

MONOTONICITY RESULTS AND INEQUALITIES FOR THE GAMMA AND INCOMPLETE GAMMA FUNCTIONS

FENG QI

(communicated by J. Pečarić)

Abstract. In the article, using the monotonicity and inequalities of the generalized weighted mean values with two parameters, we prove that the functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ are increasing in $r > 0$, $s > 0$ and $x > 0$, where $\Gamma(s)$, $\Gamma(s,x)$ and $\gamma(s,x)$ denote the gamma and incomplete gamma functions with usual notation. From this, some monotonicity results and inequalities for the gamma or incomplete gamma functions are deduced or extended, a unified proof of some known results for the gamma function is given.

1. Introduction

It is well-known that the incomplete gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z, x) = \int_x^\infty t^{z-1} e^{-t} dt, \quad \gamma(z, x) = \int_0^x t^{z-1} e^{-t} dt, \quad (1.1)$$

and $\Gamma(z, 0) = \Gamma(z)$ is called the gamma function, $\Gamma(0, x) = E_1(x)$ the exponential integral.

Some monotonicity results and inequalities for the function $\frac{\Gamma(x+\lambda)}{\Gamma(x+1)}$ and the gamma function $\Gamma(x)$ with usual notation, where $x > 0$ and $0 < \lambda < 1$ is independent of x , have been studied by many authors, cf. [2]–[14], [17]–[19], [24], [25] and [32].

Recently the author in [22] established the generalized weighted mean values $M_{p,f}(r, s; x, y)$ of a positive function f with two parameters $r, s \in \mathbb{R}$ and nonnegative

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weight $p \not\equiv 0$ for $x, y \in \mathbb{R}$ by

$$M_{p,f}(r, s; x, y) = \left(\frac{\int_x^y p(u) f^s(u) du}{\int_x^y p(u) f^r(u) du} \right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0; \quad (1.2)$$

$$M_{p,f}(r, r; x, y) = \exp \left(\frac{\int_x^y p(u) f^r(u) \ln f(u) du}{\int_x^y p(u) f^r(u) du} \right), \quad x-y \neq 0; \quad (1.3)$$

$$M_{p,f}(r, s; x, x) = f(x), \quad x = y.$$

For convenience, we write, shifting notation to suit the context,

$$M_{p,f}(r, s; x, y) = M_{p,f}(r, s) = M_{p,f}(x, y) = M_{p,f}. \quad (1.4)$$

Set $p(u) \equiv 1$, $f(u) = u$ and $x, y > 0$, then the generalized weighted mean values are reduced to the extended mean values $E(r, s; x, y)$ defined as

$$E(r, s; x, y) = \left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right]^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \quad (1.5)$$

$$E(r, 0; x, y) = \left[\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right]^{1/r}, \quad r(x-y) \neq 0; \quad (1.6)$$

$$E(r, r; x, y) = \frac{1}{e^{1/r}} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)}, \quad r(x-y) \neq 0; \quad (1.7)$$

$$E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y;$$

$$E(r, s; x, x) = x, \quad x = y.$$

REMARK 1. Many proofs of monotonicities for $E(r, s; x, y)$ and $M_{p,f}(r, s; x, y)$ have been presented by some authors, for details, please refer to [15, 21, 22, 26, 29, 30]. The logarithmic convexity of $E(r, s; x, y)$ is investigated in [23].

In this article, using the monotonicity and inequalities of the generalized weighted mean values $M_{p,f}(r, s; x, y)$, we verified that functions $\left[\frac{\Gamma(s)}{\Gamma(r)} \right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)} \right]^{1/(s-r)}$ and $\left[\frac{\gamma(s,x)}{\gamma(r,x)} \right]^{1/(s-r)}$ are increasing in $r > 0$, $s > 0$ and $x > 0$, respectively. In consequence, some monotonicity results and inequalities for the gamma or incomplete gamma functions are deduced or extended, a unified proof of some well-known results for the gamma function is provided.

2. Monotonicity Results and Inequalities

LEMMA 1 [8]. *Suppose $f(t)$ is a positive differentiable function and $p(t) \not\equiv 0$ an integrable nonnegative weight on the interval $[a, b]$, if $f'(t)$ and $\frac{f'(t)}{p(t)}$ are integrable and both increasing or both decreasing, then for all real numbers r and s , we have*

$$M_{p,f}(r, s; a, b) < E(r+1, s+1; f(a), f(b)); \quad (2.1)$$

if one of the functions $f'(t)$ or $\frac{f'(t)}{p(t)}$ is nondecreasing and the other nonincreasing, then the inequality (2.1) reverses.

THEOREM 1. For any given $x > 0$, the function $\frac{s\gamma(s,x)}{x^s}$ is decreasing in $s > 0$.

Proof. Set $p(t) = e^{-t}$, $f(t) = t$, $t \in (0, x)$ in Lemma 1, then, for $s > r > 0$, we obtain

$$\left(\frac{\int_0^x t^{s-1} e^{-t} dt}{\int_0^x t^{r-1} e^{-t} dt} \right)^{1/(s-r)} \leq \left(\frac{r}{s} \cdot \frac{x^s}{x^r} \right)^{1/(s-r)}.$$

Simplifying above inequality yields

$$\frac{s\gamma(s, x)}{x^s} \leq \frac{r\gamma(r, x)}{x^r}.$$

This implies Theorem 1. \square

LEMMA 2 [22, 31]. Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive and continuous function. Then $M_{p,f}(r, s)$ increases with both r and s .

THEOREM 2. The function $\left[\frac{\Gamma(s)}{\Gamma(r)} \right]^{1/(s-r)}$ is increasing with $r > 0$ and $s > 0$.

Proof. This follows from Lemma 2 applied to $p(u) = e^{-u}$, $f(u) = u$, $u \in (0, +\infty)$ and standard arguments. \square

COROLLARY 1. The functions $[\Gamma(r)]^{1/(r-1)}$ and $\psi(r) = \frac{\Gamma'(r)}{\Gamma(r)}$, the logarithmic derivative of the gamma function $\Gamma(r)$, are increasing in $r > 0$. Hence $\Gamma(r)$ is a logarithmically convex function in the interval $(0, +\infty)$.

REMARK 2. In [11, 17], among other things, the following monotonicity results were obtained

$$\begin{aligned} & [\Gamma(1+k)]^{1/k} < [\Gamma(2+k)]^{1/(k+1)}, \quad k \in \mathbb{N}; \\ & \left[\Gamma \left(1 + \frac{1}{x} \right) \right]^x \text{ decreases with } x > 0. \end{aligned}$$

Clearly, our Theorem 2 and Corollary 1 generalize and extend these results for the range of the argument.

COROLLARY 2. The following inequalities hold for $s > r > 0$

$$\exp [(s-r)\psi(s)] > \frac{\Gamma(s)}{\Gamma(r)} > \exp [(s-r)\psi(r)], \tag{2.2}$$

$$e^{cr} < \Gamma(r+1) < \exp [r\psi(r+1)], \tag{2.3}$$

where $c = 0.5772 \dots$ is the Euler's constant.

Proof. These follow from standard arguments and the following relationships

$$\begin{aligned} M_{p,f}(s, s) &> M_{p,f}(r, s) > M_{p,f}(r, r), \\ M_{p,f}(r, r) &> M_{p,f}(r, 0) > M_{p,f}(0, 0) \end{aligned}$$

for $s > r > 0$. \square

REMARK 3. The ratio $\frac{\Gamma(s)}{\Gamma(r)}$ has been researched by many mathematicians. W. Gautschi showed in [7] for $0 < s < 1$ and $n \in \mathbb{N}$ that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp[(1-s)\psi(n+1)]. \quad (2.4)$$

A strengthened upper bound was given by T. Erber in [6]

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}, \quad 0 < s < 1, \quad n \in \mathbb{N}. \quad (2.5)$$

J. D. Kečkić and P. M. Vasić gave in [9] the inequalities below

$$\frac{b^{b-1}}{a^{a-1}} \cdot e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} \cdot e^{a-b}, \quad 0 < a < b. \quad (2.6)$$

The following closer bounds were proved for $0 < s < 1$ and $x \geq 1$ by D. Kershaw in [10]

$$\exp\left[(1-s)\psi\left(x+s^{1/2}\right)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \quad (2.7)$$

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s-\frac{1}{4}\right)^{1/2}\right]^{1-s}. \quad (2.8)$$

Inequalities for the incomplete gamma function are given for $a > 0$ in [19, p. 526] and [32, pp. 442–443] as follows

$$a\gamma(a, a) > \gamma(a+1, a+1), \quad (2.9)$$

$$\frac{\gamma(a, a)}{\Gamma(a)} > \frac{\gamma(a+1, a+1)}{\Gamma(a+1)}, \quad (2.10)$$

$$\frac{\gamma(a, a)}{\Gamma(a)} > \frac{1}{2}. \quad (2.11)$$

More inequalities and some monotonicity results for the same quotient could be found in [12, 13, 14, 18]. Similar results can be found in [16].

It is easy to see that inequalities in (2.2) of Corollary 2 extend the range of arguments of above inequalities (2.4)–(2.8) but (2.6).

To the best of my knowledge, inequalities in (2.3) are new. In [1, 2, 3], Horst Alzer established many inequalities for the gamma function. In [24, 25] the authors found some inequalities of the incomplete gamma function by the Tchebycheff's integral inequality and Hermite-Hadamard's inequality.

LEMMA 3 [22]. *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive, increasing (or decreasing, respectively) and continuous function. Then $M_{p,f}(x, y)$ increases (or decreases, respectively) with respect to either x or y .*

THEOREM 3. For $s > r > 0$ and $x > 0$, $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ increase with either x or r and s . Therefore, $\frac{\gamma(s,x)}{x^{s-1}}$ decreases and $\frac{\Gamma(s,x)}{x^{s-1}}$ increases with $s > 0$, respectively.

Proof. The first part is a simple consequence of Lemma 2 and 3. The second part is concluded from differentiating $\frac{\gamma(s,x)}{\gamma(r,x)}$ and $\frac{\Gamma(s,x)}{\Gamma(r,x)}$ with respect to x and standard argument. \square

COROLLARY 3. The incomplete gamma functions $\gamma(r, x)$ and $\Gamma(r, x)$ are logarithmically convex with respect to $r > 0$ for $x > 0$. The function $\left[\frac{\Gamma(r,x)}{E_1(x)}\right]^{1/r}$ is increasing in $r > 0$ and $x > 0$, where $E_1(x)$ denotes the exponential integral. Therefore, the functions $\frac{\Gamma(s+\theta)}{\Gamma(r+\theta)}$, $\frac{\Gamma(s+\theta,x)}{\Gamma(r+\theta,x)}$ and $\frac{\gamma(s+\theta,x)}{\gamma(r+\theta,x)}$ are increasing with θ for $s > r > 0$ and $x > 0$.

LEMMA 4 [22]. Let $p_1(u) \not\equiv 0$ and $p_2(u) \not\equiv 0$ be nonnegative and integrable functions on the interval between x and y , $f(u)$ a positive and integrable function, the ratio $\frac{p_1(u)}{p_2(u)}$ an integrable function, $\frac{p_1(u)}{p_2(u)}$ and $f(u)$ both increasing or both decreasing. Then

$$M_{p_1,f}(r, s; x, y) \geq M_{p_2,f}(r, s; x, y) \tag{2.12}$$

If one of the functions of $f(u)$ or $\frac{p_1(u)}{p_2(u)}$ is nonincreasing and the other nondecreasing, then inequality (2.12) is reversed.

THEOREM 4. Let $g(t)$ be an integrable positive function such that $e^t g(t)$ decreasing, then

$$\frac{\gamma(s, x)}{\gamma(r, x)} \geq \frac{\int_0^x t^{s-1} g(t) dt}{\int_0^x t^{r-1} g(t) dt}, \tag{2.13}$$

$$\frac{\Gamma(s, x)}{\Gamma(r, x)} \geq \frac{\int_x^\infty t^{s-1} g(t) dt}{\int_x^\infty t^{r-1} g(t) dt}, \tag{2.14}$$

$$\frac{\Gamma(s)}{\Gamma(r)} \geq \frac{\int_0^\infty t^{s-1} g(t) dt}{\int_0^\infty t^{r-1} g(t) dt} \tag{2.15}$$

hold for $s > r > 0$ and $x > 0$. If $e^t g(t)$ is increasing, then the above inequalities reverse.

Proof. These are special cases of inequality (2.12) in Lemma 4 applied to $f(t) = t$, $p_1(t) = e^{-t}$ and $p_2(t) = g(t)$. \square

LEMMA 5 [22]. Let $p(u) \not\equiv 0$ be a nonnegative and integrable function, and $f_1(u)$ and $f_2(u)$ positive and integrable functions on the interval between x and y . If the ratio $\frac{f_1(u)}{f_2(u)}$ and $f_2(u)$ are integrable and both increasing or both decreasing, then

$$M_{p,f_1}(r, s; x, y) \geq M_{p,f_2}(r, s; x, y) \tag{2.16}$$

holds for $r, s \geq 0$ or $r \geq 0 \geq s$, and $\frac{f_1(u)}{f_2(u)} \geq 1$. The inequality (2.16) is reversed for $r, s \leq 0$ or $s \geq 0 \geq r$, and $\frac{f_1(u)}{f_2(u)} \leq 1$.

If one of the functions of $f_2(u)$ or $\frac{f_1(u)}{f_2(u)}$ is nonincreasing and the other nondecreasing, then inequality (2.16) is valid for $r, s \geq 0$ or $s \geq 0 \geq r$, and $\frac{f_1(u)}{f_2(u)} \geq 1$; the inequality (2.16) reverses for $r, s \geq 0$ or $r \geq 0 \geq s$, and $\frac{f_1(u)}{f_2(u)} \leq 1$.

THEOREM 5. Let $f(u)$ be a positive and integrable function on $(0, +\infty)$. If $\frac{f(u)}{u} > 1$ is increasing, then, for $s > r > 0$ and $x > 0$, we have

$$\frac{\Gamma(s+1)}{\Gamma(r+1)} \leq \frac{\int_0^\infty f^s(u)e^{-u}du}{\int_0^\infty f^r(u)e^{-u}du}, \quad (2.17)$$

$$\frac{\Gamma(s+1, x)}{\Gamma(r+1, x)} \leq \frac{\int_x^\infty f^s(u)e^{-u}du}{\int_x^\infty f^r(u)e^{-u}du}, \quad (2.18)$$

$$\frac{\gamma(s+1, x)}{\gamma(r+1, x)} \leq \frac{\int_0^x f^s(u)e^{-u}du}{\int_0^x f^r(u)e^{-u}du}. \quad (2.19)$$

If $\frac{f(u)}{u} < 1$ is decreasing, the above inequalities reverse for $s > r > 0$ and $x > 0$.

Proof. This is a direct consequence of Lemma 2. applied to $f_1(u) = f(u)$, $f_2(u) = u$ and $p(u) = e^{-u}$. \square

REMARK 4. Recently, using the approach by A. Laforgia and S. Sismondi in [14], some more general inequalities of the functions $\int_0^x e^{pt} dt$ and $\int_0^x e^{-pt} dt$ for $p > 0$ and $x > 0$ are obtained in [27]. For more information, please refer to [4, 28].

ADDENDUM. After submitting the final revised version of this paper the author was informed about the paper [5] where, among others, the convexity of the function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$ for $|t-s| < 1$ is verified, the best lower bound for (2.7) and the best upper bound for (2.8) are obtained, some different approach from Gautschi's in [7] is given, and several new simple inequalities for digamma function are also proved.

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Department of Mathematics
 Jiaozuo Institute of Technology
 Jiaozuo City, Henan 454000
 The People's Republic of China
 e-mail: qifeng@jzjit.edu.cn
 e-mail: qifeng618@hotmail.com

URL: <http://rgmia.vu.edu.au/qi.html>