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MAXIMUM OF GREEN'S FUNCTIONS AND LIAPUNOV INEQUALITIES FOR THE STURM–LIOUVILLE PROBLEM

LAURIE BATTLE

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Abstract. First we find the Green's function for the Sturm-Liouville problem with separated boundary conditions. Then we calculate maximum values of the Green's function to determine Liapunov-type inequalities.

1. Introduction

We consider the Sturm-Liouville problem with separated boundary conditions

$$\begin{cases} u'' + p(x)u = \lambda u \quad \text{in} \quad [a,b] \\ u(a) = \tilde{h}u'(a) \\ u(b) = \tilde{H}u'(b) \end{cases} \tag{1}$$

where $p \in L^1[a,b]$, $\|p\|_{L^1} > 0$, and $\tilde{h}, \tilde{H} \in \Re$. Assuming $\lambda = 0$ is an eigenvalue of (1), we obtain a lower bound for $\|p\|_{L^1}$ of the Liapunov type, where the bound depends on the values of $\tilde{h}$ and $\tilde{H}$ in the boundary conditions. The lower bound is determined by the maximum values of Green's functions associated with $\lambda = 0$.

The Liapunov inequality applies to the case where $\tilde{h} = \tilde{H} = 0$. It states that if $\lambda = 0$ is an eigenvalue of (1) with boundary conditions $u(a) = u(b) = 0$, then $\|p\|_{L^1} > \frac{4}{b-a}$. Proofs of Liapunov's inequality (with Dirichlet boundary conditions) may be found in the survey papers [1,3]. The same inequality was proven for antiperiodic boundary conditions [2, p.21]. We extend this result to the separated boundary conditions in the following theorem:

THEOREM. *If $\lambda = 0$ is an eigenvalue of (1) but is not an eigenvalue of $u'' = \lambda u$ with the same boundary conditions, i.e., if $\tilde{h} \neq \tilde{H} - 1$, then $\|p\|_{L^1} \geqslant C(\tilde{H}, \tilde{h})$, where $\alpha = \frac{1}{b-a}$ and*

1. $C(\tilde{H},\tilde{h}) = \left| \dfrac{\alpha\tilde{H} - \alpha\tilde{h} - 1}{\tilde{h}(\alpha\tilde{H} - 1)} \right|$ *if*

$(\tilde{H},\tilde{h}) \in \{-\frac{\sqrt{2}}{2\alpha} \leqslant \tilde{H} \leqslant \frac{1}{2\alpha},\ \tilde{h} \leqslant \min\{(3-2\sqrt{2})\tilde{H} + \frac{2\sqrt{2}-3}{\alpha},\ -\tilde{H}\}\} \cup$
$\{\tilde{H} \leqslant -\frac{\sqrt{2}}{2\alpha},\ \tilde{h} \leqslant \frac{\tilde{H}}{1-2\alpha\tilde{H}}\} \cup \{\tilde{h} \geqslant \max\{\frac{1}{\alpha} - \tilde{H},\ \frac{\tilde{H}}{1-2\alpha\tilde{H}}\}\}$

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2. $C(\tilde{H},\tilde{h}) = \left|\dfrac{\alpha\tilde{H}-\alpha\tilde{h}-1}{\tilde{H}(1+\alpha\tilde{h})}\right|$ *if*
$(\tilde{H},\tilde{h}) \in \{\tilde{H} \geqslant \frac{1}{2\alpha}, \tilde{h} \geqslant -\frac{1}{2\alpha}\} \cup \{\tilde{h} \leqslant \frac{\sqrt{2}}{2\alpha}, -\tilde{H} \leqslant \tilde{h} \leqslant (3+2\sqrt{2})\tilde{H} - \frac{1}{\alpha}\} \cup$
$\{\tilde{h} \geqslant \frac{\sqrt{2}}{2\alpha}, \tilde{h} \leqslant \frac{\tilde{H}}{1-2\alpha\tilde{H}}\} \cup \{\frac{\tilde{H}}{1-2\alpha\tilde{H}} \leqslant \tilde{h} \leqslant -\frac{1}{\alpha} - \tilde{H}\}$

3. $C(\tilde{H},\tilde{h}) = \left|\dfrac{\alpha\tilde{H}-\alpha\tilde{h}-1}{\alpha\tilde{h}\tilde{H}}\right|$ *if* $(\tilde{H},\tilde{h}) \in \{\tilde{H} \geqslant \frac{1}{2\alpha}, \tilde{h} \leqslant -\frac{1}{2\alpha}\}$,

4. $C(\tilde{H},\tilde{h}) = \left|\dfrac{4\alpha(\alpha\tilde{H}-\alpha\tilde{h}-1)}{(1-\alpha\tilde{H}-\alpha\tilde{h})^2}\right|$ *if* $(\tilde{H},\tilde{h}) \in$
$\{\max\{-\frac{1}{\alpha} - \tilde{H},\ (3-2\sqrt{2})\tilde{H} + \frac{2\sqrt{2}-3}{\alpha},\ (3+2\sqrt{2})\tilde{H} - \frac{1}{\alpha}\} \leqslant \tilde{h} \leqslant \frac{1}{\alpha} - \tilde{H}\}$.

These four regions are sketched in Fig. 4 below for $\alpha = 1$ and are labeled g_1, g_2, g_3, g_4, respectively. The definitions of $C(\tilde{H},\tilde{h})$ agree on the boundaries between regions, so that $C(\tilde{H},\tilde{h})$ is continuous. For this reason, we allow the formulas for $C(\tilde{H},\tilde{h})$ to overlap. □

By defining the function $U(s) := u(x)$, where $s = \frac{x-a}{b-a}$, we may assume that $a = 0$ and $b = 1$. In fact, if $\lambda = 0$ is an eigenvalue of (1), then $\lambda = 0$ is an eigenvalue of

$$\begin{cases} U''(s) + P(s)U(s) = \lambda U(s) & \text{in } [0,1] \\ U(0) = hU'(0) \\ U(1) = HU'(1) \end{cases} \tag{2}$$

where $P(s) = \frac{p(x)}{\alpha^2}$, $h = \alpha\tilde{h}$, $H = \alpha\tilde{H}$ $(\alpha = \frac{1}{b-a})$. This follows from the fact that $u'(x) = \alpha U'(s)$, $u''(x) = \alpha^2 U''(s)$, $u(a) = U(0)$, and $u(b) = U(1)$.

2. Lower bound for $\|P\|_{L^1}$ determined by the Green's function

Now assume $\lambda = 0$ is an eigenvalue of (1). Then $\lambda = 0$ is an eigenvalue of (2) and the eigenfunction U can be written

$$U(t) = \int_0^1 G(t,s)(-P(s)U(s))\ ds,$$

where $G(t,s)$ is the Green's function [4, p.168].

Let $0 \leqslant t_0 \leqslant 1$ be such that $|U(t_0)| = \max\{|U(t)| : 0 \leqslant t \leqslant 1\}$. Then

$$\begin{aligned} |U(t_0)| &\leqslant \int_0^1 |G(t_0,s)|\,|P(s)|\,|U(s)|\ ds \\ &\leqslant \max\{|G(t,s)| : 0 \leqslant t,s \leqslant 1\}\,|U(t_0)| \int_0^1 |P(s)|\ ds. \end{aligned}$$

Thus

$$\|P\|_{L^1} \geqslant \frac{1}{\max\{|G(t,s)| : 0 \leqslant t,s \leqslant 1\}}. \tag{3}$$

So the maximum of the Green's function determines a lower bound for $\|P\|_{L^1}$.

3. Calculation of $\max G(t,s)$

Now to solve for $G(t,s)$ for the case when $H,h<\infty$, note that $u_1(x):=x+h$ and $u_2(x):=x+H-1$ both satisfy $u''=0$, and they satisfy the boundary conditions $u_1(0)=hu_1'(0)$, $u_2(1)=Hu_2'(1)$. Then the formula for the Green's function is

$$G(t,s)=\begin{cases}\dfrac{-u_1(s)u_2(t)}{c(0)}, & s<t\\[2ex] \dfrac{-u_1(t)u_2(s)}{c(0)}, & s\geqslant t\end{cases}$$

where $c(0)=\det\begin{pmatrix}u_1(t) & u_2(t)\\ u_1'(t) & u_2'(t)\end{pmatrix}=\begin{pmatrix}t+h & t+H-1\\ 1 & 1\end{pmatrix}=h-H+1$ (see [4] p.148). So

$$G(t,s)=\begin{cases}\dfrac{(s+h)(t+H-1)}{H-h-1}, & s<t\\[2ex] \dfrac{(t+h)(s+H-1)}{H-h-1}, & s\geqslant t\end{cases}. \tag{4}$$

Since u_1 and u_2 must be linearly independent, we must have $h\neq H-1$. So we will consider all values of H, h except when $h=H-1$.

Since $G(t,s)=G(s,t)$, the maximum of G over $(t,s)\in[0,1]\times[0,1]$ will occur in the region where $s\geqslant t$. In this region, using (4)

$$G(t,s)=\frac{(t+h)(s+H-1)}{H-h-1}$$

which we will now use.

Next we observe that the maximum value of G in the triangle $0\leqslant t\leqslant s\leqslant 1$ must occur on the boundary. For if s_0 is fixed in [0,1], $\frac{\partial G}{\partial t}(t,s_0)=\frac{s_0+H-1}{H-h-1}$ is constant. So G is linear as a function of t along the line $s=s_0$ and hence the maximum of G along the line $s=s_0$ must occur at either endpoint.

In particular, the maximum along the boundary line $s=1$ must occur at an endpoint. Similarly, the maximum along the boundary line $t=0$ must occur at an endpoint, since $\frac{\partial G}{\partial s}(0,s)=\frac{h}{H-h-1}$ is constant. Along the diagonal $s=t$, $\frac{\partial G}{\partial t}(t,t)=\frac{2t+H+h-1}{H-h-1}$, which is zero if $t=\frac{1-H-h}{2}$. So the maximum along the diagonal must occur at an endpoint or at $(\frac{1-H-h}{2},\frac{1-H-h}{2})$ if $0<\frac{1-H-h}{2}<1$. In conclusion, the maximum of G over the triangle must occur at one of the points $(0,0),(1,1),(0,1)$, or at $(\frac{1-H-h}{2},\frac{1-H-h}{2})$ if $0<\frac{1-H-h}{2}<1$.

Then the maximum of G in the triangle is the maximum of the absolute value of the four values

$$G(0,0)=\frac{h(H-1)}{H-h-1}\qquad G(0,1)=\frac{hH}{H-h-1}$$

$$G(1,1)=\frac{(1+h)H}{H-h-1}\qquad G\left(\frac{1-H-h}{2},\frac{1-H-h}{2}\right)=\frac{-(1-H+h)^2}{4(H-h-1)}.$$

Each of these has the term $H-h-1$ in the denominator, which can be removed when comparing them. So define

$$g_1(H,h) := h(H-1) \qquad g_3(H,h) := hH$$
$$g_2(H,h) := (1+h)H \qquad g_4(H,h) := \frac{1}{4}(1-H+h)^2.$$

Now we must find $\max\{|g_1(H,h)|, |g_2(H,h)|, |g_3(H,h)|, |g_4(H,h)|\}$ for all values of $H,h < \infty, h \neq H-1$. But by using symmetry we need only consider the half-plane $h \geqslant -H$. The symmetry over the line $h=-H$ follows from the following calculations: $g_1(H,h) = g_2(-h,-H)$, $g_3(H,h) = g_3(-h,-H)$, and $g_4(H,h) = g_4(-h,-H)$.

We represent all possible values for (H,h) using the plane with H as the horizontal axis and h as the vertical axis. Notice that

$$H \geqslant \frac{1}{2} \iff |H| \geqslant |H-1| \iff |g_3| \geqslant |g_1|, \tag{5}$$

$$h \geqslant -\frac{1}{2} \iff |h+1| \geqslant |h| \iff |g_2| \geqslant |g_3|. \tag{6}$$

So we choose our four cases as the quadrants determined by the lines $H=\frac{1}{2}$ and $h=-\frac{1}{2}$. In each case we can eliminate one or two possibilites for the maximum by refering to (5) and (6). The four cases are as follows:

1. $H \geqslant \frac{1}{2}, h \geqslant -\frac{1}{2}$	Eliminate g_1 by (5) and g_3 by (6)
2. $H \geqslant \frac{1}{2}, h \leqslant -\frac{1}{2}$	Eliminate g_1 by (5) and g_2 by (6)
3. $H < \frac{1}{2}, h \geqslant -\frac{1}{2}$	Eliminate g_3 by (5) and (6)
4. $H < \frac{1}{2}, h < -\frac{1}{2}$	Eliminate g_3 by (5) and g_2 by (6)

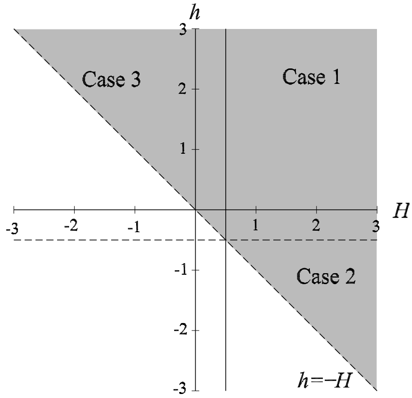

Figure 1. Symmetry over $h=-H$

As noted earlier, we need only consider the half-plane $h \geqslant -H$ because of symmetry. So case 4 is completely determined by case 1, and cases 2 and 3 can be cut in half (consider the shaded region in Fig. 1, except where $h = H - 1$.). Also recall that g_4 is considered only if $0 < \frac{1-H-h}{2} < 1$, i.e., if $-1 - H < h < 1 - H$.

1. $H \geqslant \frac{1}{2}, h \geqslant -\frac{1}{2}$

Since g_1 and g_3 are eliminated, we compare g_2 and g_4.

(a) $h > 1 - H$

Since g_4 is not considered in this region, $|g_2|$ is the maximum.

(b) $h \leqslant 1 - H$

$H \geqslant \frac{1}{2} \Longrightarrow h \leqslant 1 - H \leqslant \frac{1}{2} \Longrightarrow -\frac{1}{2} \leqslant h \leqslant \frac{1}{2}$, and

$h \geqslant -\frac{1}{2} \Longrightarrow H \leqslant 1 - h \leqslant \frac{3}{2} \Longrightarrow \frac{1}{2} \leqslant H \leqslant \frac{3}{2}$.

Then $-1 \leqslant 1 - H + h \leqslant 1 \Longrightarrow |g_4| = \frac{1}{4}(1 - H + h)^2 \leqslant \frac{1}{4}$. Now $|g_2| = |(1+h)H| = (1+h)H \geqslant \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Then $|g_2| \geqslant |g_4|$, so $|g_2|$ is the maximum.

2. $H \geqslant \frac{1}{2}, h \leqslant -\frac{1}{2}$

Since g_1 and g_2 are eliminated, we compare g_3 and g_4.

(a) $h > 1 - H$

Since g_4 is not considered in this region, $|g_3|$ is the maximum.

(b) $h \leqslant 1 - H$

$H \geqslant \frac{1}{2}$ and $h \leqslant -\frac{1}{2} \Longrightarrow 1 - H + h \leqslant 0$.

$h \leqslant 1 - H \Longrightarrow 1 - H + h \geqslant 2h$.

So $2h \leqslant 1 - H + h \leqslant 0 \Longrightarrow (1 - H + h)^2 \leqslant 4h^2$. So $|g_4| = \frac{1}{4}(1 - H + h)^2 \leqslant h^2$. Now $|g_3| = |hH| = -hH \geqslant (-h)(-h) = h^2$ since $H \geqslant -h > 0$. Then $|g_3| \geqslant |g_4|$, so $|g_3|$ is the maximum.

3. $H < \frac{1}{2}, h \geqslant -\frac{1}{2}$ (Fig. 2)

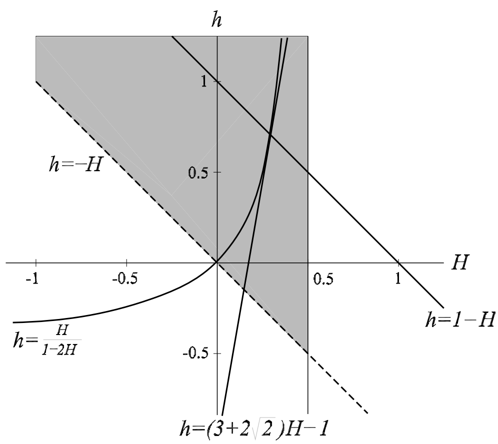

Figure 2. Case 3

Since g_3 is eliminated, we compare g_1, g_2, and g_4.

(a) $h > 1 - H$

Since g_4 is not considered in this region, we compare g_1 and g_2.

i. $H < 0$

$|g_1| = |h(H-1)| = h(1-H) = h - hH$ since $h > 1 - H > \frac{1}{2} > 0$.

$|g_2| = |H(1+h)| = -H(1+h) = -H - hH \leqslant h - hH$ since we are assuming $h \geqslant -H$. Then $|g_1| \geqslant |g_2|$, so $|g_1|$ is the maximum.

ii. $H \geqslant 0$

$|g_1| = |h(H-1)| = h(1-H)$ since $h > 1 - H > \frac{1}{2} > 0$.

$|g_2| = |H(1+h)| = H(1+h)$.

Note that $1 - 2H > 0$ since $H < \frac{1}{2}$. So

$$h \geqslant \frac{H}{1-2H} \iff h(1-2H) \geqslant H \iff h - hH \geqslant H + hH \iff$$
$$h(1-H) \geqslant H(1+h) \iff |h(H-1)| \geqslant |H(1+h)| \iff |g_1| \geqslant |g_2|.$$

So $|g_1|$ is the maximum when $h \geqslant \frac{H}{1-2H}$ and $|g_2|$ is the maximum when $h \leqslant \frac{H}{1-2H}$.

(b) $h \leqslant 1 - H$

i. $H < 0$

Since $h \geqslant -H > 0$, the argument in (a)(i) applies to this case because it requires only that $h > 0$ and not that $h > 1 - H$, and the argument shows that $|g_1| \geqslant |g_2|$.

Next we show that $|g_4| \geqslant |g_1|$. First observe that $|g_4| \geqslant |g_1|$ on the boundary of this region:

If $h = -H, |g_4(H,-H)| = \frac{1}{4}(1-2H)^2 = H^2 - H + \frac{1}{4}$, and $|g_1(H,-H)| = |-H(H-1)| = -H(1-H) = H^2 - H$. So $|g_4| \geqslant |g_1|$.

If $h = 1 - H$, $|g_4(H, 1-H)| = \frac{1}{4}(1-H+1-H)^2 = \frac{1}{4}(2(1-H))^2 = (1-H)^2$, and $|g_1(H, 1-H)| = |(1-H)(H-1)| = (1-H)^2$. So $|g_4| = |g_1|$.

If $H = 0, |g_4(0,h)| = \frac{1}{4}(1+h)^2$, and $|g_1(0,h)| = h$. So $|g_4| \geqslant |g_1|$ if $\frac{1}{4}(1+2h+h^2) - h \geqslant 0$, i.e., if $f(h) := h^2 - 2h + 1 \geqslant 0$. Now $f'(h) = 2h - 2 < 0$ for $h < 1$, but $0 \leqslant h \leqslant 1$ on this boundary line. Since f is decreasing on (0,1), $f(h) \geqslant f(1) = 0$, which shows that $|g_4| \geqslant |g_1|$.

Next we show that $|g_4| \geqslant |g_1|$ in this entire region by showing the function $g(H,h) := |g_4| - |g_1| = \frac{1}{4}(1-H+h)^2 - h(1-H)$ is nonnegative. We already know it is nonnegative on the boundary.

Now $\frac{\partial g}{\partial H}(H,h) = -\frac{1}{2}(1-H+h) + h = \frac{1}{2}(-1+H+h) \leqslant 0$ since $h \leqslant 1 - H$. Since $\frac{\partial g}{\partial H}$ does not change sign in this region and since $g \geqslant 0$ on the boundary, $g \geqslant 0$ in the region. That is, $|g_4| \geqslant |g_1|$ in the region.

ii. $H \geqslant 0$ (Fig. 3)

In this region, we make the following claim:

CLAIM. $h \geqslant \frac{H}{1-2H}$ *if and only if* $|g_1| \geqslant |g_2|$.

Proof. If $h \geqslant 0$, the argument from (a)(ii) applies because it requires only that $h > 0$ and not that $h > 1 - H$. If $h < 0$, notice that $h < \frac{H}{1-2H}$. This is true because $0 \leqslant H \leqslant \frac{1}{2}$ implies $\frac{H}{1-2H} \geqslant 0$, while $h < 0$. Now $|g_1| = |h(H-1)| = -h(1-H) = -h + hH$, and $|g_2| = |H(1+h)| = H(1+h) = H + hH \geqslant -h + hH = |g_1|$, since $h \geqslant -H$. So $|g_2| \geqslant |g_1| \iff h \leqslant \frac{H}{1-2H}$.

Now we show that $|g_4|$ is the maximum if $h \geqslant (3+2\sqrt{2})H-1$, and $|g_2|$ is the maximum if $h \leqslant (3+2\sqrt{2})H-1$ (Fig. 3):

A. Suppose $h \geqslant (3+2\sqrt{2})H-1$. First observe that $|g_4|$ is the maximum on the boundary of this region (A in Fig. 3):

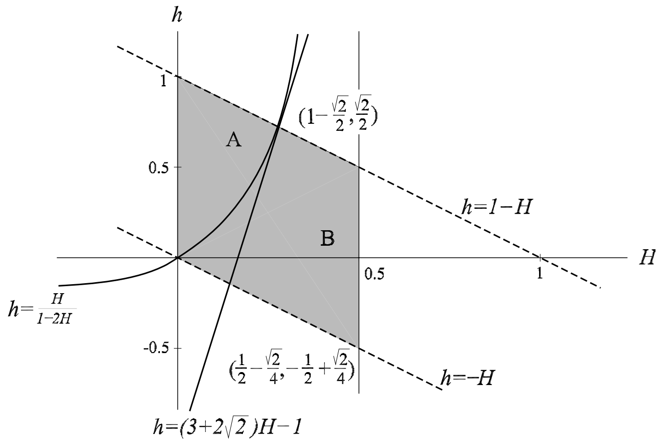

Figure 3. Case 3(b)(ii)

If $H=0$, that $|g_4| \geqslant |g_1|$ was shown in (b)(i). Now $0 \leqslant h \leqslant 1$ on this boundary line, so $h \geqslant 0 = \frac{H}{1-2H}$ since $H=0$. Then $|g_1| \geqslant |g_2|$ by the claim. So $|g_4| \geqslant |g_1| \geqslant |g_2|$.

If $h = 1-H$, we show that $h \geqslant \frac{H}{1-2H}$ on this boundary line. Now $h = 1-H \geqslant \frac{H}{1-2H}$ if $(1-H)(1-2H) \geqslant H$, i.e., if $g(H) := 2H^2-4H+1 \geqslant 0$. The intersection point of the lines $h=1-H$ and $h=(3+2\sqrt{2})H-1$ is at $H = 1-\frac{\sqrt{2}}{2}$, so $0 \leqslant H \leqslant 1-\frac{\sqrt{2}}{2}$ on this boundary line. But $g'(H) = 4H-4 < 0$ for $H<1$, so g is decreasing over $(0, 1-\frac{\sqrt{2}}{2})$. Then $g(H) \geqslant g(1-\frac{\sqrt{2}}{2}) = 0$, which proves that $h \geqslant \frac{H}{1-2H}$. Then $|g_1| \geqslant |g_2|$ by the claim.
Now $|g_4(H,1-H)| = \frac{1}{4}(2(1-H))^2 = (1-H)^2$ and $|g_1(H,1-H)| = (1-H)^2$. So $|g_4| = |g_1| \geqslant |g_2|$.

$$\text{If } h = (3+2\sqrt{2})H-1, \text{ we claim that } h \leqslant \frac{H}{1-2H} \tag{7}$$

Now $h = (3+2\sqrt{2})H-1 \leqslant \frac{H}{1-2H}$ if $((3+2\sqrt{2})H-1)(1-2H) \leqslant H$, i.e., if $g(H) := (6+4\sqrt{2})H^2-(4+2\sqrt{2})H+1 \geqslant 0$. Now $g'(H) = (12+8\sqrt{2})H-(4+2\sqrt{2}) < 0$ for $H < 1-\frac{\sqrt{2}}{2}$. Since the intersection point of the lines $h=1-H$ and $h=(3+2\sqrt{2})H-1$ occurs at $H = 1-\frac{\sqrt{2}}{2}, H \leqslant 1-\frac{\sqrt{2}}{2}$ on this boundary line. Since g' is negative over this interval, g is decreasing. So $g(H) \geqslant g(1-\frac{\sqrt{2}}{2}) = 0$. This proves (7), and then $|g_2| \geqslant |g_1|$ follows from the claim.

Now $|g_4(H,(3+2\sqrt{2})H-1)| = \frac{1}{4}((2+2\sqrt{2})H)^2 = (3+2\sqrt{2})H^2$, and $|g_2(H,(3+2\sqrt{2})H-1)| = |(1+(3+2\sqrt{2})H-1)H| = (3+2\sqrt{2})H^2$. So $|g_4| = |g_2| \geqslant |g_1|$.

If $h = -H$, then $h \leqslant 0$ since $H \geqslant 0$, and $\frac{H}{1-2H} \geqslant 0$ since $0 \leqslant H \leqslant \frac{1}{2}$. It follows that $h \leqslant \frac{H}{1-2H}$, and then $|g_2| \geqslant |g_1|$ by the claim. Now $|g_4(H,-H)| = \frac{1}{4}(1-2H)^2 = H^2 - H + \frac{1}{4}$, and $|g_2(H,-H)| = |(1-H)H| = H - H^2$. So $|g_4| \geqslant |g_2|$ if $g(H) := 2H^2 - 2H + \frac{1}{4} \geqslant 0$. On this boundary line, $0 \leqslant H \leqslant \frac{1}{2} - \frac{\sqrt{2}}{4}$. But $g'(H) = 4H - 2$ is negative for $H < \frac{1}{2}$, so g is decreasing over $(0, \frac{1}{2} - \frac{\sqrt{2}}{4})$. Then $g(H) \geqslant g(\frac{1}{2} - \frac{\sqrt{2}}{4}) = 0$. This shows that $|g_4| \geqslant |g_2| \geqslant |g_1|$.

Next observe that $|g_4|$ is the maximum in the region by considering the functions $f_1(H,h) := |g_4| - |g_1| = \frac{1}{4}(1-H+h)^2 - |h(H-1)|$, and $f_2(H,h) := |g_4| - |g_2| = \frac{1}{4}(1-H+h)^2 - |(1+h)H| = \frac{1}{4}(1-H+h)^2 - (1+h)H$. We know that $f_1, f_2 \geqslant 0$ on the boundary of this region, and we must show that $f_1, f_2 \geqslant 0$ in the entire region.

If $h \geqslant 0$, $f_1(H,h) = \frac{1}{4}(1-H+h)^2 - h(1-H)$, and $\frac{\partial f_1}{\partial H}(H,h) = -\frac{1}{2}(1-H+h)+h = \frac{1}{2}(-1+H+h) < 0$ since $h \leqslant 1-H$. If $h < 0$, $|f_1(H,h)| = \frac{1}{4}(1-H+h)^2 + h(1-H)$, and $\frac{\partial f_1}{\partial H}(H,h) = -\frac{1}{2}(1-H+h) - h = \frac{1}{2}(-1+H-3h) < \frac{1}{2}(-1+4H)$ since $h \geqslant -H$. But $(3+2\sqrt{2})H - 1 \leqslant h < 0$ implies that $H < \frac{1}{3+2\sqrt{2}} = 3 - 2\sqrt{2} < \frac{1}{4}$. So $|\frac{\partial f_1}{\partial H}(H,h)| \leqslant \frac{1}{2}(-1+4H) < 0$. Since $f_1 \geqslant 0$ on the boundary and $\frac{\partial f_1}{\partial H}$ is always negative in the region, $f_1 \geqslant 0$ in the region. That is, $|g_4| \geqslant |g_1|$ in the region.

Now $|f_2(H,h)| = \frac{1}{4}(1-H+h)^2 - (1+h)H$, so $\frac{\partial f_2}{\partial H}(H,h) = -\frac{1}{2}(1-H+h) - (1+h) = \frac{1}{2}(-3+H-3h) < \frac{1}{2}(-3+\frac{1}{2}+\frac{3}{2}) = -\frac{1}{2} < 0$ since $H < \frac{1}{2}$ and $h \geqslant -\frac{1}{2}$. Since $f_2 \geqslant 0$ on the boundary and $\frac{\partial f_2}{\partial H}$ is always negative in the region, $f_2 \geqslant 0$ in the region. That is, $|g_4| \geqslant |g_2|$ in the region.

B. Suppose $h \leqslant (3+2\sqrt{2})H - 1$.

We showed in (7) that if (H,h) is a point on the boundary line $h = (3+2\sqrt{2})H-1$, then $(3+2\sqrt{2})H - 1 < \frac{H}{1-2H}$. So in this case, $h \leqslant (3+2\sqrt{2})H - 1 < \frac{H}{1-2H}$. Then $|g_2| \geqslant |g_1|$ by the claim.

Now observe that $|g_2| \geqslant |g_4|$ on the boundary of this region (B in Fig. 3):

If $h = (3+2\sqrt{2})H - 1)$, then $|g_4| = |g_2|$ as shown in the argument following (7).

If $h = -H$, $|g_4(H,-H)| = H^2 - H + \frac{1}{4}$ and $|g_2(H,-H)| = H - H^2$ as shown in A. So $|g_2| \geqslant |g_4|$ if $f(H) := -2H^2 + 2H - \frac{1}{4} \geqslant 0$. Now $f'(H) = -4H + 2 > 0$ for $H < \frac{1}{2}$. But $\frac{1}{2} - \frac{\sqrt{2}}{4} \leqslant H \leqslant \frac{1}{2}$ on this boundary line, so f is increasing and $f(H) \geqslant f(\frac{1}{2} - \frac{\sqrt{2}}{4}) = 0$. This shows that $|g_2| \geqslant |g_4|$.

If $h = 1 - H$, $|g_4(H,1-H)| = \frac{1}{4}(2(1-H))^2 = (1-H)^2 = 1 - 2H + H^2$, and $|g_2(H,1-H)| = |(2-H)H| = 2H - H^2$. So $|g_2| \geqslant |g_4|$ if $f(H) := -2H^2 + 4H - 1 \geqslant 0$. But $1 - \frac{\sqrt{2}}{2} \leqslant H \leqslant \frac{1}{2}$ on this boundary line, and $f'(H) = -4H + 4 > 0$ for $H < 1$. Since f is increasing over $(1 - \frac{\sqrt{2}}{2}, \frac{1}{2})$, $f(H) \geqslant f(1 - \frac{\sqrt{2}}{2}) = 0$. This shows that $|g_2| \geqslant |g_4|$.

If $H = \frac{1}{2}$, $|g_4(\frac{1}{2},h)| = \frac{1}{4}(\frac{1}{2}+h)^2 = \frac{1}{16} + \frac{1}{4}h + \frac{1}{4}h^2$, and $|g_2(\frac{1}{2},h)| = (1+h)\frac{1}{2} = \frac{1}{2} + \frac{1}{2}h$. So $|g_2| \geqslant |g_4|$ if $f(h) := -\frac{1}{4}h^2 + \frac{1}{4}h + \frac{7}{16} \geqslant 0$. Now $f'(h) = -\frac{1}{2}h + \frac{1}{4} > 0$

for $h < \frac{1}{2}$. Since $-\frac{1}{2} \leqslant h \leqslant \frac{1}{2}$ on this boundary line, f is increasing. So $f(h) \geqslant f(\frac{1}{2}) = \frac{1}{2} > 0$. This shows that $|g_2| \geqslant |g_4|$.

Next observe that $|g_2| \geqslant |g_4|$ in the region by showing that the function $f(H,h) := |g_2| - |g_4| = |(1+h)H| - \frac{1}{4}(1-H+h)^2 = (1+h)H - \frac{1}{4}(1-H+h)^2$ is nonnegative. We already know that $f \geqslant 0$ on the boundary. Now $\frac{\partial f}{\partial H}(H,h) = 1+h+\frac{1}{2}(1-H+h) = \frac{3}{2} + \frac{3}{2}h - \frac{1}{2}H \geqslant \frac{3}{2} - \frac{3}{4} - \frac{1}{4}$ since $h \geqslant -\frac{1}{2}$ and $H \leqslant \frac{1}{2}$. So $|\frac{\partial f}{\partial H}(H,h)| \geqslant \frac{1}{2} > 0$. Since $f \geqslant 0$ on the boundary and $\frac{\partial f}{\partial H} \geqslant 0$ in the region, $f \geqslant 0$ in the region. That is, $|g_2| \geqslant |g_4|$ in this region.

The results of all cases where $H, h < \infty$ and $h \neq H - 1$ are illustrated in Fig. 4, assuming $[a,b] = [0,1]$. Recall that symmetry is used to determine the half-plane $h < -H$, where g_1 maps to g_2, g_3 maps to itself, and g_4 maps to itself.

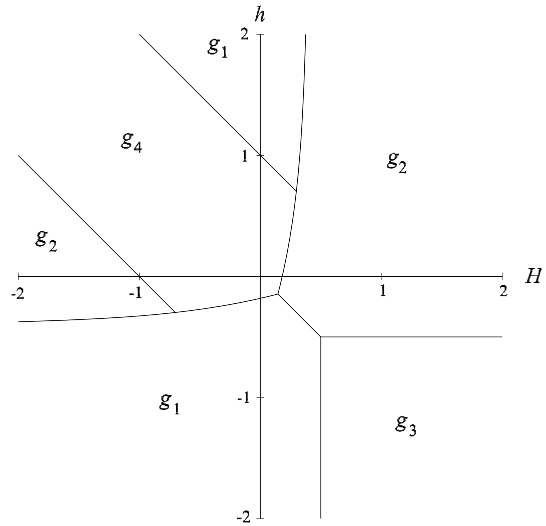

Figure 4. Maximum of the g_i

The maximum value for the case $h = 0, H = \infty$ and for the case $h = \infty, H = 0$ are as expected from the picture (Fig. 4):

1. $H = \infty$, $h = 0$

Here, the boundary conditions from (2) are $U(0) = 0, U'(1) = 0$.

The functions $u_1(x) := x$ and $u_2(x) := c$, for any $c \neq 0$, both satisfy $u'' = 0$, and they satisfy the boundary conditions $u_1(0) = 0$, $u_2'(1) = 0$. Then the Green's function (for $s \geqslant t$) is calculated to be

$$G(t,s) = \frac{-u_1(t)u_2(s)}{c(0)} = \frac{-tc}{-c} = t$$

Clearly, the maximum of $|G(t,s)| = t$ over $[0,1] \times [0,1]$ is 1, when $t = 1$ and for any $0 \leqslant s \leqslant 1$. In particular, the maximum occurs when $(t,s) = (1,1)$, which corresponds to g_2.

2. $H = 0$, $h = \infty$

Here, the boundary conditions from (2) are $U'(0) = 0, U(1) = 0$.

The functions $u_1(x) := c$, for any $c \neq 0$, and $u_2(x) := x - 1$, both satisfy $u'' = 0$, and $u_1'(0) = 0$, $u_2(1) = 0$. Then the Green's function (for $s \geqslant t$) is calculated to be

$$G(t,s) = \frac{-u_1(t)u_2(s)}{c(0)} = \frac{-c(s-1)}{c} = 1 - s$$

Clearly, the maximum of $|G(t,s)| = 1 - s$ over $[0,1] \times [0,1]$ is 1, when $s = 0$ and for any $0 \leqslant t \leqslant 1$. In particular, the maximum occurs when $(t,s) = (0,0)$, which corresponds to g_1.

This completes all cases assuming the interval is $[0,1]$. Notice how the values for $C(\tilde{H}, \tilde{h})$ in the theorem are obtained from the maximum values of the Green's functions for the interval $[0,1]$ (where $\alpha = 1 - 0 = 1$) by using (3):

$$C_0(H,h) = \begin{cases} \left|\dfrac{1}{G(0,0)}\right| = \left|\dfrac{H-h-1}{h(H-1)}\right|, & \text{if } |g_1| \text{ is the maximum} \\ \left|\dfrac{1}{G(1,1)}\right| = \left|\dfrac{H-h-1}{H(1+h)}\right|, & \text{if } |g_2| \text{ is the maximum} \\ \left|\dfrac{1}{G(0,1)}\right| = \left|\dfrac{H-h-1}{hH}\right|, & \text{if } |g_3| \text{ is the maximum} \\ \left|\dfrac{1}{G(\frac{1-H-h}{2}, \frac{1-H-h}{2})}\right| = \left|\dfrac{4(H-h-1)}{(1-H+h)^2}\right|, & \text{if } |g_4| \text{ is the maximum} \end{cases}$$

The regions described for each case in the theorem are illustrated in Fig. 4, for the interval $[0,1]$. As shown earlier, to extend this result from the interval $[0,1]$ to $[a,b]$, substitute $\alpha\tilde{h}$ for h, $\alpha\tilde{H}$ for H, and $\frac{p(x)}{\alpha^2}$ for $P(s)$, where $\alpha = \frac{1}{b-a}$ and $s = \alpha(x-a)$.

$$\int_0^1 |P(s)|\, ds \geqslant C_0(H,h) = C_0(\alpha\tilde{H}, \alpha\tilde{h}) \Longleftrightarrow \int_a^b |p(x)|\, dx \geqslant \alpha C_0(\alpha\tilde{H}, \alpha\tilde{h}) = C(\tilde{H}, \tilde{h})$$

The theorem follows from this substitution.

4. An application

One application of this result involves a vibrating wire satisfying the wave equation with elastic boundary conditions:

$$\begin{cases} u_{xx} = \rho(x) u_{tt} \\ u(0,t) = h u_x(0,t) \\ u(1,t) = H u_x(1,t) \end{cases}$$

where $\rho(x)$ is the density of the wire, $h > 0$, and $H < 0$. Using separation of variables, look for a solution of the form $u(x,t) = w(x)v(t)$. This gives the eigenvalue problem for w,

$$-w''(x) = \lambda\rho(x)w(x), w(0) = hw'(0), w(1) = Hw'(1) \tag{8}$$

Now multiply (8) by $w(x)$ and integrate both sides to get

$$\int_0^1 -w''w\ dx = \lambda \int_0^1 \rho w^2\ dx.$$

Using integration by parts, this becomes

$$\int_0^1 (w')^2\ dx - w'w|_0^1 = \lambda \int_0^1 \rho w^2\ dx$$

Then

$$\int_0^1 (w')^2\ dx - H(w'(1))^2 + h(w'(0))^2 = \lambda \int_0^1 \rho w^2\ dx \tag{9}$$

The left-hand side of (9) is nonnegative because $h > 0$ and $H < 0$. If the left-hand side equals zero, then w' must equal zero, which implies that w is constant. Then $w \equiv 0$ because $w(0) = hw'(0) = 0$ and w is constant. But $w = 0$ is not an eigenfunction, so the assumption that the left-hand side of (9) equals zero is false. So the left-hand side (and hence the right-hand side) of (9) must be strictly positive. The fact that $\rho(x) > 0$ (because it is the density) and the fact that $\lambda \int_0^1 \rho w^2\ dx > 0$ imply that $\lambda > 0$.

Let λ_0 be the least eigenvalue of (8). Note that $\lambda = 0$ is an eigenvalue of $w'' + \lambda_0\rho w = \lambda w$, so $\|\lambda_0\rho(x)\|_{L^1(0,1)} > C(H,h)$ by the theorem. Then

$$\lambda_0 \geqslant \frac{C(H,h)}{\|\rho\|_{L^1(0,1)}} = \frac{C(H,h)}{M}$$

where $M = \|\rho\|_{L^1(0,1)}$ is the total mass.

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Department of Mathematics
400 Carver Hall
Amse, Iowa 50011–2064
U.S.A.
e-mail: math@iastate,edu