Mathematical
Inequalities
& Applications
Volume 5, Number 2 (2002), 235–246

AVERAGING OPERATORS ON $l^{\{p_n\}}$ AND $L^{p(x)}$

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(*communicated by L. Pick*)

Abstract. We consider the generalized Lebesgue space $L^{p(x)}$ and its discrete analogue $l^{\{p_n\}}$, each given the appropriate Luxemburg norm. Let T_k be the averaging operator given by

$$(T_k a)_n = \frac{1}{k}(a_n + a_{n+1} + \cdots + a_{n+k-1}), a = \{a_n\} \in l^{\{p_n\}}.$$

We show that the T_k are uniformly bounded from $l^{\{p_n\}}$ into $l^{\{p_n\}}$ under certain assumptions on p_n and find a counter–example to show that T_k need not be bounded if these assumptions are not satisfied.

Moreover, we construct a bounded Lipschitz function $p(x)$ on $[0,\infty)$ such that the operator T_s given, for each

$$T_s f(x) = \frac{1}{s}\int_0^s f(t)\mathrm{d}t$$

is unbounded on $L^{p(x)}$ for all $s > 0$.

1. Introduction

The generalized Lebesgue space $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have attracted more and more interest in recent years. We refer to [5] for the establishment of the fundamental properties of these spaces, to [2] for some properties of the norm on $L^{p(x)}$, and to [4] for inequalities of Sobolev type. Further motivation for the study of these spaces is provided in [6,7] by means of mathematical models of electrorheological fluids which involve nonlinear systems of partial differential equations with coefficients of variable rate of growth.

A crucial difference between $L^{p(x)}$ and the classical Lebesgue spaces is that $L^{p(x)}$ is not, in general, invariant under translation (see [5], Ex. 2.9). Because of this, serious problems arise with regard to convolutions, the density of smooth functions in $W^{1,p(x)}$ (see [3] and [8]) and the boundedness of the Hardy–Littlewood maximal operator. With this last difficulty in mind we consider in this paper the averaging operator T_s given, for each $s > 0$, by

$$T_s f(x) = \frac{1}{s}\int_x^{x+s} f(t)\mathrm{d}t.$$

Mathematics subject classification (2000): 46E30,26D15.

Key words and phrases: Generalized Lebesque space, averaging operator.

Both authors were supported by a Royal Society Joint Project Grant.

Paper MIA-05-26

It is easy to see that for all $s > 0$ and all $q \in [1,\infty]$, T_s is bounded from the classical Lebesgue space $L^q(\mathbb{R})$ to $L^q(\mathbb{R})$, with norm independent of s. Here we show that in $L^{p(x)}$ the situation is much worse, for we construct a bounded Lipschitz function $p(x)$ such that for all $s > 0$, T_s is unbounded from $L^{p(x)}(0,\infty)$ to itself. We also consider a discrete analogue l^{p_n} of $L^{p(x)}$ and discrete versions of T_k of the form

$$T_k a = \{(T_k a)_n\}_{n\in\mathbb{Z}}, (T_k a)_n = \frac{1}{k}(a_n + a_{n+1} + \cdots + a_{n+k-1}), a = \{a_n\} \in l^{\{p_n\}},$$

where $k \in \mathbb{N}$, $a = \{a_n\}_{n\in\mathbb{Z}} \in l^{\{p_n\}}$. For a certain class of non-constant sequences $\{p_n\}$ we show that the T_k are bounded (uniformly with respect to k) from $l^{\{p_n\}}$ to itself. A counter–example is given to remove any faint hope that T_k might be bounded for all sequences $\{p_n\}$.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers and let $\{a_n\}_{n\in\mathbb{Z}}$ (or simply $\{a_n\}$) denote a sequence of real numbers defined on $\mathbb{Z}$. Let Ω be a subset of $\mathbb{R}$ and μ be a measure on Ω; $\mathscr{M}(\Omega,\mu)$ will denote the set of all μ-measurable functions defined on Ω. When $\Omega = (0,\infty)$ and μ is the Lebesgue measure on Ω we write $\mathscr{M}(\Omega,\mu) = \mathscr{M}(0,\infty)$. We fix through the paper a sequence $\{p_k\}$, $1 \leqslant p_k$ for any $k \in \mathbb{Z}$ and a μ-measurable function $p(x)$, $1 \leqslant p(x) < \infty$. We recall the definition of a Banach function space.

DEFINITION 2.1. *A linear space X, $X \subset \mathscr{M}(\Omega,\mu)$, is called a Banach function space if there exists a functional $\|.\|_X : \mathscr{M}(\Omega,\mu) \to [0,\infty]$ with the norm property satisfying:*

$$f \in X \text{ if and only if } \|f\|_X < \infty; \tag{i}$$

$$\|f\|_X = \|\,|f|\,\|_X \text{ for all } f \in M(\Omega,\mu); \tag{ii}$$

$$\text{if } 0 \leqslant f_n \nearrow f \text{ then } \|f_n\|_X \nearrow \|f\|_X; \tag{iii}$$

$$\text{if } E \subset \Omega, \mu(E) < \infty, \text{ then } \|\chi_E\|_X < \infty; \tag{iv}$$

$$\text{for any } E \subset \Omega \text{ with } \mu(E) < \infty \tag{v}$$

there is a positive constant $c(E)$ such that

$$\int_E |f(x)|\mathrm{d}x \leqslant c(E)\|f\|_X \text{ for all } f \in X.$$

DEFINITION 2.2. *Let T be a linear mapping from $\mathscr{M}(\Omega,\mu)$ into itself. We say that T is a positive operator if $Tf \geqslant 0$ μ-almost everywhere for any $f \geqslant 0$ μ-almost everywhere.*

LEMMA 2.3. *Let X, Y be Banach function spaces and $T : X \to Y$ be a positive operator which is unbounded as an operator from X to Y. Then there is $f \in X$ such that $\|f\|_X \leqslant 1$ and $\|Tf\|_Y = \infty$.*

Proof. The proof is an easy modification of the proof of Theorem 1.8, Chap.1 in [1]. $\square$

DEFINITION 2.4. *Denote for* $a = \{a_n\}$, $a_n \in \mathbb{R}$, *and for* $f(x) \in \mathscr{M}(0,\infty)$ *the Luxemburg norms by*

$$\|a\| = \inf\{\lambda > 0; \sum_{n\in\mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \leqslant 1\}$$

and

$$\|f\| = \inf\{\lambda > 0; \int_0^\infty \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\}$$

Define spaces $l^{\{p_n\}}$ *and* $L^{p(x)}(0,\infty)$ *by*

$$l^{\{p_n\}} = \{a; \|a\| < \infty\}$$

and

$$L^{p(x)}(0,\infty) = \{f \in \mathscr{M}(0,\infty); \|f\| < \infty\}.$$

It is not difficult to prove the following lemma.

LEMMA 2.5. *Both* $l^{\{p_n\}}$ *and* $L^{p(x)}(0,\infty)$ *are Banach function spaces.*

Proof. Let us prove only the case $L^{p(x)}(0,\infty)$. The properties (i), (ii), (iv) are trivial.

Prove (iii). Let $0 \leqslant f_n \nearrow f$. Assume for simplicity $\lambda := \|f\| < \infty$. The case $\lambda := \|f\| = \infty$ would be proved analogously. Set $\lambda_n := \|f_n\|$. It is not difficult to verify that λ_n is nondecreasing and $\lambda_n \leqslant \lambda$. Our aim is to prove $\lambda_n \nearrow \lambda$. Assume the contrary. Then there is a $\delta > 0$ such that $\lambda_n \leqslant \lambda - 2\delta$. Since

$$\int_0^\infty \left|\frac{f_n(x)}{v}\right|^{p(x)} dx \nearrow \int_0^\infty \left|\frac{f(x)}{v}\right|^{p(x)} dx$$

for all $v > 0$ we obtain

$$\begin{aligned} 1 &\geqslant \int_0^\infty \left|\frac{f_n(x)}{\lambda_n}\right|^{p(x)} dx \geqslant \int_0^\infty \left|\frac{f_n(x)}{\lambda_n + \delta}\right|^{p(x)} dx \\ &\geqslant \int_0^\infty \left|\frac{f_n(x)}{\lambda - \delta}\right|^{p(x)} dx \nearrow \int_0^\infty \left|\frac{f(x)}{\lambda - \delta}\right|^{p(x)} dx > 1 \end{aligned}$$

which is a contradiction.

To prove (v) it suffices to write

$$\begin{aligned} \int_E \frac{|f(x)|}{\|f\|} dx &= \int_{\{x\in E; |f(x)| \leqslant \|f\|\}} \frac{|f(x)|}{\|f\|} dx + \int_{\{x\in E; |f(x)| > \|f\|\}} \frac{|f(x)|}{\|f\|} dx \\ &\leqslant |\{x \in E; |f(x)| \leqslant \|f\|\}| + \int_{\{x\in E; |f(x)| > \|f\|\}} \left|\frac{f(x)}{\|f\|}\right|^{p(x)} dx \leqslant |E| + 1 \end{aligned}$$

which implies $\int_E |f(x)| dx \leqslant (|E| + 1)\|f\|$. $\square$

LEMMA 2.6. *Let* $\sup_{n\in\mathbb{Z}} p_n < \infty$ *and* $\operatorname{ess\,sup}_{x\in(0,\infty)} p(x) < \infty$. *Then*

$$l^{\{p_n\}} = \{a; \sum_{n\in\mathbb{Z}} |a_n|^{p_n} < \infty\}$$

and

$$L^{p(x)}(0,\infty) = \{f \in \mathscr{M}(0,\infty); \int_0^\infty |f(x)|^{p(x)} \mathrm{d}x < \infty\}.$$

Proof. Let us prove this lemma only for the space $l^{\{p_n\}}$ as the proof for the space $L^{p(x)}(0,\infty)$ would be analogous. Denote $p^* = \sup_{n\in\mathbb{Z}} p_n$ and let $\|a\| < \infty$. Then there is $\lambda > 0$ such that $\sum_{n\in\mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \leqslant 1$. Thus we can write

$$\sum_{n\in\mathbb{Z}} \frac{|a_n|^{p_n}}{(\max(1,\lambda))^{p^*}} \leqslant \sum_{n\in\mathbb{Z}} \frac{|a_n|^{p_n}}{\max(1,\lambda)^{p_n}} \leqslant \sum_{n\in\mathbb{Z}} \frac{|a_n|^{p_n}}{\lambda^{p_n}} \leqslant 1$$

which gives

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \leqslant \big(\max(1,\lambda)\big)^{p^*}.$$

On the other hand, suppose $\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \leqslant \lambda < \infty$. Without loss of generality we can assume $\lambda \geqslant 1$. Then

$$\sum_{n\in\mathbb{Z}} \Big(\frac{|a_n|}{\lambda}\Big)^{p_n} \leqslant \sum_{n\in\mathbb{Z}} \frac{|a_n|^{p_n}}{\lambda} \leqslant 1$$

and, consequently, $\|a\| \leqslant \lambda$ which finishes the proof. □

The following lemma is an easy consequence of Lemma 2.3, Lemma 2.5 and Lemma 2.6.

LEMMA 2.7. *Let* $\operatorname{ess\,sup}_{x\in(0,\infty)} p(x) < \infty$*; let* $T : L^{p(x)}(0,\infty) \to \mathscr{M}(0,\infty)$ *be positive. Then the following statements are equivalent:*

$$T \text{ is unbounded as an operator from } L^{p(x)}(0,\infty) \text{ into itself;} \tag{i}$$

$$\text{there is } f \geqslant 0 \text{ with } \int_0^\infty f(x)^{p(x)} \mathrm{d}x \leqslant 1 \text{ and } \int_0^\infty (Tf(x))^{p(x)} \mathrm{d}x = \infty. \tag{ii}$$

LEMMA 2.8. *Let* $\sup_{n\in\mathbb{Z}} p_n < \infty$ *and* T *be a linear mapping which maps the set of all real–valued sequences into itself. Let* c *be a positive constant such that*

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} \leqslant 1 \Longrightarrow \sum_{n\in\mathbb{Z}} \left|\big(Ta\big)_n\right|^{p_n} \leqslant c.$$

Then

$$\|Ta\| \leqslant \max(1,c)\|a\|.$$

Proof. Assume $0 < \|a\| < \infty$ as the other cases are clear. Then

$$\sum_{n\in\mathbb{Z}} \Big(\frac{a_n}{\|a\|}\Big)^{p_n} \leqslant 1.$$

According to the assumptions we have

$$\sum_{n\in\mathbb{Z}}\left|\left(T\left(\frac{a}{\|a\|}\right)\right)_n\right|^{p_n}\leqslant c.$$

Then

$$\sum_{n\in\mathbb{Z}}\left|\left(T\left(\frac{a}{\max(1,c)\|a\|}\right)\right)_n\right|^{p_n}\leqslant\frac{1}{\max(1,c)}\sum_{n\in\mathbb{Z}}\left|\left(T\left(\frac{a}{\|a\|}\right)\right)_n\right|^{p_n}$$
$$\leqslant\frac{c}{\max(1,c)}\leqslant 1.$$

This gives $\|T(a)\|\leqslant\max(1,c)\|a\|$ and the lemma follows. □

In an analogous way we can prove the following lemma.

LEMMA 2.9. *Let* $\operatorname{ess\,sup}_{x\in(0,\infty)}p(x)<\infty$ *and let* T *be a linear mapping from* $\mathscr{M}(0,\infty)$ *into itself. Let there exist a positive constant* c *such that*

$$\int_0^\infty|f(x)|^{p(x)}\mathrm{d}x\leqslant 1\Longrightarrow\int_0^\infty|Tf(x)|^{p(x)}\mathrm{d}x\leqslant c.$$

Then T *is a bounded linear operator on* $L^{p(x)}(0,\infty)$.

3. Boundedness of averaging operators

We adopt the notation $x^+=\max(0,x)$ for any real x.

LEMMA 3.1. *Let* $0\leqslant b_n$, $\sum_{n\in\mathbb{Z}}b_n\leqslant 1$, *let* $\varepsilon_n<1$ *and suppose that* $\varepsilon=\sum_{n\in\mathbb{Z}}\varepsilon_n^+<\infty$. *Then*

$$\sum_{n\in\mathbb{Z}}b_n^{1-\varepsilon_n}\leqslant e^{1/e}(1+\varepsilon).$$

Proof. Set

$$\mathbb{Z}_1=\{n\in\mathbb{Z};\varepsilon_n\leqslant 0\}$$
$$\mathbb{Z}_2=\{n\in\mathbb{Z}\setminus\mathbb{Z}_1;b_n>\varepsilon_n\}$$
$$\mathbb{Z}_3=\{n\in\mathbb{Z}\setminus\mathbb{Z}_1;b_n\leqslant\varepsilon_n\}.$$

Since $\mathbb{Z}_1,\mathbb{Z}_2,\mathbb{Z}_3$ are pair-wise disjoint and $\mathbb{Z}_1\cup\mathbb{Z}_2\cup\mathbb{Z}_3=\mathbb{Z}$ we can write

$$\sum_{n\in\mathbb{Z}}b_n^{1-\varepsilon_n}=\sum_{n\in\mathbb{Z}_1}b_n^{1-\varepsilon_n}+\sum_{n\in\mathbb{Z}_2}b_n^{1-\varepsilon_n}+\sum_{n\in\mathbb{Z}_3}b_n^{1-\varepsilon_n}=I_1+I_2+I_3. \tag{3.1}$$

Note that, according to the assumptions, $b_n\leqslant 1$ for all $n\in\mathbb{Z}$.
Let $n\in\mathbb{Z}_1$. Then $1-\varepsilon_n\geqslant 1$ and $b_n^{1-\varepsilon_n}\leqslant b_n$. Thus

$$I_1\leqslant\sum_{n\in\mathbb{Z}_1}b_n. \tag{3.2}$$

Let $n \in \mathbb{Z}_2$. Then $b_n > \varepsilon_n$ and, consequently, ${b_n}^{-\varepsilon_n} < \varepsilon_n^{-\varepsilon_n}$. Since $\varepsilon_n > 0$, $\varepsilon_n^{-\varepsilon_n} \leqslant e^{\frac{1}{e}}$. Thus

$$I_2 \leqslant e^{\frac{1}{e}} \sum_{n \in \mathbb{Z}_2} b_n. \tag{3.3}$$

Let $n \in \mathbb{Z}_3$. Then $0 \leqslant b_n \leqslant \varepsilon_n < 1$, which gives

$$I_3 \leqslant \sum_{n \in \mathbb{Z}_3} \varepsilon_n^{1-\varepsilon_n} \leqslant e^{\frac{1}{e}} \sum_{n \in \mathbb{Z}_3} \varepsilon_n$$

and yields with (3.1), (3.2) and (3.3),

$$\sum_{n \in \mathbb{Z}} {b_n}^{1-\varepsilon_n} \leqslant \sum_{n \in \mathbb{Z}_1} b_n + e^{\frac{1}{e}} \sum_{n \in \mathbb{Z}_2} b_n + e^{\frac{1}{e}} \sum_{n \in \mathbb{Z}_3} \varepsilon_n \leqslant e^{\frac{1}{e}}(1+\varepsilon).$$

□

LEMMA 3.2. *Let* $s \in \mathbb{Z}$ *and* $\sum_{n \in \mathbb{Z}} |p_{n+s} - p_n| \leqslant A < \infty$. *Assume* $\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1$. *Then*

$$\sum_{n \in \mathbb{Z}} |a_{n+s}|^{p_n} \leqslant e^{\frac{1}{e}}(1+A).$$

Proof. Set $b_n = |a_{n+s}|^{p_{n+s}}$ and $\varepsilon_n = 1 - \frac{p_n}{p_{n+s}}$. Then

$$\sum_{n \in \mathbb{Z}} b_n \leqslant 1, \quad \varepsilon_n < 1, \quad \sum_{n \in \mathbb{Z}} \varepsilon_n^+ \leqslant \sum_{n \in \mathbb{Z}} |\varepsilon_n| \leqslant \sum_{n \in \mathbb{Z}} |p_{n+s} - p_n| \leqslant A.$$

Using Lemma 3.1 we have

$$\sum_{n \in \mathbb{Z}} |a_{n+s}|^{p_n} = \sum_{n \in \mathbb{Z}} b_n^{1-\varepsilon_n} \leqslant e^{\frac{1}{e}}(1+A),$$

which finishes the proof. □

DEFINITION 3.3. *Let* $k \in \mathbb{Z}$, $k > 0$. *Define a linear operator* T_k *on* $l^{\{p_n\}}$ *by*

$$(T_k a)_n = \frac{1}{k}(a_n + a_{n+1} + \cdots + a_{n+k-1}).$$

LEMMA 3.4. *Let* $\sup p_n < \infty$ *and* $k \in \mathbb{Z}$, $k > 0$. *Assume for* $1 \leqslant s \leqslant k-1$ *that* $\sum_{n \in \mathbb{Z}} |p_{n+s} - p_n| \leqslant A_s < \infty$. *Then* $T_k : l^{\{p_n\}} \to l^{\{p_n\}}$ *is bounded and*

$$\|T_k\| \leqslant e^{\frac{1}{e}} + \frac{e^{\frac{1}{e}}}{k} \sum_{s=1}^{k-1} A_s.$$

Proof. Assume

$$\sum_{n \in \mathbb{Z}} |a_n|^{p_n} \leqslant 1. \tag{3.4}$$

By the Jensen inequality we have

$$I = \sum_{n\in\mathbb{Z}} |(T_k a)_n|^{p_n} = \sum_{n\in\mathbb{Z}} \Big|\frac{1}{k}(a_n + a_{n+1} + \cdots + a_{n+k-1})\Big|^{p_n}$$

$$\leqslant \frac{1}{k}\sum_{n\in\mathbb{Z}}\sum_{s=0}^{k-1} |a_{n+s}|^{p_n} = \frac{1}{k}\sum_{s=0}^{k-1}\sum_{n\in\mathbb{Z}} |a_{n+s}|^{p_n} = \frac{1}{k}\sum_{n\in\mathbb{Z}} |a_n|^{p_n} + \frac{1}{k}\sum_{s=1}^{k-1}\sum_{n\in\mathbb{Z}} |a_{n+s}|^{p_n}.$$

By Lemma 3.2 and (3.4) we obtain

$$I \leqslant \frac{1}{k} + \frac{1}{k}\sum_{s=1}^{k-1} e^{\frac{1}{e}}(1 + A_s) \leqslant e^{\frac{1}{e}} + \frac{e^{\frac{1}{e}}}{k}\sum_{s=1}^{k-1} A_s.$$

Using Lemma 2.8 we have

$$\|T_k\| \leqslant \max\Big(1, e^{\frac{1}{e}} + \frac{e^{\frac{1}{e}}}{k}\sum_{s=1}^{k-1} A_s)\Big) = e^{\frac{1}{e}} + \frac{e^{\frac{1}{e}}}{k}\sum_{s=1}^{k-1} A_s$$

as required. □

DEFINITION 3.5. *We say that the sequence* $\{p_n\}_{n\in\mathbb{Z}}$ *satisfies the condition* $\mathscr{P}$ *(write* $\{p_n\} \in \mathscr{P}$ *) if there exists a number* $p_0 \geqslant 1$ *and a sequence* $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ *such that* $\varepsilon = \sum\limits_{n\in\mathbb{Z}} |\varepsilon_n| < \infty$ *and* $p_n = p_0 + \varepsilon_n$.

Note that $\{p_n\} \in \mathscr{P}$ gives $\sup_{n\in\mathbb{Z}} p_n < \infty$.

THEOREM 3.6. *Let* $\{p_n\} \in \mathscr{P}$ *and* ε *be from Definition 3.5. Then* $\|T_k\| \leqslant e^{\frac{1}{e}}(1 + 2\varepsilon)$ *for all* $k \in \mathbb{N}$.

Proof. Let ε_n be a corresponding sequence to $\{p_n\}$ from Definition 3.5 and $\varepsilon = \sum\limits_{n\in\mathbb{Z}} |\varepsilon_n|$. Set $A_s = \sum\limits_{n\in\mathbb{Z}} |p_{n+s} - p_n|$. Clearly, $A_s = \sum\limits_{n\in\mathbb{Z}} |\varepsilon_{n+s} - \varepsilon_n| \leqslant 2\sum_{n\in\mathbb{Z}} |\varepsilon_n| = 2\varepsilon$ for any s. Thus, according to Lemma 2.3 we obtain

$$\|T_k\| \leqslant e^{\frac{1}{e}} + \frac{e^{\frac{1}{e}}}{k}\sum_{s=1}^{k-1} A_s \leqslant e^{\frac{1}{e}}(1 + 2\varepsilon). \quad \square$$

4. Counter–examples

In this section we show that if $\{p_n\} \notin \mathscr{P}$ then the operator T_k need not be bounded. Moreover, we construct a bounded Lipschitz function $p(x)$ such that T_s is unbounded on $L^{p(x)}(0,\infty)$ for all $s > 0$.

EXAMPLE 4.1. *Let* $1 < p_0 < p_1 < \infty$. *Define a sequence* $\{p_n\}_{n\in\mathbb{Z}}$ *by*

$$p_n = \begin{cases} p_0 & \text{if } n \text{ is odd} \\ p_1 & \text{if } n \text{ is even.} \end{cases}$$

Then the operator T_2 *is not bounded.*

Proof. Take any sequence $\{b_n\}_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} |b_n|^{p_1} < \infty \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |b_n|^{p_0} = \infty.$$

Set

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ b_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\sum_{n\in\mathbb{Z}} |a_n|^{p_n} = \sum_{n\in\mathbb{Z}} |a_{2n}|^{p_1} = \sum_{n\in\mathbb{Z}} |b_n|^{p_1} < \infty.$$

On the other hand,

$$\sum_{n\in\mathbb{Z}} |(T_2 a)_n|^{p_n} = \sum_{n\in\mathbb{Z}} \left|\frac{a_n + a_{n+1}}{2}\right|^{p_n} \geqslant \sum_{n\in\mathbb{Z}} \left|\frac{a_{2n-1} + a_{2n}}{2}\right|^{p_{2n-1}}$$

$$\geqslant \sum_{n\in\mathbb{Z}} \left|\frac{b_n}{2}\right|^{p_0} = \frac{1}{2^{p_0}} \sum_{n\in\mathbb{Z}} |b_n|^{p_0} = \infty,$$

which finishes the proof. □

EXAMPLE 4.2. *There is a Lipschitz function* $p(x)$ *on* $(0,\infty)$*,with*

$$1 < \inf_{x\in(0,\infty)} p(x) < \sup_{x\in(0,\infty)} p(x) < \infty,$$

such that the operator

$$T_s f(x) = \frac{1}{s}\int_x^{x+s} f(t)\mathrm{d}t$$

is unbounded from $L^{p(x)}(0,\infty)$ *to itself for any* $s > 0$.

Proof. Let $I, J \subset [0,\infty)$. Setting $l(I) = \inf I, r(I) = \sup I$ we define I to be left of J if $r(I) \leqslant l(J)$.

Define for any $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$ an interval $I_{n,k,i}$ by

$$I_{n,k,i} = \left[\frac{n(n+1)}{2} - \frac{n}{2^{k-1}} + \frac{i-1}{2^k}, \frac{n(n+1)}{2} - \frac{n}{2^{k-1}} + \frac{i}{2^k}\right).$$

Evidently,

$$|I_{n,k,i}| = \frac{1}{2^k}. \tag{4.1}$$

We now prove that the system $\mathscr{S} = \{I_{n,k,i}; n \in \mathbb{N}, k \in \mathbb{N}, i \in \{1,2,\dots,n\}\}$ is a non-overlapping covering of $[0,\infty)$. Take $I_{m,k,i}$ and $I_{n,l,j}$. If $m<n$ then

$$r(I_{m,l,j}) = \frac{m(m+1)}{2} - \frac{m}{2^{l-1}} + \frac{j}{2^l} \leqslant \frac{m(m+1)}{2} - \frac{m}{2^{l-1}} + \frac{m}{2^l} = \frac{m(m+1)}{2} - \frac{m}{2^l}$$

$$< \frac{m(m+1)}{2} \leqslant \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - n \leqslant \frac{n(n+1)}{2} - \frac{n}{2^{k-1}} + \frac{i-1}{2^k} = l(I_{n,k,i})$$

and so, $I_{m,l,j}$ is left of $I_{n,k,i}$. Assume $m=n$, $l<k$. Then the inequality $l \leqslant k-1$ gives

$$r(I_{m,l,j}) = \frac{m(m+1)}{2} - \frac{m}{2^{l-1}} + \frac{j}{2^l} \leqslant \frac{n(n+1)}{2} - \frac{n}{2^l} \leqslant \frac{n(n+1)}{2} - \frac{n}{2^{k-1}}$$

$$\leqslant \frac{n(n+1)}{2} - \frac{n}{2^{k-1}} + \frac{i-1}{2^k} = l(I_{n,k,i})$$

and $I_{m,l,j}$ is left of $I_{n,k,i}$. Assume $m=n$, $l=k$, $j<i$. Then the inequality $j \leqslant i-1$ gives immediately $r(I_{m,l,j}) \leqslant l(I_{n,k,i})$ and $I_{m,l,j}$ is left of $I_{n,k,i}$.

Thus, the system $\mathscr{S}$ is non-overlapping. It remains to prove that $\mathscr{S}$ covers $[0,\infty)$. Let $x \geqslant 0$. Fix $n \in \mathbb{N}$ such that $\frac{n(n-1)}{2} \leqslant x < \frac{n(n+1)}{2}$. Since

$$\left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\right)$$

$$= \bigcup_{k=1}^{\infty} \left[\frac{n(n-1)}{2} + \frac{n}{2} + \frac{n}{2^2} + \dots + \frac{n}{2^{k-1}}, \frac{n(n-1)}{2} + \frac{n}{2} + \frac{n}{2^2} + \dots + \frac{n}{2^k}\right)$$

$$= \bigcup_{k=1}^{\infty} \left[\frac{n(n+1)}{2} - \frac{n}{2^{k-1}}, \frac{n(n+1)}{2} - \frac{n}{2^{k-1}} + \frac{n}{2^k}\right) = \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{n} I_{n,k,i}$$

it follows that $x \in I_{n,k,i}$ for some i and k.

We now construct a function $p(x)$. Let p_0 be a fixed real number, $p_0 > \frac{17}{16}$. Let q be the function defined on $[0,\frac{1}{2}] = I_{1,1,1}$ by

$$q(x) = \begin{cases} p_0 - x & \text{on } [0,\frac{1}{16}]; \\ p_0 - \frac{1}{16} & \text{on } [\frac{1}{16},\frac{3}{16}]; \\ p_0 - \frac{1}{4} + x & \text{on } [\frac{3}{16},\frac{1}{4}]; \\ p_0 & \text{on } [\frac{1}{4},\frac{1}{2}]. \end{cases}$$

Then $1 < \inf q(x), q(x)$ is a Lipschitz function with constant 1 and $q(0) = q(\frac{1}{2}) = p_0$. Let us define functions $p_{n,k,i}$ on $I_{n,k,i}$ by

$$p_{n,k,i}(x) = p_0 + 2^{-(k-1)}(q(2^{k-1}(x - l(I_{n,k,i}))) - p_0).$$

Then $p_{n,k,i}(x)$ is Lipschitz with constant 1 and by (4.1) we have

$$p_{n,k,i}(l(I_{n,k,i})) = p_{n,k,i}(r(I_{n,k,i})) = p_0. \tag{4.2}$$

Set $p(x) = p_{n,k,i}(x)$ for $x \in I_{n,k,i}$. According to (4.2) the function $p(x)$ is continuous and, consequently, it is Lipschitz with constant 1.

Denote

$$K_{n,k,i} = \left[l(I_{n,k,i}) + \frac{1}{2}|I_{n,k,i}|, r(I_{n,k,i}) \right]$$

and

$$J_{n,k,i} = \left[l(I_{n,k,i}) + \frac{1}{8}|I_{n,k,i}|, l(I_{n,k,i}) + \frac{3}{8}|I_{n,k,i}| \right].$$

We remark that

$$|K_{n,k,i}| = \frac{1}{2^{k+1}}\ , |J_{n,k,i}| = \frac{1}{2^{k+2}} \tag{4.3}$$

and

$$p(x) = p_0 \text{ on } K_{n,k,i},\ p(x) = p_0 - \frac{1}{2^{k+3}} \text{ on } J_{n,k,i}. \tag{4.4}$$

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that

$$\sum_{i=1}^{\infty} a_i^{p_0} \leqslant 1 \text{ and } \sum_{i=1}^{\infty} a_i^{p_0-\varepsilon} = \infty \text{ for every } \varepsilon > 0. \tag{4.5}$$

An example of such a sequence is $\{a_i\}_{i=1}^{\infty}$ where

$$a_i = \left(\sum_{j=1}^{\infty} (j+2)^{-1} \ln^{-2}(j+2) \right)^{-\frac{1}{p_0}} (i+2)^{-\frac{1}{p_0}} \ln^{-\frac{2}{p_0}}(i+2).$$

Let $k \in \mathbb{N}$ be fixed. We show that $T_{2^{-k-1}}$ is not bounded from $L^{p(x)}(0,\infty)$ into itself.

First we prove that given any $c > 0$ we can find a function $f(x) \geqslant 0$ such that

$$\int_0^{\infty} (f(x))^{p(x)} \mathrm{d}x \leqslant 1 \quad \text{and} \quad \int_0^{\infty} (T_{2^{-k-1}} f(x))^{p(x)} \mathrm{d}x \geqslant c.$$

Fix $c > 0$. According to (4.5) there is $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} a_i^{p_0 - 2^{-k-3}} \geqslant 4^{p_0+1} c. \tag{4.6}$$

Set

$$f(x) = 2^{\frac{k+1}{p_0}} \sum_{i=1}^{n} a_i \chi_{K_{n,k,i}}(x).$$

Then due to (4.3), (4.4), and (4.5) we have

$$\int_0^{\infty} f(x)^{p(x)} \mathrm{d}x = \sum_{i=1}^{n} \int_{K_{n,k,i}} (2^{\frac{k+1}{p_0}} a_i)^{p_0} \mathrm{d}x = 2^{k+1} \sum_{i=1}^{n} a_i^{p_0} |K_{n,k,i}| = \sum_{i=1}^{n} a_i^{p_0} \leqslant 1.$$

Let $x \in J_{n,k,i}$. To calculate $(T_{2^{-k-1}}f)(x)$ we first denote $S_{n,k,i} = l(I_{n,k,i}) + \frac{1}{2}|I_{n,k,i}|$. Clearly, by (4.1),

$$\begin{aligned} T_{2^{-k-1}}f(x) &= 2^{k+1}\int_x^{x+2^{-k-1}} f(t)\mathrm{d}t \\ &\geqslant 2^{k+1}\int_{S_{n,k,i}}^{S_{n,k,i}+\frac{1}{8}|I_{n,k,i}|} f(t)\mathrm{d}t = 2^{k+1}2^{\frac{k+1}{p_0}}a_i\frac{1}{8}|I_{n,k,i}| = \frac{1}{4}2^{\frac{k+1}{p_0}}a_i. \end{aligned}$$

Thus, using (4.3), (4.4) and (4.6) it follows that

$$\begin{aligned} \int_0^\infty \left(T_{2^{-k-1}}f(x)\right)^{p(x)}\mathrm{d}x &\geqslant \int_0^\infty \sum_{i=1}^n \left(\frac{1}{4}2^{\frac{k+1}{p_0}}a_i\right)^{p(x)} \chi_{J_{n,k,i}}(x)\mathrm{d}x \\ &= \sum_{i=1}^n \left(\frac{1}{4}2^{\frac{k+1}{p_0}}a_i\right)^{p_0-2^{-k-3}} |J_{n,k,i}| \\ &= \sum_{i=1}^n 4^{2^{-k-3}-p_0}2^{k+1}2^{-\frac{1}{p_0}(k+1)2^{-k-3}}a_i^{p_0-2^{-k-3}}\frac{1}{2^{k+2}} \\ &\geqslant 4^{-p_0}\frac{1}{2}\,2^{-1}\sum_{i=1}^n a_i^{p_0-2^{-k-3}} \geqslant c. \end{aligned}$$

We have proved that for any $c > 0$ there is a function f with

$$\int_0^\infty f(x)^{p(x)}\mathrm{d}x \leqslant 1 \quad \text{and} \int_0^\infty (T_{2^{-k-1}}f)(x)^{p(x)}\mathrm{d}x \geqslant c.$$

Since the T_s are positive operators on $\mathscr{M}(0,\infty)$ we have according to Lemma 2.3 functions $f_n(x)$ with

$$\int_0^\infty f_k(x)^{p(x)}\mathrm{d}x \leqslant 1 \quad \text{and} \int_0^\infty (T_{2^{-k-1}}f_k)(x)^{p(x)}\mathrm{d}x = \infty.$$

Let $s > 0$. Take any k such that $\frac{1}{2^{k+1}} \leqslant s$. Set $f(x) = f_k(x)$. Thus,

$$\int_0^\infty f(x)^{p(x)}\mathrm{d}x \leqslant 1$$

and

$$\begin{aligned} \int_0^\infty (T_sf)(x)^{p(x)}\mathrm{d}x &= \int_0^\infty \left(\frac{1}{s}\int_x^{x+s} f_k(t)\mathrm{d}t\right)^{p(x)}\mathrm{d}x \\ &\geqslant \int_0^\infty \left(\frac{1}{2^{k+1}s}2^{k+1}\int_x^{x+2^{-k-1}} f_k(t)\mathrm{d}t\right)^{p(x)}\mathrm{d}x \\ &\geqslant \left(\frac{1}{2^{k+1}s}\right)^{p_0}\int_0^\infty (T_{2^{-k-1}}f)(x)^{p(x)}\mathrm{d}x = \infty. \end{aligned}$$

Thus, the operator T_s is unbounded which finishes the proof. □

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(Received May 22, 2001)

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