Mathematical
Inequalities
& Applications
Volume 5, Number 2 (2002), 225–233

A NOTE ON TWO THEOREMS OF LEINDLER

JÓZSEF NÉMETH

(*communicated by L. Leindler*)

Abstract. The theorem proved here is the generalization of two theorems of Leindler regarding embedding relations among classes of Fourier coefficients. The generalization is based on replacing the power functions by more general ones introduced by Mateljevič and Pavlovič.

1. Introduction

Several authors have studied the problems of L^1 convergence of Fourier series. See for example: [1]-[4], [9]-[16]. In connection with this topic many classes of coefficients have been defined. Some of them are listed as follows:

1. A null-sequence $\mathbf{a} := \{a_n\}$ belongs to S if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leqslant A_n$ for all n. (Telyakovskiĭ [13]).

2. A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class F_p if for some $p > 1$

$$\sum_{n=1}^{\infty} n^{-1/p} \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} < \infty. \tag{1.1}$$

(Fomin [2]).

3. A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class S_p if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and

$$\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1). \tag{1.2}$$

(Č. V. Stanojević and V. B. Stanojević [12]).

4. A null sequence $\mathbf{a} := \{a_n\}$ belongs to the class F_p^* if for some $p > 1$

$$\sum_{m=1}^{\infty} 2^{m\left(1-\frac{1}{p}\right)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} < \infty. \tag{1.3}$$

Mathematics subject classification (2000): 26D15, 42A05, 42A10.

Key words and phrases: Inequalities, embedding relations, Fourier coefficients.

This research was supported by the Hungarian National Foundation for Scientific Research under Grant No. T029094.

(Fomin [2], Leindler [6]).

5. By δ-quasi-monotone sequence we mean a null-sequence $\mathbf{a} := \{a_n\}$ such that $a_n > 0$ and $\Delta a_n \geqslant -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers.
A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class $S_p(\delta)$ if there exists a δ-quasi-monotone sequence $\{A_n\}$ satisfying $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} n\,\delta_n < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1). \tag{1.4}$$

(Tomovski [14]).

6. A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class $S_p(A)$ if there exists a null-sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p} = O(1). \tag{1.5}$$

(Leindler [6]).

Many authors have investigated the embedding relations among the classes above mentioned. See for example [2], [5], [6], [11], [15], [16]. The strongest and in certain sense the closing results in this topic are due to L. Leindler [5], [6]. Namely in [5] he proved that if $p > 1$ then

$$F_p \subseteq S_p \subseteq F_p^* \subseteq F_p, \tag{1.6}$$

furthermore he showed in [6] that if $p > 1$ then

$$S_p \subseteq S_p(\delta) \subseteq S_p(A) \subseteq F_p^*. \tag{1.7}$$

Collecting the results (1.6) and (1.7) it can be obtained the following surprising statement: if $p > 1$ then

$$F_p \equiv S_p \equiv F_p^* \equiv S_p(\delta) \equiv S_p(A).$$

In the theory of functions, in particular in problems connecting with power series and embedding theorems several authors have used functions more general than the power functions. First of all the works of H. P. Mulholland [8] and M. Mateljevič and M. Pavlovič [7] should be mentioned in this respect. The following definition is due to Mateljevič and Pavlovič.

$\Delta(q,p)$ $(q \geqslant p > 0)$ denotes the family of the nonnegative real functions $\varphi(x)$ defined on $[0,\infty)$ with the following properties: $\varphi(0) = 0$, and there exist $q \geqslant p > 0$ such that $\frac{\varphi(t)}{t^q}$ is nonincreasing and $\frac{\varphi(t)}{t^p}$ is nondecreasing on $(0,\infty)$.

Δ will denote the set of the functions $\varphi(x)$ belonging to $\Delta(q,p)$ for some $q \geqslant p > 0$.

We need some subclasses of Δ.

$\Delta^{(1)}$ and $\Delta^{(2)}$ denote the families of functions $\varphi(x)$ belonging to $\Delta(q,p)$ for some $q \geqslant p \geqslant 1$ and $q \geqslant p > 1$, respectively.

The aim of the present paper is to generalize the classes of sequences mentioned above and to prove embedding theorems similar to those of Leindler [5], [6] using $\varphi \in \Delta$ instead of x^p.

Since the relation (1.6) is not true for $p = 1$ (see for example $a_n = \frac{1}{\log n}$) the following question occurs: does the embedding relation (1.6) hold for functions like $x\log^{\alpha}(1+x)$ $(\alpha > 0)$ instead of x^p $(p > 1)$? In addition to the generalization of (1.6) we want to answer this question, too. Therefore we need one more new function-class, wider than $\Delta^{(2)}$ and narrower than $\Delta^{(1)}$ which contains the functions like $x\ \log^{\alpha}(1+x)$, too. The definition reads as follows:

$\Delta^{(3)}$ is the collection of functions $\varphi(x)$ belonging to $\Delta(q,1)$ for some $q > 1$ such that for all $0 < A$ there exists $1 < p(A) = p$ satisfying that $\frac{\varphi(x)}{x^p}$ is nondecreasing on $(0,A)$.

From the above definitions directly follows that

$$\Delta^{(2)} \subset \Delta^{(3)} \subset \Delta^{(1)} \subset \Delta. \tag{1.8}$$

Before formulating our result we give the definitions of the new modified classes of sequences building on the functions $\varphi \in \Delta$.

F_{φ} can be got by changing (1.1) to

$$\sum_{n=1}^{\infty} \overline{\varphi}\left(\frac{\sum_{k=n}^{\infty} \varphi(|\Delta a_k|)}{n}\right) < \infty, \tag{1.9}$$

where $\overline{\varphi}$ denotes the inverse of φ.

S_{φ} can be got from S_p if we only change (1.2) to

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(1). \tag{1.10}$$

F_{φ}^{*} is defined by replacing (1.3) by

$$\sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum_{n=2^m+1}^{2^{m+1}} \varphi(|\Delta a_n|)}{2^m}\right) < \infty. \tag{1.11}$$

$S_{\varphi}(\delta)$ can be defined just by changing (1.4) to

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(1). \tag{1.12}$$

$S_{\varphi}(A)$ is defined merely by replacing (1.5) by

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(1). \tag{1.13}$$

After giving the definitions of the above classes we can formulate our result which is the generalization of Leindler's theorems proved in [5] and [6] furthermore it contains the case $\varphi(x) = x\ \log^{\alpha}(1+x)$ $(\alpha > 0)$, too.

2. Result

THEOREM. *If* $\varphi \in \Delta^{(3)}$ *then the following embedding relations*

$$F_\varphi \subseteq S_\varphi \subseteq S_\varphi(\delta) \subseteq S_\varphi(A) \subseteq F_\varphi^* \subseteq F_\varphi$$

hold.

COROLLARY. *If* $\varphi \in \Delta^{(3)}$ *then the following identity holds:*

$$F_\varphi \equiv S_\varphi \equiv S_\varphi(\delta) \equiv S_\varphi(A) \equiv F_\varphi^*.$$

REMARK. It can be proved that there exists a function $\tilde{\varphi}(x) \notin \Delta^{(3)}$ such that $\tilde{\varphi}(x) \in \Delta^{(1)}$, $\lim_{x\to 0^+} \frac{\tilde{\varphi}(x)}{x} = 0$, for all $p > 1$: $\lim_{x\to 0^+} \frac{x^p}{\tilde{\varphi}(x)} = 0$ and $F_{\tilde{\varphi}}^* \not\subseteq F_{\tilde{\varphi}}$. See for example

$$\tilde{\varphi}(x) = \begin{cases} \frac{x}{|\ln x|}, & \text{if} \quad 0 < x < \frac{1}{e}, \\ ex^2, & \text{if} \quad \frac{1}{e} \leqslant x, \end{cases}$$

and $\mathbf{a} := \frac{1}{\log n}$ serves the sequence for which $\mathbf{a} \in F_{\tilde{\varphi}}^*$ and $\mathbf{a} \notin F_{\tilde{\varphi}}$.

3. Lemmas

LEMMA 1. *If* $\varphi \in \Delta(q,p)$ $(q \geqslant p > 0)$ *and* $0 \leqslant \Theta \leqslant 1,\ 1 \leqslant \eta$ *then*

$$\Theta^q \varphi(t) \leqslant \varphi(\Theta t) \leqslant \Theta^p \varphi(t) \tag{3.1}$$

and

$$\eta^p \varphi(t) \leqslant \varphi(\eta t) \leqslant \eta^q \varphi(t) \quad \textit{for} \quad t \geqslant 0. \tag{3.2}$$

If $\varphi \in \Delta^{(3)}$ *then the right side of (3.1) holds for all intervall* $(0,A)$ *where* $\frac{\varphi(t)}{t^p}$ *is nondecreasing.*

The result (3.1) is a part of Lemma 1 in [7] and (3.2) is an obvious consequence of (3.1). The last statement follows immediately from the definition of $\Delta^{(3)}$.

LEMMA 2. *If* $\varphi \in \Delta^{(1)}$ *then*

$$\overline{\varphi}\left(\sum_{i=1}^{\infty} a_i\right) \leqslant \sum_{i=1}^{\infty} \overline{\varphi}(a_i)$$

where $\overline{\varphi}$ *is the inverse of* φ *and* $a_i \geqslant 0$ *for all* i.

Proof. Since according to the definition and Lemma 1 the function φ is strictly monotonic and continuous, $\overline{\varphi}$ exists.

It is enough to prove that for all positive a and b

$$\overline{\varphi}(a+b) \leqslant \overline{\varphi}(a) + \overline{\varphi}(b) \text{ holds.} \tag{3.3}$$

Taking $x = \overline{\varphi}(a)$ and $y = \overline{\varphi}(b)$ (3.3) is equivalent to

$$\varphi(x) + \varphi(y) \leqslant \varphi(x+y). \tag{3.4}$$

Let us suppose that $y \leqslant x$. Since $\frac{\varphi(t)}{t}$ is nondecreasing we have

$$\varphi(x+y) = \frac{\varphi(x+y)}{x+y}(x+y) \geqslant \frac{\varphi(x)}{x}(x+y) = \varphi(x) + \varphi(x)\frac{y}{x} \geqslant \varphi(x) + \varphi(y)$$

which gives (3.4) and this completes the proof of Lemma 2.

LEMMA 3. *If $\varphi \in \Delta^{(3)}$ and $\{b_i\}$ is a bounded sequence of positive numbers then for any i*

$$\sum_{m=0}^{i} 2^m \overline{\varphi}\left(\frac{b_i}{2^m}\right) \leqslant K \cdot 2^i \overline{\varphi}\left(\frac{b_i}{2^i}\right), \tag{3.5}$$

where K is independent of $\{b_i\}$ and i.

[In (3.5) and later in the sequel K denotes a positive constant, not necessarily the same on any two occurrences].

Proof. Let $A > 0$ such that $\varphi(\frac{A}{2}) > B$ where $b_i \leqslant B$ for all i, and $p > 1$ satisfying that $\frac{\varphi(t)}{t^p}$ is nondecreasing on $(0, A)$. First we show that for all $x \in \left(0; \frac{A}{2}\right)$

$$2\varphi(x) \leqslant \varphi(2^{1/p}x) \tag{3.6}$$

holds. Indeed, using Lemma 1 for $\Theta = 2^{-1/p}$

$$\varphi(x) = \varphi\left(\frac{1}{2^{1/p}} \cdot 2^{1/p}x\right) \leqslant \left(2^{-\frac{1}{p}}\right)^p \varphi(2^{1/p}x) = \frac{1}{2}\varphi(2^{1/p}x)$$

which gives (3.6).

From (3.6) we get

$$2^{1/p}x \geqslant \overline{\varphi}(2\varphi(x)) \tag{3.7}$$

and taking $t = \varphi(x)$ we have

$$2^{1/p}\overline{\varphi}(t) \geqslant \overline{\varphi}(2t) \tag{3.8}$$

if $t \in (0, \varphi(\frac{A}{2}))$.

Using (3.8) we obtain that for arbitrary i

$$\frac{2^{m+1}\overline{\varphi}\left(\frac{b_i}{2^{m+1}}\right)}{2^m\overline{\varphi}\left(\frac{b_i}{2^m}\right)} \geqslant 2^{1-\frac{1}{p}} > 1, \tag{3.9}$$

which immediately gives (3.5).

LEMMA 4. *Let $\{c_n\}$ be a δ-quasi-monotone sequence with*

$$\sum_{n=1}^{\infty} n\,\delta_n < \infty.$$

If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty}(n+1)|\Delta c_n| < \infty$.

This result can be found in [10].

LEMMA 5. *If R_n is a nonnegative monotonically decreasing sequence such that*

$$\sum_{n=1}^{\infty} R_n < \infty \tag{3.10}$$

then there exists a monotone decreasing sequence $\{A_n\}$ such that for any $n \geqslant 1$

$$R_n \leqslant A_n, \tag{3.11}$$

$$A_n \leqslant K A_{2n}, \tag{3.12}$$

$$\sum_{k=1}^{\infty} A_k < \infty. \tag{3.13}$$

This statement can be found in the proof of Theorem of L. Leindler in [5].

4. Proofs

The kernel of the proofs in our statements is the same as in those of Leindler, we had to modify the method only in the points which need changes because of replacing the functions x^p by $\varphi(x)$.

First we prove the embedding relation

$$F_\varphi \subseteq S_\varphi.$$

If $\mathbf{a} \in F_\varphi$ then setting

$$R_n := \overline{\varphi}\left(\frac{\sum_{k=n}^{\infty} \varphi|\Delta a_k|}{n}\right),$$

by (1.9) we get

$$\sum_{n=1}^{\infty} R_n < \infty \tag{4.1}$$

and since $R_n \downarrow$ therefore the sequence $\{R_n\}$ satisfies the conditions Lemma 5, and so there exists $\{A_n\}$ with (3.11)–(3.13).

Using these inequalities we get that if $2^i \leqslant n < 2^{i+1}$

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} &= \sum_{k=1}^{n} \frac{k\varphi(R_k) - (k+1)\varphi(R_{k+1})}{\varphi(A_k)} \\
&\leqslant \sum_{m=0}^{i} \sum_{k=2^m}^{2^{m+1}-1} [k\varphi(R_k) - (k+1)R_{k+1}] \frac{1}{\varphi(A_k)} \\
&\leqslant \sum_{m=0}^{i} 2^m \varphi(R_{2^m}) \cdot \frac{1}{\varphi(A_{2^{m+1}})} = I.
\end{aligned}$$

Using (3.11), (3.12) and Lemma 1 and (1.8) we have that

$$I \leqslant K \cdot \sum_{m=0}^{i} 2^m \varphi(A_{2^m}) \cdot \frac{1}{\varphi(A_{2^m})} \leqslant Kn$$

and this proves that $\mathbf{a} \in S_\varphi$ holds, that is $F_\varphi \subseteq S_\varphi$.

Next we prove the relation $S_\varphi \subseteq S_\varphi(\delta)$. This relation is obviously follows from the definition taking $\delta_k = k^{-3}$.

Now we show the embedding $S_\varphi(\delta) \subseteq S_\varphi(A)$. Since $\mathbf{a} \in S_\varphi(\delta)$ there exists a δ-quasi-monotone sequence $\{A_n\}$ with $\sum_{n=1}^{\infty} n\,\delta_n < \infty$. Applying Lemma 4 we get that $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$. At the same time the estimation (1.13) is automatically satisfied by $\mathbf{a} \in S_\varphi(\delta)$ because of (1.10). Thus $S_\varphi(\delta) \subseteq S_\varphi(A)$ is proved.

Next we prove the embedding relation

$$S_\varphi(A) \subseteq F_\varphi^*.$$

The first part of the proof is the same as in [6].

Setting

$$D_m := \sum_{n=2^m}^{2^{m+1}} |\Delta A_n|.$$

By $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$ we obtain that

$$\sum_{m=0}^{\infty} 2^m D_m < \infty. \tag{4.2}$$

Since $A_n \to 0$ thus

$$A_{2^m} = \sum_{n=2^m}^{\infty} \Delta A_n \leqslant \sum_{n=m}^{\infty} D_n.$$

Utilizing the last inequality and (4.2) we get that

$$\sum_{m=1}^{\infty} 2^m A_{2^m} \leqslant \sum_{m=1}^{\infty} 2^m \sum_{n=m}^{\infty} D_n = \sum_{n=1}^{\infty} D_n \sum_{m=1}^{n} 2^m \leqslant 2 \sum_{n=1}^{\infty} 2^n D_n < \infty. \tag{4.3}$$

Now we define one more sequence $\{C_m\}$ as follows:

$$C_m := A_{2^m} + D_m \quad \text{for all} \quad m \geqslant 1.$$

If $2^m < k \leqslant 2^{m+1}$ then

$$A_k = A_{2^m} - \sum_{n=2^m}^{k} \Delta A_n \leqslant A_{2^m} + \sum_{n=2^m}^{k-1} |\Delta A_n| \leqslant A_{2^m} + D_m = C_m.$$

Using this estimation we have the following inequality

$$\begin{aligned}\sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum_{n=2^m+1}^{2^{m+1}} \varphi(|\Delta a_n|)}{2^m}\right) &= \sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum_{n=2^m+1}^{2^{m+1}} \frac{\varphi(|\Delta a_n|)}{\varphi(A_n)} \varphi(A_n)}{2^m}\right) \\ &\leqslant \sum_{m=1}^{\infty} 2^m \overline{\varphi}\left[\frac{\sum_{n=2^m+1}^{2^{m+1}} \frac{\varphi(|\Delta a_n|)}{\varphi(A_n)}}{2^m} \varphi(C_m)\right] = I.\end{aligned} \tag{4.4}$$

Since $\mathbf{a} \in S_\varphi(A)$ we get by (1.13) that the sum in the bracket is $\varphi(C_m) \cdot O(1)$, that is

$$I \leqslant \sum_{m=1}^{\infty} 2^m \overline{\varphi}(K\varphi(C_m)) = II. \tag{4.5}$$

By Lemma 1, using (1.8) we have that

$$II \leqslant K \cdot \sum_{m=1}^{\infty} 2^m C_m. \tag{4.6}$$

Taking into account (4.2) and (4.3) we get that the right hand side of (4.6) is finite which by (4.4)–(4.6) proves that $\mathbf{a} \in F_\varphi^*$.

Herewith the embedding relation

$$S_\varphi(A) \subseteq F_\varphi^*$$

is also proved.

Finally we show the embedding statement $F_\varphi^* \subseteq F_\varphi$

$$\begin{aligned}\sum_{n=2}^{\infty} \overline{\varphi}\left(\frac{\sum_{k=n}^{\infty} \varphi(|\Delta a_k|)}{n}\right) &\leqslant \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \overline{\varphi}\left(\frac{\sum_{k=n}^{\infty} \varphi(|\Delta a_k|)}{2^m}\right) \\ &\leqslant \sum_{m=0}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum_{k=2^m+1}^{\infty} \varphi(|\Delta a_k|)}{2^m}\right) = I^*.\end{aligned} \tag{4.7}$$

By using Lemma 2, and (1.8) we have that

$$I^* \leqslant \sum_{m=0}^{\infty} 2^m \sum_{i=m}^{\infty} \overline{\varphi}\left[\sum_{k=2^i+1}^{2^{i+1}} \frac{\varphi(|\Delta a_k|)}{2^m}\right] = I^{**}. \tag{4.8}$$

By changing the order of summation we get from (4.8) that

$$I^{**} = \sum_{i=0}^{\infty} \sum_{m=0}^{i} 2^m \overline{\varphi}\left[\frac{1}{2^m} \sum_{k=2^i+1}^{2^{i+1}} \varphi(|\Delta a_k|)\right]. \tag{4.9}$$

Since the boundedness of the sequence $\gamma_i := \sum_{k=2^i+1}^{2^{i+1}} \varphi(|\Delta a_k|)$ follows from (1.11), we can use Lemma 3 with $\{\gamma_i\}$ in place of $\{b_i\}$, so we obtain that

$$I^{**} \leqslant K \sum_{i=0}^{\infty} 2^i \overline{\varphi} \left[\frac{\sum_{k=2^i+1}^{2^{i+1}} \varphi(|\Delta a_k|)}{2^i} \right]. \tag{4.10}$$

If $\mathbf{a} \in F_\varphi^*$ then the sum in (4.10) is finite thus from (4.7)–(4.10) it follows that the first sum in (4.7) is also finite, that is $\mathbf{a} \in F_\varphi$.

Thus the relation

$$F_\varphi^* \subseteq F_\varphi$$

is proved.

The proof of Theorem is complete.

REFERENCES

[1] R. P. BOAS JR., *Integrability theorems for trigonometric transforms*, Springer–Verlag, Ergebnisse 38, Berlin, (1967).

[2] G. A. FOMIN, *A class of trigonometric series*, Mat. Zametki **23** (1978), 213–222.

[3] J. W. GARRETT AND Č. V. STANOJEVIĆ, *Necessary and sufficient conditions for L^1 convergence of trigonometric series*, Proc. Amer. Math. Soc. **60** (1976), 68–71.

[4] A. N. KOLMOGOROV, *Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue*, Bull. Acad. Polon. Sci. (A), Sci. Math. (1923), 83–86.

[5] L. LEINDLER, *On the equivalence of classes of Fourier coefficients*, Math. Inequal. Appl. **3** (2000), 45–50.

[6] L. LEINDLER, *A note on some classes of real sequences*, Math. Inequal. Appl. **4** (2001), 53–58.

[7] M. MATELJEVIČ AND M. PAVLOVIČ, *L^p-behavior of power series with positive coefficients and Hardy spaces*, Proc. Amer. Math. Soc. **87** (1983), 309–316.

[8] H. P. MULHOLLAND, *Concerning the generalization of the Young-Hausdorff theorem and the Hardy-Littlewood theorems on Fourier constants*, Proc. London Math. Soc. **35** (1933), 257–293.

[9] S. SIDON, *Hinreichende Bedingungen für den Fourier–character einer trigonometrischen Reihe*, J. London Math. Soc. **14** (1939), 158–160.

[10] N. SINGH AND K. M. SHARMA, *Integrability of trigonometric series*, J. Indian Math. Soc. **49** (1985), 31–38.

[11] Č. V. STANOJEVIĆ, *Classes of L^1-convergence of Fourier-Stieltjes series*, Proc. Amer. Math. Soc. **82** (1981), 209–215.

[12] Č. V. STANOJEVIĆ AND V. B. STANOJEVIĆ, *Generalizations of the Sidon-Telyakovskiǐ theorem*, Proc. Amer. Math. Soc. **101** (1987), 679–684.

[13] S. A. TELYAKOVSKIǏ, *On a sufficient condition of Sidon for integrability of trigonometric series*, Mat. Zametki (Russian) **14** (1973), 317–328.

[14] Ž. TOMOVSKI, *An application of the Hausdorff-Young inequality*, Math. Inequal. Appl. **1** (1998), 527–532.

[15] Ž. TOMOVSKI, *A note on some classes of Fourier coefficients*, Math. Inequal. Appl. **2** (1999), 15–18.

[16] S. Z. A. ZENEI, *Integrability of trigonometric series*, Tamkang J. Math. **21** (1990), 295–301.

(Received July 31, 2001)

Bolyai Institute
University of Szeged
Aradi vértanúk tere 1.
H-6720 Hungary
e-mail: nemethj@math.u-szeged.hu