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AN EXTENSION OF UCHIYAMA'S RESULT ASSOCIATED WITH AN ORDER PRESERVING OPERATOR INEQUALITY

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Abstract. Let A, B and C be positive invertible operators and also let r, s and t be non-negative real numbers such that $t \geqslant s$ and $(r,t) \neq (0,0)$. Then the following (I) and (II) hold and follows from each other.

(I) If $A^t \ll B^t \nabla_\lambda C^t$ (i.e., $\log A^t \leqslant \log(B^t \nabla_\lambda C^t))$ for all $t \geqslant 0$, then

$$f(t) = \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \quad \text{is an increasing function of} \quad t.$$

(II) If $A^t \gg B^t !_\lambda C^t$ (i.e., $\log A^t \geqslant \log(B^t !_\lambda C^t))$ for all $t \geqslant 0$, then

$$h(t) = \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \quad \text{is a decreasing function of} \quad t,$$

where $B\nabla_\lambda C$ and $B!_\lambda C$ are the arithmetic mean and the harmonic mean respectively. In particular we have

(I') *If* $A^t \ll B^t \nabla_\lambda C^t$, then

$$A^{\frac{r}{2}}(B^s \nabla_\lambda C^s)A^{\frac{r}{2}} \leqslant \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

(II') *If* $A^t \gg B^t !_\lambda C^t$, then

$$A^{\frac{r}{2}}(B^s !_\lambda C^s)A^{\frac{r}{2}} \geqslant \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

These are extensions of the recent results in Uchiyama [12].

1. Introduction

A capital letter means a bounded linear operator on a Hilbert space. The following order preserving operator inequalities are given in [5].

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THEOREM F (FURUTA INEQUALITY).

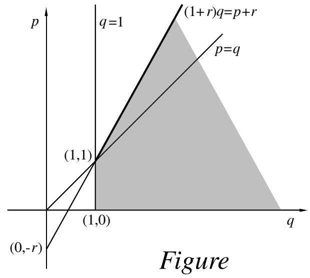
If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



Figure

Mean theoretic proofs in [2][8] are given and one page proof in [6]. The domain drawn for p, q and r in the Figure is the best possible one for (i) and (ii) of Theorem F in [11]. The following Theorem F_1 is the essential part of Theorem F.

THEOREM F_1. *If* $A \geqslant B \geqslant 0$, *then for the following inequalities hold:*

($\star$-1) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant B^{1+r}$$

($\star$-2) $$A^{1+r} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$$

for $p \geqslant 1$ *and* $r \geqslant 0$.

We recall that the arithmetic mean $B\nabla_\lambda C$ for positive operators B and C is defined by

$$B\nabla_\lambda C = \lambda B + (1-\lambda)C \quad \text{for} \quad \lambda \in [0,1]$$

and also the harmonic mean $B!_\lambda C$ for positive invertible operators B and C is defined by

$$B!_\lambda C = (\lambda B^{-1} + (1-\lambda)C^{-1})^{-1} \quad \text{for} \quad \lambda \in [0,1]$$

and if B and C are not invertible positive operators, $B!_\lambda C$ is defined by the weak limit of $(B+\varepsilon)!_\lambda(C+\varepsilon)$ as $\varepsilon \to +0$ (cf. [10]) and the following inequality always holds:

$$B\nabla_\lambda C \geqslant B!_\lambda C. \tag{1}$$

Also we recall that the *chaotic order* $A \gg B$ ([3][4][7] and [9]) is defined by $\log A \geqslant \log B$ for positive invertible operators A and B and it is well known that $A \geqslant B > 0$ yields $A \gg B$, that is, $A \gg B$ is weaker than the usual order $A \geqslant B > 0$.

Very recently, Uchiyama [12] has proved the following nice result.

THEOREM U [12]. *Let* A, B *and* C *be positive invertible operators and also let* r, s *and* t *be non-negative real numbers such that* $t \geqslant s$ *and* $(r,t) \neq (0,0)$. *If* $A \leqslant B!_\lambda C$, *then*

$$A^{\frac{r}{2}}(B^s\nabla_\lambda C^s)A^{\frac{r}{2}} \leqslant \{A^{\frac{r}{2}}(B^t\nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

Uchiyama [12] has pointed out an interesting view of Theorem F_1 as follows. In fact, put $B = C$ and $s = 1$ in Theorem U. Then we have

$$0 \leqslant A \leqslant B \quad \text{ensures} \quad A^{1+r} \leqslant A^{\frac{r}{2}}BA^{\frac{r}{2}} \leqslant (A^{\frac{r}{2}}B^tA^{\frac{r}{2}})^{\frac{1+r}{t+r}} \quad \text{for} \quad t \geqslant 1 \quad \text{and} \quad r \geqslant 0$$

that is, we have $(\star\text{-}1)$ in Theorem F_1.

In this paper we give an extension of Theorem U, that is, we obtain stronger result than one of Theorem U under weaker hypothesis than one of Theorem U.

2. Statement of results

THEOREM 1. *Let A, B and C be positive invertible operators and also let r, s and t be non-negative real numbers such that $t \geqslant s$ and $(r,t) \neq (0,0)$. Then the following* (I) *and* (II) *hold and follows from each other.*

(I) *If $A^t \ll B^t \nabla_\lambda C^t$ (i.e., $\log A^t \leqslant \log(B^t \nabla_\lambda C^t)$) for all $t \geqslant 0$, then*

$$f(t) = \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

is an increasing function of t.

(II) *If $A^t \gg B^t !_\lambda C^t$ (i.e., $\log A^t \geqslant \log(B^t !_\lambda C^t)$) for all $t \geqslant 0$, then*

$$h(t) = \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

is a decreasing function of t.

We need the following Theorem A to prove Theorem 1.

THEOREM A [4][7] (cf.[1]). *Let A an B be positive invertible oprators. Then the following* (i) *and* (ii) *are equivalent.*

(i) *$B \gg A$ (i.e., $\log B \geqslant \log A$.)*

(ii) *For any fixed $s \geqslant 0$,*
$F(t,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^tA^{\frac{r}{2}})^{\frac{s+r}{t+r}}A^{\frac{-r}{2}}$ is an increasing function of both $t \geqslant s$ and $r \geqslant 0$.

Proof of Theorem 1. First of all, since x^α is operator concave function for $\alpha \in [0,1]$, we have

$$S^\alpha \nabla_\lambda T^\alpha \leqslant (S \nabla_\lambda T)^\alpha \quad \text{for} \quad \alpha \in [0,1] \quad \text{and} \quad S,T \geqslant 0. \tag{2}$$

(I). Applying Theorem A to $A \ll (B^t \nabla_\lambda C^t)^{\frac{1}{t}}$, we obtain $F(t_1,r) \leqslant F(t,r)$ for $t \geqslant t_1 \geqslant s \geqslant 0$ and $r \geqslant 0$, that is,

$$\{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)^{\frac{t_1}{t}}A^{\frac{r}{2}}\}^{\frac{s+r}{t_1+r}} \leqslant \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)^{\frac{t}{t}}A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}. \tag{3}$$

On the other hand, by using (2) we have the following (4) since $\frac{t_1}{t} \in [0,1]$

$$(B^{t_1} \nabla_\lambda C^{t_1}) \leqslant (B^t \nabla_\lambda C^t)^{\frac{t_1}{t}}. \tag{4}$$

Multiplying both sides of (4) by $A^{\frac{r}{2}}$, we have the following (5) by (3) and (4) by Löwner-Heinz inequality since $\frac{s+r}{t_1+r} \in [0,1]$,

$$\begin{aligned} f(t_1) =& \{A^{\frac{r}{2}}(B^{t_1} \nabla_\lambda C^{t_1})A^{\frac{r}{2}}\}^{\frac{s+r}{t_1+r}} \\ \leqslant& \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} = f(t). \end{aligned} \tag{5}$$

for $t \geqslant t_1 \geqslant s \geqslant 0$ and $r \geqslant 0$, so the proof of (I) is complete.

(I) $\Longrightarrow$ (II). Replacing A by A^{-1}, B by B^{-1} and C by C^{-1} in (I) of Theorem 1, and we recall

$$A^{-t} \ll B^{-t}\nabla_\lambda C^{-t} \quad \text{is equivalent to} \quad A^t \gg (B^{-t}\nabla_\lambda C^{-t})^{-1} = B^t !_\lambda C^t.$$

Then by (I)

$$f(t) = \{A^{\frac{-r}{2}}(B^{-t}\nabla_\lambda C^{-t})A^{\frac{-r}{2}}\}^{\frac{s+r}{t+r}}$$

is an increasing function of t, so that taking inverse of the right hand side of $f(t)$,

$$\begin{aligned} g(t) &= \{A^{\frac{r}{2}}(B^{-t}\nabla_\lambda C^{-t})^{-1}A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \\ &= \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \end{aligned}$$

is a decreasing function of t, so the proof of (I) $\Longrightarrow$ (II) is complete. The reverse implication (II) $\Longrightarrow$ (I) is easily shown by the same way as (I) $\Longrightarrow$ (II).

Whene the proof of Theorem 1 is complete.

COROLLARY 2. *Under the same assumption of Theorem 1, the following* (I) *and* (II) *hold and follows from each other.*

(I) *If* $A^t \ll B^t\nabla_\lambda C^t$ *for all* $t \geqslant 0$, *then*

$$A^{\frac{r}{2}}(B^s\nabla_\lambda C^s)A^{\frac{r}{2}} \leqslant \{A^{\frac{r}{2}}(B^t\nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

(II) *If* $A^t \gg B^t !_\lambda C^t$ *for all* $t \geqslant 0$, *then*

$$A^{\frac{r}{2}}(B^s !_\lambda C^s)A^{\frac{r}{2}} \geqslant \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

Proof. We have only to put $t = s$ in Theorem 1.

LEMMA 1. *Let B and C be positive opoerators. Then we have the following* (i) *and* (ii).

(i) $$B !_\lambda C \leqslant (B^t\nabla_\lambda C^t)^{\frac{1}{t}} \quad \textit{for all } t \geqslant 1$$

(ii) $$(B !_\lambda C)^t \leqslant (B^t\nabla_\lambda C^t) \quad \textit{for all } t \in [0,1].$$

Proof. (i) In the case $t \geqslant 1$:

$$B !_\lambda C \leqslant B\nabla_\lambda C \leqslant (B^t\nabla_\lambda C^t)^{\frac{1}{t}}$$

since the first inequality always holds by (1) and the second one follows by (2).

(ii) In the case $t \in [0,1]$:

$$(B !_\lambda C)^t = (B^{-1}\nabla_\lambda C^{-1})^{-t} \leqslant (B^{-t}\nabla_\lambda C^{-t})^{-1} = B^t !_\lambda C^t \leqslant B^t\nabla_\lambda C^t$$

since the first inequality follows by (2) and by taking inverses of both sides, and the second one always holds by (1).

PROPOSITION 3. *Let A, B and C be positive invertible operators. If $A \ll B!_\lambda C$, then $A^t \ll B^t \nabla_\lambda C^t$ for all $t \geqslant 0$.*

Proof. Since the chaotic order "$\ll$" is weaker than the usual order "$\leqslant$", (i) and (ii) of lemma 1 imply the following (6):

$$(B!_\lambda C)^t \ll (B^t \nabla_\lambda C^t) \quad \text{for all} \quad t \geqslant 0 \tag{6}$$

and the hypothesis $A \ll B!_\lambda C$ yields the following (7):

$$A^t \ll (B!_\lambda C)^t \quad \text{holds for all} \quad t \geqslant 0, \tag{7}$$

so that we have $A^t \ll B^t \nabla_\lambda C^t$ for all $t \geqslant 0$ by (6) and (7).

COROLLARY 4. *Under the same assumption of Theorem 1, the following* (I) *and* (II) *hold and follows from each other.*

(I) *If $A \ll B!_\lambda C$, then*

$$f(t) = \{A^{\frac{r}{2}}(B^t \nabla_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

is an increasing function of t.

(II) *If $A \gg B\nabla_\lambda C$, then*

$$h(t) = \{A^{\frac{r}{2}}(B^t !_\lambda C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$$

is a decreasing function of t.

Proof.

(I). Proof follows by Theorem 1 and Proposition 3.

(II). Replacing A by A^{-1}, B by B^{-1} and C by C^{-1} in (I), we rcall that

$$A^{-1} \ll B^{-1}!_\lambda C^{-1} \quad \text{is equivalent to} \quad A \gg (B^{-1}!_\lambda C^{-1})^{-1} = B\nabla_\lambda C$$

(I) ensures that $h(t)$ is a decreasing function of t by the same way as one in the proof (I) $\longrightarrow$ (II) in Theorem 1.

REMARK 1. We remark that (I) of Corollary 4 implies Theorem U since $A \leqslant B!_\lambda C$ yields $A \ll B!_\lambda C$ and $f(s) \leqslant f(t)$ for $t \geqslant s \geqslant 0$, and also (II) of Corollary 4 yields [12, Proposition 2.5] by the same way.

COROLLARY 5. *Let B and C be positive invertible operators and also let r, s and t be non-negative real numbers such that $t \geqslant s$ and $(r,t) \neq (0,0)$. Then the following* (I) *and* (II) *hold and follows from each other.*

(I) $f(t) = \{(B!_\lambda C)^{\frac{r}{2}}(B^t \nabla_\lambda C^t)(B!_\lambda C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$ *is an increasing function of t.*

(II) $h(t) = \{(B\nabla_\lambda C)^{\frac{r}{2}}(B^t !_\lambda C^t)(B\nabla_\lambda C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$ *is a decreasing function of t.*

Proof. We have only to put $A = B!_\lambda C$ and $A = B\nabla_\lambda C$ respectively in (I) and (II) of Corollary 4.

COROLLARY 6. *Let B and C be positive invertible operators and also let r,s and t be non-negative real numbers such that $t \geqslant s$ and $(r,t) \neq (0,0)$. Then the following* (I) *and* (II) *hold and follows from each other.*

(I) $(B!_\lambda C)^{\frac{r}{2}}(B^s\nabla_\lambda C^s)(B!_\lambda C)^{\frac{r}{2}} \leqslant \{(B!_\lambda C)^{\frac{r}{2}}(B^t\nabla_\lambda C^t)(B!_\lambda C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}$.

(II) $\{(B\nabla_\lambda C)^{\frac{r}{2}}(B^t!_\lambda C^t)(B\nabla_\lambda C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \leqslant (B\nabla_\lambda C)^{\frac{r}{2}}(B^s!_\lambda C^s)(B\nabla_\lambda C)^{\frac{r}{2}}$.

Proof. We have only to put $t=s$ in Corollary 5.

REMARK 2. Corollary 6 is shown in [12, Corollary 2.7].

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