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ON THE KY FAN INEQUALITY AND RELATED INEQUALITIES I

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Abstract. Refinements of the inequalities of Ky Fan [3], Wang and Wang [16], Sándor and Trif [12], and Sándor [14] are obtained. Generalizations and new proofs of some of these inequalities are also included.

1. Introduction and notation

Let $x = (x_1, x_2, \ldots, x_n)$ be an n-tuple of positive numbers. The unweighted arithmetic, geometric and harmonic means of x, denoted by A_n, G_n and H_n, respectively, are defined as follows

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

Assume that $x_i < 1$, $1 \leqslant i \leqslant n$ and define $x' := 1 - x = (1 - x_1, 1 - x_2, \ldots, 1 - x_n)$. Throughout the sequel the symbols A'_n, G'_n, and H'_n will stand for the unweighted arithmetic, geometric and harmonic means of x'.

A remarkable new counterpart of the inequality $G_n \leqslant A_n$ has been published in [3] (see page 5).

THEOREM A. *If* $0 < x_i \leqslant 1/2$, *for all* $i = 1, 2, \ldots, n$, *then*

$$\frac{G_n}{G'_n} \leqslant \frac{A_n}{A'_n} \tag{1}$$

with equality only if all the x_i *are equal.*

This result, commonly referred to as the Ky Fan inequality, has stimulated an interest of many researchers. New proofs, improvements and generalizations of the inequality (1) have been found. For more details the interested reader is referred to [1], [9], [10], [12], and [13]. The most recent proof of (1) (see [16]) utilizes some results that are obtained in [11].

W.-L. Wang and P.-F. Wang [15] have established a counterpart of the classical inequality $H_n \leqslant G_n$. Their result reads as follows.

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THEOREM B. *If* $0 < x_i \leqslant 1/2$, *for all* $i = 1, 2, \dots, n$, *then*

$$\frac{H_n}{H'_n} \leqslant \frac{G_n}{G'_n}. \tag{2}$$

Weighted arithmetic, geometric and harmonic means of x with weights $w = (w_1, w_2, \dots, w_n)$, $w_i \geqslant 0$, $1 \leqslant i \leqslant n$ with $w_1 + w_2 + \cdots + w_n = 1$, denoted by $A(x;w)$, $G(x;w)$ and $H(x;w)$ are defined in the usual way

$$A(x;w) = \sum_{i=1}^{n} w_i x_i, \quad G(x;w) = \prod_{i=1}^{n} x_i^{w_i}, \quad H(x;w) = \frac{1}{\sum_{i=1}^{n} w_i x_i}.$$

Other means used in this paper are the weighted identric and logarithmic means $I(x;w)$ and $L(x;w)$. Both means admit integral representations that are included below. Let

$$E_{n-1} = \{(u_1, \dots, u_{n-1}) : u_i \geqslant 0, \;\; i = 1, 2, \dots, n, \;\; u_1 + \cdots + u_{n-1} \leqslant 1\}$$

be the Euclidean simplex and let $\mu(u)$, $u \in E_{n-1}$ be a probability measure on E_{n-1}. Define $d\mu(u) = \mu(u) du_1 \dots du_{n-1}$. The weights w_i are the natural weights, i.e.,

$$w_i = \int_{E_{n-1}} u_i d\mu(u), \tag{3}$$

$i = 1, 2, \dots, n$, where $u_n = 1 - u_1 - \cdots - u_{n-1}$. The weighted identric mean $I(x;w)$ of x is defined as follows ([8])

$$I(x;w) = \exp\left[\int_{E_{n-1}} \ln(u \cdot x) d\mu(u)\right], \tag{4}$$

where $u \cdot x = u_1 x_1 + \cdots + u_n x_n$ is the inner product of u and x. Recently Sándor and Trif [12] have obtained a new refinement of the Ky Fan inequality.

THEOREM C. *Let* $0 < x_i \leqslant 1/2$, *for all* $i = 1, 2, \dots, n$. *Then*

$$\frac{G(x;w)}{G'(x;w)} \leqslant \frac{I(x;w)}{I'(x;w)} \leqslant \frac{A(x;w)}{A'(x;w)}, \tag{5}$$

where $G(x;w)$ *and* $A(x;w)$ *are the weighted geometric and arithmetic means, respectively. The weights* $w = (w_1, \dots, w_n)$ *are the natural weights of the probability measure* $\mu(u)$.

Another interesting result connecting weighted harmonic and arithmetic means was obtained by Sándor (see [10], [14]).

THEOREM D. *If* $0 < x_i \leqslant 1/2$, *for* $i = 1, 2, \dots, n$, *then*

$$\frac{1}{H'(x;w)} - \frac{1}{H(x;w)} \leqslant \frac{1}{A'(x;w)} - \frac{1}{A(x;w)} \tag{6}$$

with equality if and only if $x_1 = \ldots = x_n$.

Alzer [2] has obtained a refinement of the inequality (6) for the unweighted means

$$\frac{1}{H_n'} - \frac{1}{H_n} \leqslant \frac{1}{G_n'} - \frac{1}{G_n} \leqslant \frac{1}{A_n'} - \frac{1}{A_n}.$$

This paper is organized as follows. A generalization and a refinement of the Ky Fan inequality are obtained in Section 2. A new proof of the Wang and Wang inequality (2) is presented in Section 3. The main result of this section also provides a refinement of the inequality (2). Two refinements of the inequality (5) are derived in Section 4. The underlying probability measure is the Dirichlet measure. The last section of this paper deals with a refinement of the inequality (6). We shall demonstrate that the quantity $1/L'(x;w) - 1/L(x;w)$ interpolates the inequality in question. Here

$$L(x;w) = \left[\int_{E_{n-1}} (u \cdot x)^{-1} d\mu(u) \right]^{-1} \tag{7}$$

is the weighted logarithmic mean of x (see [8]).

2. A generalization and refinement of Ky Fan's inequality

Before we state and prove the main result of this section, let us introduce more notation. The unweighted power mean of order p $(p \in \mathbf{R})$ of x, denoted by M_p, is defined as

$$M_p(x) = \begin{cases} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p}, & p \neq 0 \\ \left(\prod_{i=1}^{n} x_i \right)^{1/n}, & p = 0. \end{cases}$$

The following result, which is due to K. Menon (see, e.g., [5], p. 284), will be used in the proof of Theorem 1.

LEMMA. *Let* $a > 0$, $t \in \mathbf{R}$. *If the numbers* b_i, $1 \leqslant i \leqslant n$, *satisfy the inequality* $b_i \geqslant a/t$, *then*

$$\sum_{i=1}^{n} \frac{1}{(a+b_i)^t} \geqslant \frac{n}{\left[a + \left(\prod_{i=1}^{n} b_i \right)^{1/n} \right]^t}. \tag{8}$$

THEOREM 1. *Let* $0 < t \leqslant 1$ *and let* $0 < x_i \leqslant t/(1+t)$, *for* $i = 1, 2, \ldots, n$. *Then the following inequality is valid*

$$\frac{G_n}{G_n + G_n'} \leqslant M_t. \tag{9}$$

REMARK. Under the assumptions of Theorem 1, the following inequality

$$\frac{G_n}{G'_n} \leqslant \frac{G_n}{(G_n+G'_n)A'_n} \leqslant \frac{M_t}{A'_n} \leqslant \frac{A_n}{A'_n} \tag{10}$$

holds true. The first inequality in (10) follows from the Ky Fan inequality (1) and the third one is an obvious consequence of $M_t \leqslant A_n$ $(t \leqslant 1)$.

Proof of Theorem 1. Let $b_i = 1/x_i - 1$, $1 \leqslant i \leqslant n$, and let $a = 1$. Then $0 < x_i < t/(1+t)$ implies $b_i \geqslant a/t$. Making use of (8) we obtain

$$\frac{1}{n}\sum_{i=1}^{n} x_i^t \geqslant \left(\frac{G_n}{G_n+G'_n}\right)^t.$$

Since $t > 0$, the assertion (9) follows. □

3. A refinement of the inequality of Wang and Wang

For later use we introduce the following notation. Let $x = (x_1,\ldots,x_n)$ with $x_i > 0$, $1 \leqslant i \leqslant n$. We define a vector $a = (a_1,\ldots,a_n)$ where $a_i = x_1\cdots x_{i-1}x_{i+1}\cdots x_n$, $1 \leqslant i \leqslant n$ and put $S_n = a_1+\ldots+a_n$. The unweighted arithmetic mean of a is $A_n(a) = S_n/n$. Similarly, the unweighted identric and geometric means of a will be denoted by $I_n(a)$ and $G_n(a)$, respectively. Also, let

$$J_n = \frac{I_n(a)}{G_n(a)}H_n, \tag{11}$$

where H_n is the unweighted harmonic mean of x.

PROPOSITION 1. *J_n is a mean of the vector x. Moreover, it interpolates the harmonic-geometric mean inequality, i.e.,*

$$H_n \leqslant J_n \leqslant G_n. \tag{12}$$

Proof. Opening statement is an immediate consequence of (12). The first inequality in (12) follows from the well-known result $G_n(a) \leqslant I_n(a)$. For the proof of the second inequality in (12) we use the identity

$$\frac{H_n}{G_n(a)} = \frac{G_n}{A_n(a)} \tag{13}$$

and the inequality $I_n(a) \leqslant A_n(a)$. We shall now establish formula (13). We have

$$H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} = \frac{n(x_1\cdot\ldots\cdot x_n)}{S_n} = \frac{G_n^n}{S_n/n} = \frac{G_n^n}{A_n(a)}.$$

It is easy to see that $G_n(a) = G_n^{n-1}$. Thus

$$\frac{H_n}{G_n(a)} = \frac{G_n^n}{G_n^{n-1}A_n(a)} = \frac{G_n}{A_n(a)}. \quad \square$$

The main result of this section is contained in the following.

THEOREM 2. *Let* $0 < x_i \leqslant 1/2$, *for* $i = 1, 2, \ldots, n$. *Then*

$$\frac{H_n}{H'_n} \leqslant \frac{J_n}{J'_n} \leqslant \frac{G_n}{G'_n}. \tag{14}$$

Proof. It follows from (13) that

$$A_n(a) = \frac{G_n(a) G_n}{H_n}$$

and also, after replacing x by $x' = 1 - x$ and a by $a' = 1 - a$, that

$$A'_n(a) = \frac{G'_n(a) G'_n}{H'_n}.$$

Hence

$$\frac{A_n(a)}{A'_n(a)} = \frac{H'_n}{H_n} \frac{G_n(a)}{G'_n(a)} \frac{G_n}{G'_n}. \tag{15}$$

In order to prove the inequality (14) we need the following one

$$\frac{G_n(a)}{G'_n(a)} \leqslant \frac{I_n(a)}{I'_n(a)} \leqslant \frac{A_n(a)}{A'_n(a)}. \tag{16}$$

This follows from (5) by using the Lebesgue measure $\mu(u) = (n-1)!$. It is well-known that the weights $w = (w_1, \ldots, w_n)$ of the Lebesgue measure are all equal, i.e., $w_i = 1/n$, $1 \leqslant i \leqslant n$. Also, since $0 < x_i \leqslant 1/2$, $0 < a_i \leqslant (1/2)^{n-1} \leqslant 1/2$ for $n \geqslant 2$. Application of (15) to the last term in (16) completes the proof. □

4. Refinements of the inequality (5)

The goal of this section is to obtain two refinements of the inequality (5). These results are derived for the underlying probability measure being the Dirichlet measure. For the reader's convenience, we include below a definition of this measure. Also, a concept of the Dirichlet average of a function is given.

Let $b = (b_1, \ldots, b_n)$ where $b_i > 0$ for $i = 1, \ldots, n$. The Dirichlet measure μ on the Euclidean simplex E_{n-1} is defined as follows

$$\mu(u) = \frac{1}{B(b)} \prod_{i=1}^{n} u_i^{b_i - 1},$$

where $B(b)$ is the multivariate beta function in variables b_i, $u = (u_1, \ldots, u_n)$ with $(u_1, \ldots, u_{n-1}) \in E_{n-1}$ and $u_n = 1 - b_1 - \ldots - b_{n-1}$ (see [4], 4.4-1). The weights $w = (w_1, \ldots, w_n)$ of μ are $w_i = b_i/c$, $1 \leqslant i \leqslant n$, where $c = b_1 + \ldots + b_n$.

Let f be an integrable function on the convex hull of $x = (x_1, \ldots, x_n)$. Dirichlet average $F(x; \mu)$ of f is defined as

$$F(x; \mu) = \int_{E_{n-1}} f(u \cdot x) d\mu(u) \tag{17}$$

(see [4], 5.2-1).

Let K be an interval containing $x_1, \ldots, x_n$. If $f : K \to \mathbf{R}$ is a convex function, then

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant F(x;\mu) \leqslant \sum_{i=1}^{n} w_i f(x_i) \tag{18}$$

with the inequalities reversed if f is concave on K.

Two refinements of this inequality have been obtained in [7] (Theorem 4.1 and Corollary 4.2). Let $\lambda > 0$. Define $b_\lambda = (b_1, \ldots, b_{n-1}, b_n + \lambda)$ and denote the associated Dirichlet measure by μ_λ. Further, let $v = (v_1, \ldots, v_n)$ be the weights of μ_λ, i.e., $v_i = b_i/(c+\lambda)$, $1 \leqslant i \leqslant n-1$ and $v_n = (b_n + \lambda)/(c+\lambda)$. Also, let $\alpha = c/(c+\lambda)$, $\beta = 1-\alpha$ and let $y = (\alpha x_1 + \beta x_n, \ldots, \alpha x_n + \beta x_n)$. We shall utilize the following result ([7]).

LEMMA. *Let $f : K \to \mathbf{R}$ be a concave function. Then*

$$\sum_{i=1}^{n} v_i f(x_i) \leqslant \alpha F(x;\mu) + \beta f(x_n) \leqslant F(x;\mu_\lambda) \leqslant F(y;\mu) \leqslant f\left(\sum_{i=1}^{n} v_i x_i\right) \tag{19}$$

with the inequalities reversed if f is a convex function on K.

In what follows we will write $G(x;v)$, $I(x;v)$ and $A(x;v)$ for the weighted geometric, identric and arithmetic means, respectively of x with weights v.

THEOREM 3. *Let $0 < x_i \leqslant 1/2$ for $i = 1, \ldots, n$. Then*

$$\begin{aligned}\frac{G(x;v)}{G'(x;v)} &\leqslant \left[\frac{I(x;w)}{I'(x;w)}\right]^{\alpha}\left(\frac{x_n}{1-x_n}\right)^{\beta} \\ &\leqslant \frac{I(x;v)}{I'(x;v)} \leqslant \frac{I(y;w)}{I'(y;w)} \leqslant \frac{A(x;v)}{A'(x;v)}.\end{aligned} \tag{20}$$

Proof. Let $f(t) = \ln t - \ln(1-t)$, $0 < t \leqslant 1/2$. Clearly function f is concave on the stated domain. Making use of (17) and (4) we see that the middle term in (19) can be written as

$$F(x;\mu_\lambda) = \int_{E_{n-1}} \ln(u \cdot x)d\mu_\lambda(u) - \int_{E_{n-1}} \ln(1 - u \cdot x)d\mu_\lambda(u).$$

Since $1 - u \cdot x = u \cdot (1-x) = u \cdot x'$,

$$\begin{aligned}F(x;\mu_\lambda) &= \int_{E_{n-1}} \ln(u \cdot x)d\mu_\lambda(u) - \int_{E_{n-1}} \ln(u \cdot x')d\mu_\lambda(u) \\ &= \ln I(x;v) - \ln I'(x;v) = \ln \frac{I(x;v)}{I'(x;v)}.\end{aligned}$$

In a similar fashion one can rewrite the remaining terms in (19) to obtain the desired inequality (20). □

5. A refinement of the inequality (6)

The following result will be utilized when proving the main result of this section.

LEMMA ([6]). *Let* $f : K \to \mathbf{R}$ *be a convex function and let* $x = (x_1, \ldots, x_n)$ *with* $x_i \in K$ *for* $i = 1, \ldots, n$. *Further, let* $w = (w_1, \ldots, w_n)$ *are the natural weights of the probability measure* μ *on* E_{n-1}. *Then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant \int_{E_{n-1}} f(u \cdot x) d\mu(u) \leqslant \sum_{i=1}^{n} w_i f(x_i) \tag{21}$$

with the inequalities reversed if f *is a concave function.*

This result provides a refinement of the classical Jensen's inequality.

In what follows, $L(x;w)$ will stand for the weighted logarithmic mean of x with weights w (see (7)).

We have

THEOREM 4. *Let* $0 < x_i \leqslant 1/2$ *for* $i = 1, \ldots, n$ *and let the weights* $w = (w_1, \ldots, w_n)$ *be defined as in (3). Then*

$$\frac{1}{H'(x;w)} - \frac{1}{H(x;w)} \leqslant \frac{1}{L'(x;w)} - \frac{1}{L(x;w)} \leqslant \frac{1}{A'(x;w)} - \frac{1}{A(x;w)}. \tag{22}$$

Proof. In order to prove (22) we define $f(t) = 1/t - 1/(1-t)$, $0 < t \leqslant 1/2$. Clearly function f is convex on the stated domain. Use of (21) together with (7) gives

$$\frac{1}{A(x;w)} - \frac{1}{A'(x;w)} \leqslant \frac{1}{L(x;w)} - \frac{1}{L'(x;w)} \leqslant \frac{1}{H(x;w)} - \frac{1}{H'(x;w)}.$$

Hence the desired inequality follows. □

We close this section with the following.

Open Problem. Prove or disprove that under the assumptions of Theorem 4 the following inequality

$$\frac{1}{G'(x;w)} - \frac{1}{G(x;w)} \leqslant \frac{1}{L'(x;w)} - \frac{1}{L(x;w)} \tag{23}$$

is valid.

Alzer ([1], Theorem 9.1) has established the following result

$$\frac{1}{H'(x;w)} - \frac{1}{H(x;w)} \leqslant \frac{1}{G'(x;w)} - \frac{1}{G(x;w)}. \tag{24}$$

If the inequality (23) holds true, then this together with (24) will provide a refinement of the first inequality in (22).

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