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SPHERICALLY SYMMETRIC FUNCTIONS WITH A CONVEX SECOND DERIVATIVE AND APPLICATIONS TO EXTREMAL PROBABILISTIC PROBLEMS

IOSIF PINELIS

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Abstract. We describe the class of all functions $\varphi\colon [0,\infty) \to \mathbb{R}$ for which the second derivative $g''_\varphi(x;y,y)$ of the spherically symmetric function $g_\varphi(x) := \varphi(|x|)$ in the direction of y is convex in x, where x and y are vectors in a Hilbert space H and $|\cdot|$ is the norm in H. Applications to extremal probabilistic problems are given.

1. Introduction

The convexity of the second derivative of generalized moment functions in extremal probabilistic problems was apparently used first, even though implicitly, by Whittle [13], who obtained the following Khinchine-type inequality:

$$\mathsf{E}\left|\sum_{i=1}^{n} \varepsilon_i a_i\right|^p \leqslant \mathsf{E}|\gamma|^p, \tag{1}$$

where $p \geqslant 3$, the a_i's are any real numbers such that $\sum_{i=1}^n a_i^2 = 1$, γ is a standard normal random variable (r.v.), and the ε_i's are independent Rademacher r.v.'s, so that $\mathsf{P}(\varepsilon_i = 1) = \mathsf{P}(\varepsilon_i = -1) = 1/2$. Inequality (1) is exact in the sense that, for any given $p \geqslant 3$, $\mathsf{E}|\gamma|^p$ is the exact upper bound on $\mathsf{E}\left|\sum_{i=1}^n \varepsilon_i a_i\right|^p$ over all a_i's with $\sum_{i=1}^n a_i^2 = 1$; this follows from the central limit theorem.

The only property of the moment function $g(t) = |t|^p$ that is essential for (1) is that its second derivative is convex if $p \geqslant 3$. Thus, one has the following generalization of (1):

$$\mathsf{E}g\left(\sum_{i=1}^{n} \varepsilon_i a_i\right) \leqslant \mathsf{E}g(\gamma) \tag{2}$$

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for all $g \in C^2_{\mathsf{conv}}(\mathbb{R})$; here and in what follows, $C^2_{\mathsf{conv}}(\mathbb{R})$ denotes the set of all functions $g\colon \mathbb{R} \to \mathbb{R}$ whose second derivative g'' is finite and convex. Cf. Eaton [3, 4] and Pinelis [7].

In [7], inequality (2) was extended to spherically symmetric generalized moment functions g defined on a Hilbert space. Here we offer further extensions involving Hilbert-space-valued multilinear and even multiaffine forms in independent Rademacher r.v.'s $\varepsilon_1, \dots, \varepsilon_n$ (known also as Rademacher "chaoses"), in place of the linear form $\sum_{i=1}^n \varepsilon_i a_i$.

Cox and Kemperman [2, (3.14)] obtained the following result:

$$\mathsf{E}g(X+Y) \geqslant \mathsf{E}g(X) + \mathsf{E}g(Y), \tag{3}$$

where X and Y are independent mean zero r.v.'s and g is a function in $C^2_{\mathsf{conv}}(\mathbb{R})$ such that $g(0) = g'(0) = g''(0) = 0$. The conditions $g'(0) = g''(0) = 0$ are not in fact needed here.

Utev [12] extended inequality (3) to the case of Banach-space-valued summands. In the same paper [12], he also obtained a number of other exact comparison inequalities for generalized moment functions with a convex second derivative for Hilbert- and Banach-space-valued r.v.'s.

Of all generalized moment functions g defined on a Hilbert space H, the spherically symmetric functions, i.e. the functions of the norm — of the form $g_\varphi(x) := \varphi(|x|)$ — naturally are of particular interest.

The structure of this paper is as follows.

In Section 2, notation and definitions are introduced.

In Section 3, various necessary and sufficient conditions are stated for a spherically symmetric function to have a convex second derivative.

In Section 4, applications to extremal probabilistic problems are described.

Section 5 is devoted to the proofs.

2. Notation and definitions

Let V be any vector space over $\mathbb{R}$.

Let $C^2_{\text{weak}}(V)$ stand for the set of all functions $g\colon V \to \mathbb{R}$ such that the first two derivatives of the function $\mathbb{R} \ni t \mapsto g(x+ty)$ at $t=0$, denoted by $g'(x;y)$ and $g''(x;y,y)$ respectively, exist and are finite for all x and y in V.

One can generalize the class $C^2_{\mathsf{conv}}(\mathbb{R})$, defined above in the Introduction, at least in two ways, as follows.

Let $C^2_{\mathsf{conv},1}(V)$ stand for the set of all functions $g \in C^2_{\text{weak}}(V)$ such that the mapping $\mathbb{R} \ni t \mapsto g''(x+ty;y,y)$ is convex, for each pair $(x,y) \in V \times V$.

Let $C^2_{\mathsf{conv},2}(V)$ stand for the set of all functions $g \in C^2_{\text{weak}}(V)$ such that the mapping $V \ni x \mapsto g''(x;y,y)$ is convex, for each $y \in V$. In other words, $C^2_{\mathsf{conv},2}(V)$ is the set of all functions $g \in C^2_{\text{weak}}(V)$ such that the mapping $\mathbb{R} \ni t \mapsto g''(x+tz)(y,y)$ is convex, for each triple $(x,y,z) \in V \times V \times V$.

It is obvious that $C^2_{\mathsf{conv},2}(V) \subseteq C^2_{\mathsf{conv},1}(V)$ for any V.

Let H be a real Hilbert space of dimension $\geqslant 1$ with a scalar product (xy) and the corresponding norm $|x| := \sqrt{(xx)}$, where x and y are in H.

In Section 3, we shall describe the classes $C^2_{\mathsf{conv,sph},1}(H)$ and $C^2_{\mathsf{conv,sph},2}(H)$ of all functions $\varphi\colon [0,\infty)\to\mathbb{R}$ for which the spherically symmetric function $g_\varphi\colon H\to\mathbb{R}$ defined by

$$g_\varphi(x) := \varphi(|x|) \quad \forall x\in H$$

belongs to $C^2_{\mathsf{conv},1}(H)$ and $C^2_{\mathsf{conv},2}(H)$, respectively.

Clearly, for $H=\mathbb{R}$, the classes $C^2_{\mathsf{conv,sph},1}(H)$ and $C^2_{\mathsf{conv,sph},2}(H)$ are one and the same class; it will be denoted by $C^2_{\mathsf{conv,sph}}(\mathbb{R})$.

Note that the functions belonging to the classes $C^2_{\mathsf{conv,sph},1}(H)$, $C^2_{\mathsf{conv,sph},2}(H)$, and $C^2_{\mathsf{conv,sph}}(\mathbb{R})$ are defined on $[0,\infty)$ and not on H or on $\mathbb{R}$.

For any real-valued function, defined in neighborhood of a point $t\in\mathbb{R}$, let

$$f(t+) := \lim_{s\downarrow t} f(s); \quad f'_+(t) := \lim_{s\downarrow t}\frac{f(s)-f(t)}{s-t}$$

(if these limits exist);

$$D_+f(t) := \liminf_{s\downarrow t}\frac{f(s)-f(t)}{s-t}\big(\in[-\infty,\infty]\big);$$
$$D^0_+f(t) := f(t); \quad D^k_+f(t) := D_+D^{k-1}_+f(t)\ \forall k=1,2,\dots$$

if D^{k-1}_+f assumes only finite values in a right-hand side neighborhood of t. D_+ is known as the lower right Dini derivative; see e.g. page 56 in [5]. Similarly defined are the left-hand side versions $f(t-), f'_-, D_-$, and D^k_-.

Note that if φ belongs to $C^2_{\mathsf{conv,sph},1}(H)$ or $C^2_{\mathsf{conv,sph},2}(H)$, then it belongs to $C^2_{\mathsf{conv,sph}}(\mathbb{R})$, and so, the second derivative φ'' is finite and convex on $(0,\infty)$, whence

$$D_+\varphi=\varphi', \quad D^2_+\varphi=\varphi'', \quad \text{and} \quad D^3_+\varphi=(\varphi'')'_+ \text{ on } (0,\infty).$$

Moreover, one has the following simple proposition.

PROPOSITION 1. *A function $\varphi\colon[0,\infty)\to\mathbb{R}$ belongs to $C^2_{\mathsf{conv,sph}}(\mathbb{R})$ iff it is continuous at 0 and twice continuously differentiable on $(0,\infty)$, its second derivative φ'' is convex on $(0,\infty)$, and, finally, there exist the limits*

$$\varphi'(0):=\varphi'(0+)=0; \quad \varphi''(0):=\varphi''(0+)\in\mathbb{R}; \quad (\varphi'')'_+(0)\in[0,\infty).$$

For any $\varphi\in C^2_{\mathsf{conv,sph}}(\mathbb{R})$ and any $t>0$, let

$$\begin{aligned} h_2(t) &:= h_{2,\varphi}(t) := t\varphi''(t)-\varphi'(t);\\ h_3(t) &:= h_{3,\varphi}(t) := t^2(\varphi'')'_+(t)-2h_2(t);\\ h_4(t) &:= h_{4,\varphi}(t) := t^3D^4_+\varphi(t). \end{aligned}$$

In view of Proposition 1, for any $\varphi\in C^2_{\mathsf{conv,sph}}(\mathbb{R})$,

$$h_2(0) := h_2(0+) = 0, \tag{4}$$

$$(h_2)'_+(t) = t(\varphi'')'_+(t)\geqslant 0 \quad \forall t\geqslant 0, \tag{5}$$

and so, h_2 is nondecreasing on $[0,\infty)$ and

$$h_2(t) \geqslant 0 \quad \forall t \geqslant 0; \tag{6}$$

furthermore, for any $t \geqslant 0$,

$$h_2(t) = 0 \quad \text{iff} \quad h_2(s) = 0 \, \forall s \in [0,t]. \tag{7}$$

3. Characterizations of spherically symmetric functions with a convex second derivative

The following is the basic result.

THEOREM 1. *Assume that* $\dim H \geqslant 2$*, to avoid the trivial case* $H = \mathbb{R}$*. Then* $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ *iff* $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$ *and the differential inequality*

$$3h_4(t)h_2(t) \geqslant h_3(t)^2 \quad \forall t \in (0,\infty) \tag{8}$$

is true.

If for some $t > 0$ one has $h_4(t) = \infty$, it is assumed that inequality (8) holds for such a value of t; at that, it does not matter whether $h_2(t) = 0$ or not (cf. (6)); nor does the value of $h_3(t)$ matter in such a case.

REMARK 1. Suppose that $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$ and the differential inequality (8) is true for all $t \in (0,\infty) \setminus F$, where F is any finite set. Then it is still true that $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$. This is clear from the proof of Theorem 1, given below, in Section 5; review, in particular, the proof of the “if” part of Lemma 6 there.

REMARK. Theorem 1 will hold if in the definitions of h_2, h_3, and h_4 one replaces D_+ everywhere by D_-.

REMARK. It is obvious that, if $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$, then $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$. On the other hand, if $\dim H = 1$, then not all functions $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ satisfy (8); e.g., consider the functions $\varphi(t) := (t-a)^3_+$ for $a > 0$; here and in what follows, $t_+ := \max(0,t)$ for all real t. Cf. Corollary 2 below.

REMARK. Obviously, instead of the “spherically symmetric” functions $g_\varphi(x) = \varphi(|x|)$, one may as well consider the “elliptically symmetric” functions of the form $\varphi(|x|_A)$, where $|x|_A := \sqrt{(xAx)}$ and A is any nonnegative-definite self-adjoint linear operator in H. For the necessity of (8), one would then have to require that $\dim A(H) \geqslant 2$, instead of $\dim H \geqslant 2$.

COROLLARY 1. *For any* H*, the power function* $\varphi_p(t) := t^p$ *belongs to* $C^2_{\mathsf{conv,sph,2}}(H)$ *iff* $p = 0$ *or* $p = 2$ *or* $p \geqslant 3$.

Utev [12] showed that $\varphi_p \in C^2_{\mathsf{conv,sph,2}}(H)$ for $p \in \{2,4,6\} \cup [8,\infty)$.

COROLLARY 2. *For any H with $\dim H \geqslant 2$ and any $a > 0$, the function $\varphi_{a,p}(t) := (t-a)_+^p$ belongs to $C^2_{\mathsf{conv,sph,2}}(H)$ iff $p \geqslant 7/2$.*

Using Corollaries 1 and 2 and the fact that $C^2_{\mathsf{conv,sph,2}}(H)$ is a convex cone, one can find many other examples of functions in $C^2_{\mathsf{conv,sph,2}}(H)$. Thus, the functions defined for $t > 0$ by the expressions

$$a + bt^2 + \int_3^\infty t^p \mu(dp) \quad \text{and} \quad a + bt^2 + \int_{0+}^\infty (t-s)_+^{7/2} \nu(ds) \tag{9}$$

belong to $C^2_{\mathsf{conv,sph,2}}(H)$, where a and b are any real numbers and μ and ν are any measures such that $\int_3^\infty t^p \mu(dp) < \infty \ \forall t \in (0,\infty)$ and ν is finite.

The first of the two expressions (9) contains such examples as $e^t - t$, $\cosh t$, $e^{t\sqrt{t}} - t\sqrt{t}$, and

$$\begin{cases} \dfrac{t^\beta - t^\alpha}{\ln t} & \text{if } t \in (0,\infty) \setminus \{1\} \\ \beta - \alpha & \text{if } t = 1, \end{cases} \tag{10}$$

where $3 \leqslant \alpha < \beta$; example (10) corresponds to $\mu(dp) := I\{\alpha < p < \beta\}dp$; as usual, $I\{A\}$ denotes the indicator of an assertion A, so that $I\{A\} = 1$ if A is true and $I\{A\} = 0$ otherwise. Note that expression (10) asymptotically behaves as $\dfrac{t^\alpha}{|\ln t|}$ when $t \downarrow 0$ and as $\dfrac{t^\beta}{\ln t}$ when $t \to \infty$. A more interesting and pure (without $\ln t$) example of similar behavior of functions in $C^2_{\mathsf{conv,sph,2}}(H)$ is given below after Corollary 3.

Concerning the second of the two expressions in (9), note that in place of $7/2$ there, one could take any $p \geqslant 7/2$. However, that would not produce new examples, because for any $p > 7/2$ and any $s > 0$

$$(t-s)_+^p = \int_{0+}^\infty (t-u)_+^{7/2} \nu_{s,p}(du),$$

where

$$\nu_{s,p}(du) := \frac{\Gamma(p+1)}{\Gamma(9/2)\Gamma(p-7/2)}\,(u-s)_+^{p-9/2} du.$$

It is possible to solve differential inequality (8) “explicitly” for φ. The “solution” is given by the key identity (48). This leads to the following “parametric” representation of $C^2_{\mathsf{conv,sph,2}}(H)$.

THEOREM 2. *Let $\varphi\colon [0,\infty) \to \mathbb{R}$ and $\dim H \geqslant 2$.*
Then $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ iff for all $t \geqslant 0$

$$\varphi(t) = a + bt^2 + cI\{t > t_0\} \int_{t_0+}^t du \cdot (t^2 - u^2) \exp \int_{t_0+\varepsilon}^u r_K(s)ds, \tag{11}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \geqslant 0$, $t_0 \in [0,\infty]$, $\varepsilon > 0$, and

$$r_K(t) := \frac{3K(t)}{t(t^2 - K(t))}, \quad t > t_0, \tag{12}$$

for some nondecreasing *function* $K\colon (t_0,\infty)\to\mathbb{R}$ *such that*

$$\forall t>t_0 \quad 0\leqslant K(t)<t^2; \tag{13}$$

also, if $t_0\in(0,\infty)$*, then* K *must satisfy the additional condition*

$$\int_{t_0+}^{t_0+\varepsilon} r_K(s)ds=\infty. \tag{14}$$

In particular, (11) implies that $\varphi(t)=a+bt^2$ for all $t\in[0,t_0]$. In the case $t_0=\infty$, (11) must be understood simply as $\varphi(t)=a+bt^2$ for all $t\geqslant 0$. Concerning the integral in (11) under the exp sign (for which it may happen that $u<t_0+\varepsilon$) and other such integrals, we understand an integral of the form $\int_t^s f(u)du$ with $s<t$ as $\left(-\int_s^t f(u)du\right)$.

The following two propositions complement the above theorem.

PROPOSITION 2. *Representation (11) is essentially unique. Suppose that* φ *may be represented in the form (11). Then*

$$a=\varphi(0); \quad b=\frac{\varphi''(0)}{2}. \tag{15}$$

Now further assume that c *is nonzero in (11). Then*

$$t_0=\sup\{t>0\colon h_2(t)=0\}; \tag{16}$$

$$K(t)=t^2\,\frac{t^2D_+^3\varphi(t)-2t\varphi''(t)+2\varphi'(t)}{t^2D_+^3\varphi(t)+t\varphi''(t)-\varphi'(t)} \quad \forall t>t_0; \tag{17}$$

moreover, the denominator $t^2D^3\varphi_+(t)+t\varphi''(t)-\varphi'(t)$ *is strictly positive for all* $t>t_0$*; if* $\{t>0\colon h_2(t)=0\}=\emptyset$*, then* $\sup\{t>0\colon h_2(t)=0\}:=0$*; as to* ε*, it may be any positive real number; however, the value of* c *is uniquely determined by the choice of* $\varepsilon>0$ *provided that* $t_0<\infty$*. Thus, for every* $\varepsilon>0$*, one has a parametrization of the set* $C^2_{\mathsf{conv,sph,2}}(H)$ *of functions* φ *by means of the set of the 5-tuples* (a,b,c,t_0,K)*.*

PROPOSITION 3. *Suppose that* $\varphi\in C^2_{\mathsf{conv,sph}}(\mathbb{R})$*,* $K(t)$ *is defined by (17), and* t_0 *is defined by (16); in particular,* $t_0:=\infty$ *if* $h_2=0$ *on* $(0,\infty)$*. Then* $\varphi\in C^2_{\mathsf{conv,sph,2}}(H)$ *iff* K *is non-decreasing on* (t_0,∞)*.*

While Theorem 1 is convenient for checking whether a given function is in $C^2_{\mathsf{conv,sph,2}}(H)$, Theorem 2 is more suitable for construction of functions in $C^2_{\mathsf{conv,sph,2}}(H)$ possessing particular properties. The following useful corollary is our first illustration of this point.

COROLLARY 3. *Let* $\varphi\in C^2_{\mathsf{conv,sph,2}}(H)$*. Then there exists a nondecreasing sequence* $(\psi_m)_{m=1}^\infty$ *in* $C^2_{\mathsf{conv,sph,2}}(H)$ *approximating* φ *in the sense that, for every* m*,* $\psi_m=\varphi$ *on* $[0,m]$ *and such that* $\psi_m(t)\sim c_mt^3$ *as* $t\to\infty$*, for some sequence* $(c_m)_{m=1}^\infty$ *in* $(0,\infty)$*.*

Corollary 3 will be used in this paper to prove Theorem 6 below.

EXAMPLE 1. A nontrivial example of a function in $C^2_{\text{conv,sph,2}}(H)$ is given by

$$\varphi_{K\equiv 1}(t) := (4t^2+11)\sqrt{(t^2-1)_+} - (9t^2+6)\ \arctan\sqrt{(t^2-1)_+}\,,$$

corresponding to the following choice of parameters in (11):

$$a=b=0, \quad t_0=1, \quad \text{and} \quad K=1 \text{ on } (1,\infty).$$

This function asymptotically behaves as $4t^3$ when $t\to\infty$ and as $k\cdot(t-1)^{7/2}$ when $t\downarrow 1$, where $k:=\dfrac{96\sqrt{2}}{35}$; cf. Corollaries 1 and 2. Its third derivative, $\varphi'''_{K\equiv 1}(t)$, rapidly increases from 0 to 24 when t increases from 1 to ∞.

Also, note that the function $\varphi_p(t)=t^p$ in $C^2_{\text{conv,sph,2}}(H)$ with $p\geqslant 3$ corresponds to $a=b=t_0=0$ and $K(t)=\alpha t^2$ $\forall t>0$ with $\alpha\in[0,1)$ (so that (13) holds), where the correspondence between $p\geqslant 3$ and $\alpha\in[0,1)$ is given by $p=\dfrac{3}{1-\alpha}$.

For any $\varphi\in C^2_{\text{conv,sph}}(\mathbb{R})$ one has the Taylor expansion

$$\varphi(u)=\varphi(0)+\varphi''(0)\frac{u^2}{2}+\frac{1}{6}\int_{t\geqslant 0}(u-t)^3_+\,d(f'')'_+(t) \tag{18}$$

for all $u\geqslant 0$; cf. (3.2) in [7]. This means that any $\varphi\in C^2_{\text{conv,sph}}(\mathbb{R})$ can be represented as the limit of *linear* combinations with nonnegative coefficients of functions of the following three types: (i) the constants; (ii) $u\mapsto au^2$, $a\in\mathbb{R}$, and (iii) $u\mapsto b(u-t)^3_+$, $t\geqslant 0$, $b\geqslant 0$. (Speaking somewhat loosely, one can say that these three types of functions constitute the extreme rays of this convex cone of functions, $C^2_{\text{conv,sph}}(\mathbb{R})$; this cone is actually closed with respect to the pointwise convergence; cf. the proof of Proposition A.1 in [7]). Thus, the simple to obtain linear representation (18) is very useful, as it allows one to reduce any comparison inequality between r.v.'s ξ and η of the type $\mathsf{E}\varphi(|\xi|)\leqslant\mathsf{E}\varphi(|\eta|)$ for all $\varphi\in C^2_{\text{conv,sph}}(\mathbb{R})$ to the same inequality but only for the extreme functions φ, of the three above types; for example, see the proof of Lemma 3.1 in [7].

Unlike (18), representation (11) is highly *nonlinear* (even though monotonic) in K, while $C^2_{\text{conv,sph,2}}(H)$ is still a convex cone of functions. Hence, some natural and interrelated questions arise here.

Open problem

- Is there a tractable *linear* representation for the functions $\varphi\in C^2_{\text{conv,sph,2}}(H)$ if $\dim H\geqslant 2$?
- What are then the extreme rays of the convex cone $C^2_{\text{conv,sph,2}}(H)$? What do they have to do with functions like the one described in Example 1?

Class $C^2_{\text{conv,sph,1}}(H)$ is much easier to describe that $C^2_{\text{conv,sph,2}}(H)$:

THEOREM 3. $C^2_{\text{conv,sph,1}}(H)=C^2_{\text{conv,sph}}(\mathbb{R})$.

4. Applications

4.1. Generalized moment comparison inequalities with generalized moment functions in $C^2_{\mathsf{conv,sph},1}(H)$ (As we have just stated in Theorem 3, $C^2_{\mathsf{conv,sph},1}(H) = C^2_{\mathsf{conv,sph}}(\mathbb{R})$.)

Let us denote any vector $(t_1,\dots,t_n)\in\mathbb{R}^n$ by the same bold-faced letter, $\mathbf{t}$, and let

$$\mathbf{t}^{(i)} := (t_1,\dots,t_{i-1},t_{i+1},\cdots,t_n)$$

for all $i\in\{1,\dots,n\}$; similarly, with letters ε and ξ in place of t.

Let us refer to a function $M\colon\mathbb{R}^n\to V$ as *multiaffine* if it is affine in each of its n arguments; that is, if the function $M_{i,\mathbf{t}^{(i)}}\colon\mathbb{R}\to V$ defined by

$$M_{i,\mathbf{t}^{(i)}}(t_i) := M(\mathbf{t})$$

is affine: for every $i\in\{1,\dots,n\}$ and every $\mathbf{t}^{(i)}\in\mathbb{R}^{n-1}$ there are vectors $a_{i,\mathbf{t}^{(i)}}$ and $b_{i,\mathbf{t}^{(i)}}$ in V such that

$$M(\mathbf{t}) = M_{i,\mathbf{t}^{(i)}}(t_i) = a_{i,\mathbf{t}^{(i)}} + t_i b_{i,\mathbf{t}^{(i)}} \quad \forall t_i\in\mathbb{R}.$$

It is easy to see that a function $M\colon\mathbb{R}^n\to V$ is multiaffine iff M admits a representation of the form

$$M(\mathbf{t}) = \sum_{J\subseteq\{1,\dots,n\}}\Big(\prod_{i\in J} t_i\Big)c_J \quad \forall\mathbf{t}\in\mathbb{R}^n,$$

where $\prod\limits_{i\in\emptyset} t_i := 1$ and $c_J\in V\ \ \forall J\subseteq\{1,\dots,n\}$.

In the special case when $c_J\neq 0$ only if $\operatorname{card} J = 1$, the multiaffine form $M(\mathbf{t})$ is just a linear form (in $t_1,\dots,t_n$) with values in V.

Let $\varepsilon_1,\dots,\varepsilon_n$ be independent Rademacher r.v.'s, as in the Introduction.

Let $\xi_1,\dots,\xi_n$ be independent (not necessarily identically but) symmetrically distributed real-valued r.v.'s such that

$$\mathsf{E}\xi_1^2 = \cdots = \mathsf{E}\xi_n^2 = 1.$$

In particular, the ξ_i's may be independent standard normal r.v.'s. Let

$$\boldsymbol{\varepsilon} := (\varepsilon_1,\dots,\varepsilon_n) \quad\text{and}\quad \boldsymbol{\xi} := (\xi_1,\dots,\xi_n).$$

THEOREM 4. *Let a function $M\colon\mathbb{R}^n\to V$ be multiaffine. Then*

$$\mathsf{E}g(M(\boldsymbol{\varepsilon})) \leqslant \mathsf{E}g(M(\boldsymbol{\xi})) \quad \textit{whenever}\quad g\in C^2_{\mathsf{conv},1}(V). \tag{19}$$

If $M\colon\mathbb{R}^n\to V$ is multiaffine, then $M(\boldsymbol{\varepsilon})$ is the sum of so-called *Rademacher "chaoses"* of different degrees.

Theorems 4 and 3 imply the following.

THEOREM 5. *Let a function* $M: \mathbb{R}^n \to H$ *be multiaffine. Then*

$$\mathsf{E}\varphi(|M(\varepsilon)|) \leqslant \mathsf{E}\varphi(|M(\boldsymbol{\xi})|) \quad \textit{whenever} \quad \varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R}). \tag{20}$$

The latter result was obtained in [7] in the special case when $M(\mathbf{t})$ is a linear form (in $t_1 \dots, t_n$); in that case, obviously, $|M(\mathbf{t})|^2$ is an arbitrary nonnegative-definite quadratic form in $t_1 \dots, t_n$.

4.2. Generalized moment comparison inequalities with generalized moment functions in $C^2_{\mathsf{conv,sph,2}}(H)$. In this subsection, we shall assume that the Hilbert space H is separable. This condition is imposed in order that the sum of any H-valued r.v.'s be a r.v.

THEOREM 6. *Let* $X_1, \dots, X_n$ *be independent zero-mean* H*-valued r.v.'s, with the sum* $S := X_1 + \dots + X_n$. *Then one has the following general and exact version of the Rosenthal [11] lower bound:*

$$\mathsf{E}\varphi(|S|) \geqslant \sum_{i=1}^{n} \mathsf{E}\varphi(|X_i|) \tag{21}$$

for any generalized moment function $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ *with* $\varphi(0) = 0$*; cf. the Cox-Kemperman inequality (3). In particular,*

$$\mathsf{E}|S|^p \geqslant \sum_{i=1}^{n} \mathsf{E}|X_i|^p \quad \forall p \geqslant 3; \tag{22}$$

$$\mathsf{E}(|S| - a)_+^p \geqslant \sum_{i=1}^{n} \mathsf{E}(|X_i| - a)_+^p \quad \forall p \geqslant \frac{7}{2} \quad \forall a > 0. \tag{23}$$

In [12], inequality (22) was obtained for $p \geqslant 8$.

THEOREM 7. *Let* G *be a finite Borel measure on* H *such that* $\int_H x G(dx) = 0$. *Let* $\mathscr{X}(G)$ *be the set of all finite sequences* $(X_1, \dots, X_n)$ *of independent zero-mean* H*-valued r.v.'s such that*

$$\sum_{i=1}^{n} \mathsf{P}(X_i \in A) = G(A) \tag{24}$$

for all Borel $A \subseteq H \setminus \{0\}$*; here the length* n *of the sequence is not fixed. Let* $\mathscr{X}_{\text{ident}}(G)$ *be the set of all the sequences* $(X_1, \dots, X_n) \in \mathscr{X}(G)$ *with identically distributed* $X_1, \dots, X_n$. *Finally, let* $P_G :=$ *Pois*(G) *be the compound Poisson distribution in* H *with the Lévy measure* G*, so that the characteristic functional of* P_G *is given by*

$$\int_H e^{i(ux)} P_G(dx) = \exp \int_H (e^{i(ux)} - 1) G(dx)$$

for all $u \in H$. *Then*

$$\sup_{\mathscr{X}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^{n} X_i\right|\right) = \sup_{\mathscr{X}_{\text{ident}}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^{n} X_i\right|\right) = \int_H \varphi(|x|) P_G(dx) \tag{25}$$

for all $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$. *In particular,*

$$\sup_{\mathscr{X}(G)} \mathsf{E}\left|\sum_{i=1}^n X_i\right|^p = \sup_{\mathscr{X}_{\mathrm{ident}}(G)} \mathsf{E}\left|\sum_{i=1}^n X_i\right|^p = \int_H |x|^p P_G(dx) \quad \forall p \geqslant 3. \tag{26}$$

These results hold if in (24) the equality sign is replaced by " $\leqslant$ *".*

Relation (26) was obtained in [12] for $p \geqslant 8$.

Theorem 7 takes its origin in Prokhorov [10], where $H = \mathbb{R}$. This line of inquiry was developed in [8], [6], [12], and [9]. The proof, below, of relations (21) and (25) is based on results of [12].

5. Proofs

In what follows, $f\colon [0,\infty) \to \mathbb{R}$ and $[a,b) \subseteq [0,\infty)$. Lemmas 1–4 below are elementary and essentially well known. They will be given here for easy reference. Lemmas 1 and 2 will be given with no proofs.

If a reader is willing to assume that f is a restriction to $[0,\infty)$ of a smooth enough function, defined on $\mathbb{R}$, then Lemmas 1–4 are not needed; such an additional assumption would also make the reading a little easier at a few places below. However, then such subtle examples as Example 1 on page 13 would be lost, and also that would probably make it more difficult to approach the open problem stated on page 13.

LEMMA 1. *If* f *is continuous on* $[a,b)$, *then it is convex on* $[a,b)$ *iff* f'_+ *is defined, finite, right-continuous, and nondecreasing on* $[a,b)$.

LEMMA 2. *Let* $c \in (a,b)$ *and let* f *be continuous on* $[a,b]$ *and convex on* $[a,c)$ *and on* $(c,b]$. *Then* f *is convex on* $[a,b]$ *iff* $f'_-(c) \leqslant f'_+(c)$.

Now one can prove Proposition 1.

Proof of Proposition 2. Let $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$, so that the second derivative g'' of the function $g(\cdot) := \varphi(|\cdot|)$ is finite and convex, and hence continuous, on $\mathbb{R}$. Then $\varphi''[= g'']$ is convex on $(0,\infty)$, and

$$\exists \varphi''(0+) = g''(0) \in \mathbb{R}. \tag{27}$$

Therefore, one can extend φ'' from $(0,\infty)$ to $[0,\infty)$ by the formula $\varphi''(0) := \varphi''(0+)$. Next, by Lemma 1,

$$(g'')'_+(0) = \lim_{t\downarrow 0} \frac{\varphi''(t) - \varphi''(0)}{t} = (\varphi'')'_+(0) \in \mathbb{R},$$

and, analogously,

$$(g'')'_-(0) = \lim_{t\uparrow 0} \frac{\varphi''(|t|) - \varphi''(0)}{t} = -(\varphi'')'_+(0).$$

It follows that

$$(\varphi'')'_+(0) = \frac{(g'')'_+(0) - (g'')'_-(0)}{2} \in [0,\infty),$$

in view of Lemma 2. Since g is even, one has $g'(0) = 0$, and so,

$$g''(0) = \lim_{t \downarrow 0} \frac{\varphi'(t)}{t}, \tag{28}$$

so that $\varphi'(0+) = 0$. The "only if" part is proved.

To prove the "if" part, observe that, by the l'Hospital rule,

$$g'(0) = \lim_{t \to 0} \frac{\varphi(|t|) - \varphi(0)}{t} = \varphi'(0) = 0;$$
$$g''(0) = \lim_{t \to 0} \frac{\varphi'(|t|)}{|t|} = \varphi''(0) = \lim_{t \to 0} g''(t).$$

It remains to use the above equalities, the equality $(g)''_{\pm}(0) = \pm(\varphi'')'_{+}(0)$, and Lemma 2. $\square$

LEMMA 3. *Let $f = u_1 u_2 + u_3$ on $[a,b)$ for some functions u_1, u_2, and u_3, such that u_1 is nondecreasing or continuous (or, more generally, such that $u_1(t-) \leqslant u_1(t)$ $\forall t \in (a,b)$), u_2 is nonnegative and continuous, and u_3 is continuous. Or, more generally, assume that $f(t-) \leqslant f(t)$ $\forall t \in (a,b)$. Then f is nondecreasing on $[a,b)$ iff $D_+ f \geqslant 0$ on $[a,b)$.*

Proof. This statement is also essentially well known; cf., e.g., Theorem 3.4.4 (or Exercise 1 on page 76) in [5]. A short proof is obtained by checking that $D_+ f \geqslant 0$ implies that $\sup\{t \in [c,b) : f(s) \geqslant f(c) - \varepsilon(s-c)\ \forall s \in [c,t]\} = b$ for any $c \in [a,b)$ and any $\varepsilon > 0$. $\square$

LEMMA 4. *Let $f(t) = u(u_1(t))u_2(t) + u_3(t)$ for all $t \in [a,b)$, where function u is nondecreasing on $[a,b)$, function u_1 is strictly increasing on $[a,b)$, function u_2 is nonnegative on $[a,b)$, and $\forall i \in \{1,2,3\}$ $\exists (u_i)'_+(a) \in \mathbb{R}$. Then*

$$D_+ f(a) \geqslant u(u_1(a))(u_2)'_+(a) + (u_3)'_+(a);$$

moreover, if $(D_+ u)(u_1(a)) < \infty$ or $(u_1)'_+(a) u_2(a) > 0$, then

$$D_+ f(a) = (D_+ u)(u_1(a))(u_1)'_+(a) u_2(a) + u(u_1(a))(u_2)'_+(a) + (u_3)'_+(a).$$

Proof. This is proved essentially the same way as for usual derivatives. $\square$

LEMMA 5. $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ *iff* $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$ *and $g_\varphi(x;y,y)$ is convex in x on every straight line not through 0, for every $y \in H$.*

Proof. One only needs to check the "if" part. To that end, it suffices to show that the second derivative $g''_\varphi(x;y,y)$ is continuous in x.

Calculations yield

$$g''_\varphi(x;y,y) = \varphi''(|x|)\frac{(xy)^2}{|x|^2} + \frac{\varphi'(|x|)}{|x|}\ \frac{|x|^2|y|^2 - (xy)^2}{|x|^2} \tag{29}$$

if $x \neq 0$. This and Proposition 1 imply the continuity of $g''_\varphi(x;y,y)$ in $x \neq 0$.

Further, (29) implies that, for $x \neq 0$ and $y \neq 0$,

$$g''_\varphi(x;y,y) = \left(\tau\varphi''(|x|) + (1-\tau)\frac{\varphi'(|x|)}{|x|}\right)|y|^2, \tag{30}$$

where

$$\tau := \frac{(xy)^2}{|x|^2|y|^2},$$

so that $0 \leqslant \tau \leqslant 1$. Taking now (27) and (28) into account, one has

$$\lim_{x\to 0} g''_\varphi(x;y,y) = \varphi''(0)|y|^2 = g''_\varphi(0;y,y). \tag{31}$$

Therefore, $g''_\varphi(x;y,y)$ is continuous in x at $x=0$ as well, provided that $y \neq 0$. Finally, $g''_\varphi(x;y,y) = 0$ for all x if $y = 0$. $\square$

Take any vectors x, y, and z in H. Then one can write

$$y = \alpha x + \beta z + u \tag{32}$$

for some α and β in $\mathbb{R}$ and some $u \in H$ such that

$$(xu) = (zu) = 0.$$

Let

$$\begin{aligned}
\Delta &:= \Delta(x,z) := \sqrt{|x|^2|z|^2 - (xz)^2};\\
A &:= A(x,z) := |x|^4\left((xz)^2h_4(|x|) + \Delta^2\cdot h_3(|x|)\right);\\
B &:= B(x,z) := (xz)^4h_4(|x|) + 6(xz)^2\Delta^2\cdot h_3(|x|) + 3\Delta^4\cdot h_2(|x|);\\
C &:= C(x,z) := (xz)|x|^2\left((xz)^2h_4(|x|) + 3\Delta^2\cdot h_3(|x|)\right);\\
U &:= U(x,z) := |x|^2|u|^2\left((xz)^2h_3(|x|) + \Delta^2\cdot h_2(|x|)\right).
\end{aligned}$$

LEMMA 6. *Let* $\dim H \geqslant 2$. *Then* $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$ *iff* $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$ *and* U, A, B, *and* $AB - C^2$ *are nonnegative for all* $z \neq 0$ *and* $x \neq 0$ *in* H *such that* $h_4(|x|) < \infty$.

Proof. Let $x_0 \neq 0$ and $z \neq 0$ in H be such that

$$0 \notin \{x_0 + sz \colon s \in \mathbb{R}\}. \tag{33}$$

Let

$$f(t) := f(x_0,y,z;t) := g''_\varphi(x_0 + tz; y, y), \tag{34}$$

where $y \in H$ and $t \in \mathbb{R}$.

Rewrite (29) as

$$f(t) = (xy)^2\frac{h_2(|x|)}{|x|^3} + |y|^2\frac{\varphi'(|x|)}{|x|}; \tag{35}$$

here and to the end of the proof of Lemma 6,

$$x := x_0 + tz, \tag{36}$$

so that $x \neq 0$ for any $t \in \mathbb{R}$, in view of (33). On the other hand, for any nonzero $x \in H$ there exist nonzero x_0 and z in H and $t \in \mathbb{R}$ such that both (33) and (36) hold; this follows because $\dim H \geqslant 2$.

For all $s > 0$,

$$\left(\frac{\varphi'(s)}{s}\right)' = \frac{h_2(s)}{s^2}; \quad (h_2)'_+(s) = \frac{h_3(s) + 2h_2(s)}{s}; \quad D_+h_3(s) = \frac{h_4(s)}{s}; \tag{37}$$

here we used Lemma 4. Now, using (37) and again Lemma 4, one has for all $t \in \mathbb{R}$

$$\begin{aligned} |x|^7 D_+^2 f(t) &= H(x,y,z) - (h_3(|x|) + 2h_2(|x|)) \cdot (xy)^2(xz)^2 \\ &+ (h_3(|x|) - h_2(|x|)) \cdot \left(|x|^2\left(4(xy)(xz)(yz) + (xy)^2|z|^2 + (xz)^2|y|^2\right) - 5(xy)^2(xz)^2\right) \\ &+ h_2(|x|) \cdot |x|^4 \left(2(yz)^2 + |y|^2|z|^2\right), \\ &\text{where} \quad H(x,y,z) \geqslant 0; \end{aligned} \tag{38}$$

if, in addition, one has $D_+^4\varphi(|x|) < \infty$ or $(xy)(xz) \neq 0$, then

$$H(x,y,z) = h_4(|x|) \cdot (xy)^2(xz)^2; \tag{39}$$

note that $h_4(|x|) < \infty$ iff $D_+^4\varphi(|x|) < \infty$. When (39) holds, then, using (32), one can rewrite (38) as

$$|x|^7 D_+^2 f(t) = A\alpha^2 + B\beta^2 + 2C\alpha\beta + U, \quad \forall t \in \mathbb{R}; \tag{40}$$

this is the crucial observation in the proof of Theorem 1.

If $\varphi \in C^2_{\mathsf{conv,sph,2}}(H)$, then, by Lemmas 5 and 1, one has $D_+^2 f(t) \geqslant 0$ for all $t \in \mathbb{R}$, and then the "only if" part of Lemma 6 follows from (40), because, for any x and z in H, one may take $\alpha, \beta \in \mathbb{R}$ and $u \in H$ arbitrarily in (32) to thus set y.

Let us now prove the "if" part. Let $y \in H$. Consider the following two cases: (i) for all $t \in \mathbb{R}$, $(xy) = (x_0y) + t(zy) = 0$ and (ii) there is at most one number $t_0 \in \mathbb{R}$ such that $(x_0y) + t_0(zy) = 0$. In Case (i), $(xy) = (zy) = 0$, and so, one may take $u = y$ in (32), whence by (38) one has $D_+^2 f(t) \geqslant |x|^{-7}U \geqslant 0 \ \forall t \in \mathbb{R}$. In Case (ii), one has (40) for all $t \in \mathbb{R} \setminus \{t_0\} \setminus \{t_1\}$, where $t_1 := -(x_0z)/|z|^2$ – because for such t, one has $(xy)(xz) \neq 0$.

Thus, in either case,

$$\exists t_0 \in \mathbb{R}\ \exists t_1 \in \mathbb{R}\ \forall t \in \mathbb{R} \setminus \{t_0\} \setminus \{t_1\} \quad D_+^2 f(t) \geqslant 0. \tag{41}$$

Note that, for any $t \in \mathbb{R}$, $|x_0 + sz|$ increases in $s \in [t, \infty)$ if $(xz) \geqslant 0$, where x is given by (36) as before, and $|x_0 + sz|$ decreases in $s \in [t, t+\delta)$ for some $\delta > 0$ if $(xz) < 0$. It now follows from (29) and (34) that

$$f'_\pm(t) = (\varphi'')'_{\text{sign}}(|x|)\frac{(xy)^2(xz)}{|x|^3} + F(x,y,z),$$

where sign $:= \pm$ if $(xz) \geqslant 0$ and sign $:= \mp$ if $(xz) < 0$; $F(x,y,z)$ is some expression in x, y, and z.

Thus, in all cases $f_+'(t) \geqslant f_-'(t)$ for all $t \in \mathbb{R}$ — since $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$, and so, φ'' is convex. Now, in view of Lemmas 1–3, (41) implies that f is convex on $\mathbb{R}$. Thus, $g_\varphi(x;y,y)$ is convex in x on every straight line not through 0. It remains to use Lemma 5.

LEMMA 7. *If $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$, then functions h_2, h_3, and h_4 are nonnegative on $(0,\infty)$.*

Proof. By (6), $h_2 \geqslant 0$ on $(0,\infty)$. Next, it follows from Proposition 1 and Lemma 1 that $D_+^3\varphi = (\varphi'')_+'$ is nondecreasing on $(0,\infty)$. Therefore, $D_+^4\varphi \geqslant 0$, whence $h_4 \geqslant 0$ and, in view of (37), $D_+h_3 \geqslant 0$ on $(0,\infty)$. Again by Proposition 1, $h_3(0+) = 0$, and so, by Lemma 3, $h_3 \geqslant 0$ on $(0,\infty)$.

LEMMA 8. *For any $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$, any nonzero $x \in H$, and any $z \in H$, A, B, and U are nonnegative.*

Proof. This follows from Lemma 7.

LEMMA 9. *Suppose that* $\dim H \geqslant 2$, $x \in H \setminus \{0\}$, *and* $h_4(|x|) < \infty$. *Then*

$$(\forall z \neq 0\; AB - C^2 \geqslant 0) \quad \textit{iff} \quad 3h_4(|x|)h_2(|x|) \geqslant h_3(|x|)^2.$$

Proof. By the continuity and the condition $\dim H \geqslant 2$, $AB - C^2 \geqslant 0$ for all $z \neq 0$ iff $AB - C^2 \geqslant 0$ for all z such that

$$0 < (xz)^2 < |x|^2|z|^2.$$

Assuming the latter conditions, one can see that

$$\frac{AB - C^2}{|x|^4\Delta^6} = p\langle^2 + q\langle + r,$$

where

$$\langle := \langle\,(x,z) := \frac{(xz)^2}{|x|^2|z|^2 - (xz)^2};$$

$$p := h_3(|x|)h_4(|x|); \quad q := 3\big(h_2(|x|)h_4(|x|) - h_3(|x|)^2\big); \quad r := 3h_2(|x|)h_3(|x|).$$

The condition $\dim H \geqslant 2$ implies that $\forall x \in H \setminus \{0\}\ \forall t > 0\ \exists z \in H\quad \langle\,(x,z) = t$. Hence, $AB - C^2 \geqslant 0$ for all $z \neq 0$ iff $pt^2 + qt + r \geqslant 0$ for all $t > 0$ iff $p \geqslant 0$, $r \geqslant 0$, and ($q \geqslant 0$ or $q^2 - 4pr \leqslant 0$). Now, $q^2 - 4pr = 3(3a - b)(a - 3b)$ and $q = 3(a - b)$, where

$$a := h_2(|x|)h_4(|x|) \quad \text{and} \quad b := h_3(|x|)^2.$$

Therefore, ($q \geqslant 0$ or $q^2 - 4pr \leqslant 0$) iff $3a \geqslant b$ iff $3h_4(|x|)h_2(|x|) \geqslant h_3(|x|)^2$. It remains to notice that $p \geqslant 0$ and $r \geqslant 0$, by Lemma 7. □

Proof of Theorem 1. If $\varphi \in C^2_{\text{conv,sph},2}(H)$ and $\dim H \geqslant 2$, then $\varphi \in C^2_{\text{conv,sph}}(\mathbb{R})$, and, by virtue of Lemmas 6 and 9, condition (8) is satisfied. This proves the "only if" part of the theorem. The "if" part follows from Lemmas 8, 9, and 6. □

Proof of Corollary 1. This follows from Theorem 1, because, for $\varphi := \varphi_p$, one has

$$3h_4(t)h_2(t) - h_3(t)^2 = 2p^3(p-2)^2(p-3)t^{2p-2}$$

for $t > 0$. □

Proof of Corollary 2. W.l.o.g., $a = 1$, because $\varphi_{a,p}(t) = a^p\varphi_{1,p}(t/a)$ if $a > 0$. For $\varphi := \varphi_{1,p}$, $p := r + 7/2$, and $s > 0$, one has

$$\begin{aligned}&\frac{8}{\left(r+\frac{7}{2}\right)^2 s^{2r+1}}\Big(3h_4(1+s)h_2(1+s) - h_3(1+s)^2\Big)\\ &\quad = 150r + 220r^2 + 104r^3 + 16r^4 + 255s + 902rs + 980r^2s + 424r^3s + 64r^4s\\ &+ 445s^2 + 1518rs^2 + 1556r^2s^2 + 648r^3s^2 + 96r^4s^2 + 285s^3 + 994rs^3 + 1052r^2s^3\\ &\qquad + 440r^3s^3 + 64r^4s^3 + 63s^4 + 228rs^4 + 256r^2s^4 + 112r^3s^4 + 16r^4s^4,\end{aligned}$$

which is nonnegative if $r \geqslant 0$, i.e., if $p \geqslant 7/2$. Hence, by Theorem 3, $\varphi_{a,p} \in C^2_{\text{conv,sph},2}(H)$ provided that $a > 0$ and $p \geqslant 7/2$. On the other hand, for $\varphi := \varphi_{1,p}$ one has

$$\lim_{s\downarrow 0} \frac{3h_4(1+s)h_2(1+s) - h_3(1+s)^2}{s^{2p-6}} = 2p^2(p-1)^2(p-2)(p-7/2).$$

Hence, if $\varphi_{1,p} \in C^2_{\text{conv,sph},2}(H)$, then either $p \geqslant 7/2$ or $p \leqslant 2$. But $\varphi_{1,p} \in C^2_{\text{conv,sph},2}(H)$ implies $\varphi_{1,p} \in C^2_{\text{conv,sph}}(\mathbb{R})$, so that, by Proposition 1, $\varphi''_{1,p}$ must be convex, whence $p \geqslant 3$. Thus, if $\varphi_{1,p} \in C^2_{\text{conv,sph},2}(H)$, then $p \geqslant 7/2$. □

Proof of Theorem 2.

"only if" part

Since $\varphi \in C^2_{\text{conv,sph},2}(H)$, one has $\varphi \in C^2_{\text{conv,sph}}(\mathbb{R})$, and so, the conditions described in Proposition 1 hold. Let us define a and b as in (15). W.l.o.g., $a = b = 0$; otherwise, we would just deal with the function $\varphi(t) - a - bt^2$ (instead of $\varphi(t)$), which is easily seen to be in $C^2_{\text{conv,sph},2}(H)$ iff φ is in $C^2_{\text{conv,sph},2}(H)$. Also we have that $\varphi'(0) = 0$ and φ'' is convex. Thus,

$$\varphi(0) = \varphi'(0) = \varphi''(0) = 0; \tag{42}$$

$$(\varphi'')'_+ \geqslant 0; \quad \varphi'' \geqslant 0 \tag{43}$$

on $(0, \infty)$. Next,

$$h_2(s) = s^2\left(\frac{\varphi'(s)}{s}\right)' \quad \forall s > 0. \tag{44}$$

Thus, if $h_2(t) = 0$ for some $t > 0$, then, according to (7), $h_2 = 0$ on $(0, t]$, and so, $\exists k \in \mathbb{R}\ \forall s \in (0, t]\ \ \varphi'(s) = ks$, whence $\exists k_1 \in \mathbb{R}\ \forall s \in (0, t]\ \ \varphi(s) = ks^2/2 + k_1$.

Now, by (42), one has $\varphi = 0$ on $[0,t]$. We see that if $h_2(t) = 0$ for some $t > 0$, then $h_2 = 0$ on $[0,t]$ and $\varphi = 0$ on $[0,t]$. Vice versa, if $\varphi = 0$ on $[0,t]$ for some $t > 0$, then obviously $h_2 = 0$ on $[0,t]$.

Let t_0 be defined by (16). If $t_0 = \infty$, then $\varphi = 0$ on $[0,\infty)$; in this case (11) trivially takes place with $a = b = c = 0$.

For the rest of the proof of the "only if" part, assume that $0 \leqslant t_0 < \infty$. Then both functions h_2 and φ, being continuous, are strictly positive on (t_0,∞) and zero on $[0,t_0]$. Hence, the continuous functions φ' and φ'' are also zero on $[0,t_0]$.

Let now, for all $t > t_0$,

$$r(t) := D_+ \ln \frac{h_2(t)}{t^2} = D_+ \ln\left(\left(\frac{\varphi'(t)}{t}\right)'\right) = \left(\ln\left(\left(\frac{\varphi'(t)}{t}\right)'\right)\right)'_+. \tag{45}$$

Using Lemma 4, one has

$$r(t) = \frac{h_3(t)}{t h_2(t)} \quad \text{and} \quad D_+ r(t) = \frac{h_4(t)h_2(t) - 3h_3(t)h_2(t) - h_3(t)^2}{t^2 h_2(t)^2} \tag{46}$$

for all $t > t_0$.

Let next

$$K(t) := \frac{t^3 r(t)}{3 + t r(t)} \quad \forall t > t_0. \tag{47}$$

This definition is correct, since it follows from (46) and Lemma 7 that $r(t) \in [0,\infty)$ for all $t > t_0$. In view of (47), this is equivalent to (13).

Now it is easy to see that

$$r = r_K$$

on (t_0,∞), where r_K is defined by (12).

The key fact in the proof of Theorem 2 is that for all $t > t_0$,

$$D_+ K(t) = \frac{t}{(3 + t r(t))^2} \, \frac{3h_4(t)h_2(t) - h_3(t)^2}{h_2(t)^2}; \tag{48}$$

here we used Lemma 4 again. To check (45), one may also use (46).

Identity (48) implies that K is nondecreasing on (t_0,∞) iff $3h_4h_2 - h_3^2 \geqslant 0$ on (t_0,∞).

According to (46) and Proposition 1, $r(t+) = r(t) \geqslant r(t-)$ for all $t > t_0$. It now follows from (45) that $\forall \varepsilon > 0 \ \exists c_1 > 0 \ \forall t > t_0$

$$\left(\frac{\varphi'(t)}{t}\right)' = c_1 \exp \int_{t_0+\varepsilon}^{t} r(s)\,ds. \tag{49}$$

If $t_0 > 0$, then, as we saw, $\varphi = 0$ on $[0,t_0]$. By (42), $\dfrac{\varphi'(t)}{t} \to \varphi''(0) = 0$ as $t \downarrow 0$. Also, we saw that the continuous on $[0,\infty)$ function φ' is zero on $[0,t_0]$.

Hence, $\frac{\varphi'(t)}{t} \to 0$ as $t \downarrow t_0$, whether $t_0 = 0$ or $t_0 > 0$. Therefore, (49) and (42) imply that for all $t > t_0$

$$\frac{\varphi'(t)}{t} = c_1 \int_{t_0+}^{t} du \cdot \exp \int_{t_0+\varepsilon}^{u} r(s)ds; \tag{50}$$

$$\begin{aligned} \varphi(t) &= c_1 \int_{t_0+}^{t} vdv \cdot \int_{t_0+}^{v} du \cdot \exp \int_{t_0+\varepsilon}^{u} r(s)ds \\ &= c_1 \int_{t_0+}^{t} du \cdot \left(\exp \int_{t_0+\varepsilon}^{u} r(s)ds \right) \int_{u}^{t} vdv \\ &= c \int_{t_0+}^{t} du \cdot (t^2 - u^2) \exp \int_{t_0+\varepsilon}^{u} r(s)ds, \end{aligned} \tag{51}$$

where $c := c_1/2 > 0$. Thus, one has (11) (with $a = b = 0$).

Using (50), one has for all $t > t_0$

$$\frac{1}{c_1} \varphi''(t) = \int_{t_0+}^{t} du \cdot \exp \int_{t_0+\varepsilon}^{u} r(s)ds + t \exp \int_{t_0+\varepsilon}^{t} r(s)ds; \tag{52}$$

$$\frac{1}{c_1} (\varphi'')'_+(t) = 2 \exp \int_{t_0+\varepsilon}^{t} r(s)ds + tr(t+). \tag{53}$$

Assume now that $t_0 > 0$. Recall that φ'' is continuous and $\varphi''(t_0) = 0$. Then (52) implies that $\exp \int_{t_0+\varepsilon}^{t} r(s)ds \to 0$ as $t \downarrow t_0$. This proves (14). The "only if" part of the theorem is thus completely proved.

"if" part

Suppose that all that follows the word "iff" in the statement of Theorem 2 is true. We want to show that then $\varphi \in C^2_{\mathsf{conv,sph},2}(H)$. W.l.o.g., $a = b = 0$ and $t_0 < \infty$. Then (51), (50), (52), and (53) hold for all $t > t_0$ with $r := r_K$ and $c_1 := 2c$. Note that (12) and (13) imply $r(t) \geqslant 0$ for $t > t_0$. Hence, (53) implies that $(\varphi'')'_+ > 0$ on (t_0, ∞), and then (5) implies that h_2 is strictly increasing on (t_0, ∞). Next, (52) and (14) imply that $\varphi''(t_0+) = 0$, whether $t_0 = 0$ or $t_0 > 0$. Also, (52) yields $\varphi'(t_0+) = 0$. Hence, $h_2(t_0+) = 0$. Since h_2 is strictly increasing on (t_0, ∞), it is strictly positive on (t_0, ∞). Then, as we saw, one has (48), whence $3h_4h_2 - h_3^2 \geqslant 0$ on (t_0, ∞). On the other hand, on the interval $(0, t_0)$ (if non-empty), the function φ and all its derivatives are 0, as well as the function $3h_4h_2 - h_3^2$. Hence, (8) holds $\forall t \in (0, \infty) \setminus \{t_0\}$. Thus, in view of Remark 1, it remains to verify that $\varphi \in C^2_{\mathsf{conv,sph}}(\mathbb{R})$.

As has been shown, $h_2 > 0$ and $3h_4h_2 \geqslant h_3^2$ on (t_0, ∞), whence $h_4(t) = t^3 D^4_+ \varphi(t) \geqslant 0$ for all $t > t_0$, and so, $D^4_+ \varphi \geqslant 0$ on (t_0, ∞). This implies that φ'' is convex on (t_0, ∞). In addition, $\varphi'' = 0$ on $(0, t_0)$ and $(\varphi'')'_+(t_0) \geqslant 0$ (by virtue of (53) and the nonnegativity of r on (t_0, ∞)). It follows that φ'' is convex on $(0, \infty)$.

Since φ'' is also finite on $(0, \infty)$, one has $(\varphi'')'_+(t_0) < \infty$. Finally, as has been shown, $\varphi'(t_0+) = \varphi''(t_0+) = 0$. Hence, $\varphi'(0+) = \varphi''(0+) = 0$ if $t_0 = 0$. If $t_0 > 0$, then the equalities $\varphi'(0+) = \varphi''(0+) = 0$ are trivial, since $\varphi = 0$ on $(0, t_0)$. Now it remains to refer to Proposition 1.

Theorem 2 is completely proved. □

Proof of Propositions 2 and 3. These follow from the above reasoning, most of which can be turned back. In particular, it follows from (47) and (45) that for all $t > t_0$

$$K(t) = t^2\,\frac{tD_+h_2(t) - 2h_2(t)}{tD_+h_2(t) + h_2(t)},$$

whence one has (17); the denominator in (17) coincides with $tD_+h_2(t) + h_2(t)$ and hence is strictly positive for all $t > t_0$, according to (5), (6), and (16). □

Proof of Corollary 3. In representation (11), replace K by K_m defined by

$$K_m(t) := K(t) \wedge K(m) \quad \forall t > t_0,$$

to obtain ψ_m, in place of φ, for every natural m. Thus, $K_m = \text{const}$ on $[m, \infty)$. It is easy to see that the resulting sequence $(\psi_m)_{m=1}^{\infty}$ satisfies all the requirements stated in Corollary 3; cf. Example 1 on page 13. □

Proof of Theorem 3. Essentially, this theorem follows from [7, Lemma 3.1]. Alternatively, in (32) and thence in (40), one may set $z = y$ and thus $\alpha = 0$, $\beta = 1$, and $u = 0$. Now it remains to use Lemma 8. □

Proof of Theorem 4. This follows from Lemma 3.2 of [7]; cf. the proof of Theorem 2.3 therein; the condition that the function f in Lemma 3.2 of [7] be even is actually not needed, because the distributions of both the ε_i's and the ξ_i's are symmetric, so that $\mathsf{E}f(\varepsilon_i) = \mathsf{E}f_{\text{symm}}(\varepsilon_i)$ and $\mathsf{E}f(\xi_i) = \mathsf{E}f_{\text{symm}}(\xi_i)$, where $f_{\text{symm}}(t) := (f(t) + f(-t))/2$. □

Proof of Theorem 6. Theorem 2 of [12] implies that

$$\mathsf{E}g(S) \geqslant \sum_{i=1}^{n} \mathsf{E}g(X_i)$$

for any function $g \in \mathscr{F}_2(H)$ with $g(0) = 0$ such that $\mathsf{E}|g(X_i)| < \infty \ \forall i$, where $\mathscr{F}_2(H)$ is the class of all functions $g \in C^2_{\text{weak}}(H)$ satisfying the following conditions:

(i) g is twice Fréchet-differentiable with a continuous second derivative g'' (in fact, one seems to need here something different: that the map $H \times H \ni (x,y) \mapsto g''(x;y,y)$ be bounded on all bounded sets in $H \times H$; recall that bounded sets in an infinite-dimensional Hilbert spaces need not be compact);

(ii) $g''(x;y,y)$ is convex in $x \in H$ for every $y \in H$;

(iii) $\exists\ c_g \in (0,\infty) \quad |g(x+y)| \leqslant c_g \cdot (1 + |g(x)|)(1 + |g(y)|)$ for all x and y in H.

Thus, to prove (21), it suffices to check conditions (i)–(iii) for the functions $g_m\colon H \to \mathbb{R}$ (in place of g) defined by

$$g_m(x) := \psi_m(|x|) \quad \forall x \in H,$$

where ψ_m are as in Corollary 3.

$\Big($The condition $\mathsf{E}|g(X_i)| < \infty\ \ \forall i$ can be circumvented here. Indeed, if $\varphi(t) = bt^2$ for all real t, then (21) is a trivial equality. Hence, it suffices to prove (21) only for such $\varphi \in C^2_{\mathsf{conv,sph,2}}(H) \left[\subseteq C^2_{\mathsf{conv,sph}}(\mathbb{R})\right]$ that $a = b = 0$ in (11), so that one has (42), whence, in view of Proposition 1, φ is convex, and so, $\mathsf{E}\varphi(|S|) \geqslant \mathsf{E}\varphi(|X_i|)\ \ \forall i$, by Jensen's inequality. $\Big)$

That the map $H \times H \ni (x,y) \mapsto g''_m(x;y,y)$ is bounded on all bounded sets in $H \times H$ follows from (29), (31), and Proposition 1; note that one has (31) *uniformly* in y over all y in any bounded set. This verifies (i).

That g_m satisfies conditions (ii) and (iii) for all m is obvious.

Thus, (21) is proved. Now (22) and (23) follow by Corollaries 1 and 2. □

Proof of Theorem 7. The inequality

$$\sup_{\mathscr{X}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^n X_i\right|\right) \leqslant \int_H \varphi(|x|)P_G(dx)$$

is not difficult to deduce from Theorem 6; cf. the proof of Theorem 4 in [12]. The inequality

$$\int_H \varphi(|x|)P_G(dx) \leqslant \sup_{\mathscr{X}_{\text{ident}}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^n X_i\right|\right)$$

follows from the analogue of the Fatou lemma for convergence in distribution [1, Theorem 5.3] if one considers, for $n \geqslant G(H)$, the i.i.d. r.v.'s $X_1,\dots,X_n$ with the common distribution determined by the condition

$$\mathsf{P}(X_i \in A) := \frac{1}{n}G(A) \quad \text{for all Borel } A \subseteq H \setminus \{0\};$$

cf. [8]. Finally, the inequality

$$\sup_{\mathscr{X}_{\text{ident}}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^n X_i\right|\right) \leqslant \sup_{\mathscr{X}(G)} \mathsf{E}\varphi\left(\left|\sum_{i=1}^n X_i\right|\right)$$

is trivial. □

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Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931
e-mail: ipinelis@math.mtu.edu