbf{AP}) = |||\mathbf{A}|||$. Thus $\mathbf{P}$ "adjusts" $\mathbf{A}$ to the matrix $\mathbf{AP}$, which is $|||\cdot|||$-radial. We call the matrix norm $|||\cdot|||$ (spectrally) *weak adjustable* if for any $\mathbf{A} \in \mathscr{M}_n \setminus \mathscr{R}_{|||\cdot|||}$, there is a $\mathbf{P} \in \mathscr{M}_n$ such that (i) $\rho(\mathbf{P}) = |||\mathbf{P}||| = 1$ and (ii) $\rho(\mathbf{A}) < \rho(\mathbf{AP})$. Thus $\mathbf{P}$ "adjusts" $\mathbf{A}$ to the matrix $\mathbf{AP}$ with a bigger spectral radius. Observe that for an adjustable norm $|||\cdot|||$ and for any $\mathbf{A} \in \mathscr{M}_n \setminus \mathscr{R}_{|||\cdot|||}$ we have

$$\rho(\mathbf{AP}) = |||\mathbf{A}||| \cdot |||\mathbf{P}||| > \rho(\mathbf{A})\rho(\mathbf{P}).$$

Therefore we have the following:

COMMENT. *Weak adjustability of any matrx norm is implied by its adjustability.*

THEOREM 2. *$\mathscr{R}_{|||\cdot|||}$ is a maximal $\mathscr{S}$-family if and only if $|||\cdot|||$ is a weak adjustable matrix norm on $\mathscr{M}_n$.*

Proof. If $\mathbf{A} \notin \mathscr{R}_{|||\cdot|||}$, choose $\mathbf{P} \in \mathscr{R}_{|||\cdot|||}$ for matrix $\mathbf{A}$ and get $\rho(\mathbf{AP}) > \rho(\mathbf{A}) = \rho(\mathbf{P})\rho(\mathbf{A})$, so that $\mathbf{A}$ cannot be added to $\mathscr{R}_{|||\cdot|||}$ and leave it submultiplicative. The converse is similar. If no $\mathbf{A} \in \mathscr{M}_n \setminus \mathscr{R}_{|||\cdot|||}$ can be added to it, then each such $\mathbf{A}$ must have a $\mathbf{P} \in \mathscr{R}_{|||\cdot|||}$ such that $|||\mathbf{P}||| = 1$ and such that $\rho(\mathbf{AP}) > \rho(\mathbf{A})\rho(\mathbf{P}) = \rho(\mathbf{A})$, the contradiction to submultiplicativity, which means that $\mathbf{A}$ can be weakly adjusted by some matrix from $\mathscr{R}_{|||\cdot|||}$. □

An example of a not weakly adjustable norm is the Frobenius norm or more general any not unit preserving norm, i.e. such that $|||\mathbf{I}_n||| > 1$, in which $\mathbf{I}_n \in \mathscr{M}_n$ is the identity matrix. Namely for every matrix $\mathbf{P}$ such that $\rho(\mathbf{P}) = |||\mathbf{P}||| = 1$ we have

$$\rho(\mathbf{I}_n\mathbf{P}) = \rho(\mathbf{P}) = 1 = \rho(\mathbf{I}_n) \quad \text{and} \quad |||\mathbf{I}_n\mathbf{P}||| = |||\mathbf{P}||| = 1 < |||\mathbf{I}_n|||$$

which means that the identity matrix $\mathbf{I}_n$ cannot be adjusted in either sense.

3. Main results

In order that Theorem 2 be interesting, we should identify some (weak) adjustable matrix norms. We will show that the property holds for any matrix norm induced by a vector norm satisfying the parallelogram law, i.e. by $\|.\|$ such that for all $\mathbf{x}, \mathbf{y} \in C^n$ the following identity holds

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2),$$

or equivalently by $\|.\|$ derived from an inner product ([3], Problem 4, p. 263).

THEOREM 3. *Each matrix norm induced by a vector norm satisfying the parallelogram law is adjustable.*

Proof. Let $|||\cdot|||$ be a matrix norm induced by a vector norm $\|\cdot\|$ satisfying the parallelogram law and assume (without loss of generality) that $|||\mathbf{A}||| = \max_{\|z\|=1} \|\mathbf{Az}\| = 1 = \|\mathbf{Ax}\|$, $\|\mathbf{x}\| = 1$, and $\mathbf{y} = \mathbf{Ax}$. Further, let $\mathscr{B}_1 = \{\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$ and $\mathscr{B}_2 = \{\mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ be two unitary bases of C^n w.r.t. $\|\cdot\|$, i.e. $\mathscr{B}_1$ and $\mathscr{B}_2$ consist, respectively, of vectors of lengths ($\|\cdot\|$) one and orthogonal in the inner product giving $\|\cdot\|$. Further, let $\mathbf{P}$ be the invertible matrix that transforms the basis $\mathscr{B}_2$ to $\mathscr{B}_1$. The proof follows by observing that $\mathbf{P}$ is an isometry (w.r.t. $\|\cdot\|$), i.e. $\rho(\mathbf{P}) = |||\mathbf{P}||| = 1$, $\mathbf{APy} = \mathbf{y}$, and then

$$1 = |||\mathbf{A}||| = \|\mathbf{Ax}\| = \|\mathbf{APy}\| \leqslant |||\mathbf{AP}||| \leqslant |||\mathbf{A}||| = 1.$$

□

Note that for any positive definite matrix $\mathbf{W} \in \mathscr{M}_n$, we may define an inner product $\langle\ ,\ \rangle$ on $\mathscr{C}^n \times \mathscr{C}^n$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^*\mathbf{Wx}$ and define the corresponding norm by $\|\mathbf{x}\|_w = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$. Every inner product for $\mathscr{C}^n$ occurs in this way. The matrix norm induced by $\|\cdot\|_w$ has a form $|||\mathbf{A}|||_w = |||\mathbf{W}^{1/2}\mathbf{A}\mathbf{W}^{-1/2}|||_2$, in which $|||\cdot|||_2$ stands for the spectral norm (cf. [3], Thm. 5.6.7, p. 296). Further, note that if $\mathscr{F}$ is an $\mathscr{S}$-family (respectively, a maximal $\mathscr{S}$-family), then so is $\mathbf{T}^{-1}\mathscr{F}\mathbf{T} = \{\mathbf{T}^{-1}\mathbf{A}\mathbf{T} : \mathbf{A} \in \mathscr{F}\}$ for a fixed similarity by the nonsingular matrix $\mathbf{T} \in \mathscr{M}_n$. Therefore, Theorem 3 can be proved by proving it for the spectral norm only. This may be proven by observing that every matrix $\mathbf{A}$ can be adjusted by its unitary polar factor to a radial matrix $(\mathbf{AA}^*)^{1/2}$ such that $|||\mathbf{A}|||_2 = |||(\mathbf{AA}^*)^{1/2}|||_2$.

COROLLARY. *Let $\|\cdot\|$ be induced by a vector norm $\|\cdot\|$ satisfying the parallelogram law. Then any matrix $\mathbf{A}$ admits a decomposition $\mathbf{A} = \mathbf{RP}$, in which $\mathbf{R}$ is $\|\cdot\|$-radial, $\|\mathbf{A}\| = \|\mathbf{R}\|$, and $\mathbf{P}$ is an isometry (w.r.t $\|\cdot\|$).*

We observe that for the standard Euclidian vector norm, the polar decomposition is an example of such a decomposition.

Not only matrix norms induced by vector norms satisfying the parallelogram law are adjustable as is shown in the next theorem. For such norms let us use the followimg notation. Given a vector norm $\|\cdot\|$ on C^n, denote $\mathscr{E}_{\|\cdot\|}$ the set of all *extreme* points of the unit ball of the norm $\|\cdot\|$. For particular cases we have:

(i) $\mathscr{E}_{\|\cdot\|_2}$ is the set of all unit vectors,

(ii) $\mathscr{E}_{\|\cdot\|_1} = \{\varepsilon \mathbf{e}_k : |\varepsilon| = 1,\ k = 1, 2, \ldots, n\}$

(iii) $\mathscr{E}_{\|\cdot\|_\infty} = \{\mathbf{u} : |u_j| = 1,\ j = 1, 2, \ldots, n\}$.

THEOREM 4. *If for any $\mathbf{y}$ with $\|\mathbf{y}\| = 1$ and any $\mathbf{x} \in \mathscr{E}_{\|\cdot\|}$ there is $\mathbf{P}$ such that*

$$\rho(\mathbf{P}) = \|\mathbf{P}\| = 1 \quad \text{and} \quad \mathbf{Py} = \mathbf{x},$$

then the induced norm $\|\cdot\|$ is adjustable.

Proof. Take $\mathbf{A}$ with $\|\mathbf{A}\| = 1$. Then there is $\mathbf{x} \in \mathscr{E}_{\|\cdot\|}$ such that $\|\mathbf{Ax}\| = 1$. Let $\mathbf{y} = \mathbf{Ax}$. By assumption there is $\mathbf{P}$ such that

$$\rho(\mathbf{P}) = \|\mathbf{P}\| = 1 \quad \text{and} \quad \mathbf{Py} = \mathbf{x},$$

Then since $\mathbf{APy} = \mathbf{Ax} = \mathbf{y}$, we have $\rho(\mathbf{AP}) \geqslant 1$. Therefore

$$1 \leqslant \rho(\mathbf{AP}) \leqslant \|\mathbf{AP}\| \leqslant \|\mathbf{A}\| \cdot \|\mathbf{P}\| = \|\mathbf{A}\| = 1$$

which shows adjustability of the norm $\|\cdot\|$. □

The special choices of matrices $\mathbf{P}$ from Theorem 4 for norms mentioned above are presented in the following.

REMARK.

(i) For $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$ there is a unitary (isometry) matrix $\mathbf{P}$ such that $\mathbf{Py} = \mathbf{x}$. Clearly $\rho(\mathbf{P}) = \|\mathbf{P}\| = 1$.

(ii) For $\|\mathbf{y}\|_1 = 1$ and $\mathbf{x} = \varepsilon\,\mathbf{e}_k$ with $|\varepsilon| = 1$, let $\mathbf{P} = \mathbf{xu}^*$ where $\mathbf{u}$ is the vector of signum of complex conjugate of entries of $\mathbf{y}$. Then $\mathbf{Py} = \mathbf{x}$ and $\mathbf{Px} = \mathbf{u}_k\mathbf{x}$. Clearly $\|\mathbf{P}\| = \rho(\mathbf{P}) = 1$.

(iii) Let $\|\mathbf{y}\|_\infty = 1$ and $\mathbf{x} \in \mathscr{E}_{\|\cdot\|_\infty}$. There is an index k with

$$1 = \|\mathbf{y}\|_\infty = \max_{j=1,2,\cdots,n} |y_j| = |y_k|.$$

Let $\mathbf{P} = \frac{\bar{y}_k}{|y_k|}\mathbf{x}\,\mathbf{e}_k^*$. Then $\mathbf{Py} = \mathbf{x}$, $\mathbf{Px} = \frac{\bar{y}_k}{|y_k|}x_k\,\mathbf{x}$, so that $\|\!|\mathbf{P}|\!\| = \rho(\mathbf{P}) = 1$. □

Note that unit preserving norms that are not induced are not necessarily adjustable. Let $\|\!|\mathbf{A}|\!\| = \max_i \|\!|\mathbf{A}|\!\|_i, i = 1, \infty$, i.e. the maximum of the row and the column norms. $\mathscr{R}_{\|\!|\cdot|\!\|}$ is the intersection $\mathscr{R}_{\|\!|\cdot|\!\|_1} \cap \mathscr{R}_{\|\!|\cdot|\!\|_\infty}$, and these two radial sets are not equal. For example,

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathscr{R}_{\|\!|\cdot|\!\|_\infty} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin \mathscr{R}_{\|\!|\cdot|\!\|_1}.$$

Similarly, in $\mathscr{M}_n$, $(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) \in \mathscr{R}_{\|\!|\cdot|\!\|_\infty}$ and $(\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) \notin \mathscr{R}_{\|\!|\cdot|\!\|_1}$. From Theorem 2 it follows that $\|\!|\cdot|\!\|$ is not weakly adjustable.

4. Questions

We close by mentioning some questions raised by the present work.

(a) Is weak adjustability of a matrix norm equivalent to adjustability; i.e. is every weak adjustable norm also adjustable? There are many interesting sub-questions here, including what subsets of the set $\{\mathbf{P} \in \mathscr{M}_n : \rho(\mathbf{P}) = \|\!|\mathbf{P}|\!\| = 1\}$ may suffice for (weak) adjustability. Correspondingly, what alternate notions of adjustability might be useful?

(b) Is every induced matrix norm adjustable? In particular, a specific approach to this question raises the more precise question whether for the matrix norm $\|\!|\cdot|\!\|$ induced by $\|\cdot\|$ (or even a general matrix norm with subordinate $\|\cdot\|$) and any $\mathbf{A} \in \mathscr{M}_n$ such that $\|\!|\mathbf{A}|\!\| = 1$ with $\|\mathbf{y}\| = \|\mathbf{x}\| = 1$, $\mathbf{y} = \mathbf{Ax}$, does there exist $\mathbf{P} \in \mathscr{M}_n$ such that (i) $\rho(\mathbf{P}) = \|\!|\mathbf{P}|\!\| = 1$ and (ii) $\mathbf{Py} = \mathbf{x}$?

(c) Is every maximal $\mathscr{S}$-family that contains no nonzero matrix with spectral radius 0 actually an $\mathscr{R}_{\|\!|\cdot|\!\|}$, i.e. the radial set for some induced matrix norm $\|\!|\cdot|\!\|$?

(d) It is known that the isometry group does not characterise a vector norm. The radial set $\mathscr{R}_{\|\!|\cdot|\!\|}$ of an induced matrix norm $\|\!|\cdot|\!\|$ may be viewed as an expansion of the isometry group of the inducing vector norm $\|\cdot\|$. Does the radial set characterize an induced matrix norm, i.e. is $\mathscr{R}_{\|\!|\cdot|\!\|_1} = \mathscr{R}_{\|\!|\cdot|\!\|_2}$ if and only if $\|\!|\cdot|\!\|_1 = \|\!|\cdot|\!\|_2$ for induced norms $\|\!|\cdot|\!\|_1$, $\|\!|\cdot|\!\|_2$?

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