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COMPACT OPERATORS BETWEEN REAL INTERPOLATION SPACES

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Abstract. We show necessary and sufficient conditions for compactness of operators acting between real interpolation spaces.

1. Introduction

This paper deals with Interpolation Theory, a branch of Analysis where inequalities play an important role (see [1], [2], [3] or [14]). We shall investigate compact operators between real interpolation spaces.

Let (A_0, A_1) and (B_0, B_1) be Banach couples and let T be a linear operator such that $T : A_0 \longrightarrow B_0$ is compact and $T : A_1 \longrightarrow B_1$ is bounded. In 1992, Cwikel [9] proved that under these assumptions the interpolated operator by the real method $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ is also compact. Here $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$. A novel approach to Cwikel's theorem was given by Cobos, Kühn and Schonbek in [6] (see also the papers by Cobos and Peetre [7] and by Cobos and Persson [8]). Their proof is based on relationship between real interpolation space $(A_0, A_1)_{\theta,q}$ and vector valued ℓ_q spaces modelled on the sum $A_0 + A_1$ and on the intersection $A_0 \cap A_1$. First compactness results of Lions and Peetre [10] are used to check that the composition of T with certain operators P_n are compact and then it is shown that the extended operator can be approximated by compact operators in the operator norm.

The aim of the present paper is to analyze how far are these arguments from being optimal. We show that compactness of the interpolated operator is equivalent to convergence of TP_n to the extended operator together with the fact that $T : A_0 \cap A_1 \longrightarrow (B_0, B_1)_{\theta,q}$ is compact. Here the real interpolation space is viewed as a J-space. When $(A_0, A_1)_{\theta,q}$ is realized as a K-space we establish a similar result but using now operators P_nT and compactness of $T : (A_0, A_1)_{\theta,q} \longrightarrow B_0 + B_1$. Our proofs are based on inequalities involving norms of operators P_n and some other projections acting between vector valued spaces.

As a motivation for conditions used in this paper we should mention an unpublished result by A. Persson (see [11], Chapter VI, Thm. 4) characterizing precompact subspaces of K-spaces (see also [12]).

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Recently Schonbek has investigated in [13] a similar question to the one studied here in Section 3 but for the complex interpolation method .

2. Preliminaries

Given any Banach couple (A_0, A_1) and any positive number t, the J- and the K-functional are defined by

$$J(t,a) = \max\{\|a\|_{A_0}\ ,\ t\|a\|_{A_1}\}\ ,\quad a \in A_0 \cap A_1$$

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\ ,\ a_i \in A_i\}\ ,\quad a \in A_0 + A_1.$$

Let $1 \leqslant q \leqslant \infty$ and $0 < \theta < 1$. The real interpolation space $(A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$, such that $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) with $\{u_m\}_{m\in\mathbf{Z}} \subset A_0 \cap A_1$ and $\left(\sum_{m=-\infty}^{\infty}\left(2^{-\theta m}J(2^m, u_m)\right)^q\right)^{\frac{1}{q}} < \infty$. The norm on $(A_0, A_1)_{\theta,q}$ is

$$\|a\|_{\theta,q} = \inf\left\{\left(\sum_{m=-\infty}^{\infty}\left(2^{-\theta m}J(2^m, u_m)\right)^q\right)^{\frac{1}{q}}\ :\ a = \sum_{m=-\infty}^{\infty} u_m\right\}.$$

This is the real method realized as a J-space. It can be equivalently defined in terms of the K-functional

$$(A_0, A_1)_{\theta,q} = \left\{a \in A_0 + A_1\ :\ \|a\|_{\theta,q} = \left(\sum_{m=-\infty}^{\infty}\left(2^{-\theta m}K(2^m, a)\right)^q\right)^{\frac{1}{q}} < \infty\right\}.$$

Note that the J- and the K-norm are not equal although we denote then by the same symbol. They are only equivalent. This will not cause any confusion because we will point out the norm we are using in each case. Full details on the real method can be found in the books by Bennett and Sharpley [1], Bergh and Löfström [2], Brudnyi and Krugljak [3] or Triebel [14].

Next we recall the classical compactness result established by Lions and Peetre in [10] (see also [2], 3.8).

Lions-Peetre Lemma . *Let (A_0, A_1) be a Banach couple and let B be a Banach space. Let $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$ and assume that T is a linear operator.*

(i) If $T : A_0 \longrightarrow B$ is compact and $T : A_1 \longrightarrow B$ is bounded, then $T : (A_0, A_1)_{\theta,q} \longrightarrow B$ is compact.

(ii) If $T : B \longrightarrow A_0$ is compact and $T : B \longrightarrow A_1$ is bounded, then $T : B \longrightarrow (A_0, A_1)_{\theta,q}$ is compact.

We refer to [4] and [5] for recent results on this lemma.

Given any sequence $\{W_m\}_{m\in\mathbf{Z}}$ of Banach spaces and any sequence $\{\lambda_m\}_{m\in\mathbf{Z}}$ of non-negative numbers, we put

$$\ell_q(\lambda_m W_m) = \left\{ w = \{w_m\}_{m\in\mathbf{Z}} : w_m \in W_m \quad \text{and} \right.$$
$$\left. \|w\|_{\ell_q(\lambda_m W_m)} = \left(\sum_{m=-\infty}^{\infty} (\lambda_m \|w_m\|_{W_m})^q \right)^{\frac{1}{q}} < \infty \right\}.$$

3. Compact operators between J-spaces

In this section the real interpolation space $(A_0,A_1)_{\theta,q}$ is realized as a J-space. We shall work with vector valued sequence spaces modelled on the intersection. We denote by G_m the Banach space $A_0 \cap A_1$ endowed with the norm $J(2^m,\cdot)$ and we consider the space $\ell_q(2^{-\theta m}G_m)$. Relationship between the real interpolation space $(A_0,A_1)_{\theta,q}$ and the space $\ell_q(2^{-\theta m}G_m)$ is given by the operator $\pi : \ell_q(2^{-\theta m}G_m) \longrightarrow (A_0,A_1)_{\theta,q}$ defined by $\pi\{u_m\} = \sum_{m=-\infty}^{\infty} u_m$, which is a metric surjection.

Besides π, the following operators will be useful to work with sequences

$$P_n\{u_m\} = \{\ldots 0,0,u_{-n},u_{-n+1},\ldots,u_{n-1},u_n,0,0,\ldots\}$$
$$Q_n^+\{u_m\} = \{\ldots 0,0,u_{n+1},u_{n+2},\ldots\}$$
$$Q_n^-\{u_m\} = \{\ldots u_{-n-2},u_{-n-1},0,0,\ldots\}$$

Here $n \in \mathbf{N}$. They satisfy the next three properties:

I) Any of these operators acting on $\ell_1(G_m)$, on $\ell_1(2^{-m}G_m)$ or on $l_q(2^{-\theta m}G_m)$ has norm 1.

II) If I stands for the identity operator, then

$$I = P_n + Q_n^+ + Q_n^- \quad \text{for each} \quad n \in \mathbf{N}.$$

III) For each $n \in \mathbf{N}$,

$$\|Q_n^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} = 2^{-(n+1)} = \|Q_n^-\|_{\ell_1(2^{-m}G_m),\ell_1(G_m)}$$

and

$$\|P_n\|_{\ell_1(G_m)+\ell_1(2^{-m}G_m),\ell_1(G_m)\cap\ell_1(2^{-m}G_m)} \leqslant 2^n.$$

The following result characterizes compact operators between J-spaces.

THEOREM 3.1. *Let (A_0,A_1) and (B_0,B_1) be Banach couples and let T be a linear operator such that $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ are bounded. Then for any $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, the interpolated operator by the real method $T : (A_0,A_1)_{\theta,q} \longrightarrow (B_0,B_1)_{\theta,q}$ is compact if and only if the following conditions hold*

(a) $T : A_0 \cap A_1 \longrightarrow (B_0,B_1)_{\theta,q}$ *is compact.*

(b) $\sup\left\{ \|T(\sum_{|m|>n} u_m)\|_{\theta,q} \ : \ \|\{u_m\}\|_{\ell_q(2^{-\theta m}G_m)} \leqslant 1 \right\} \longrightarrow 0$ *as* $n \to \infty$.

Proof. Assume that (a) and (b) hold. For each $n \in \mathbf{N}$, the operator πP_n is bounded from $\ell_1(2^{-mi}G_m)$ into $A_0 \cap A_1$. So, for $i = 0, 1$, $T\pi P_n : \ell_1(2^{-mi}G_m) \longrightarrow (B_0, B_1)_{\theta,q}$ is compact. Using Lions-Peetre Lemma /(ii) and taking into account that

$$(\ell_1(G_m)\,,\ \ell_1(2^{-m}G_m))_{\theta,q} = l_q(2^{-\theta m}G_m) \tag{1}$$

we derive that

$$T\pi P_n : \ell_q(2^{-\theta m}G_m) \longrightarrow (B_0, B_1)_{\theta,q}$$

is compact for any $n \in \mathbf{N}$. Since assumption (b) means that $\{T\pi P_n\}_{n\in\mathbf{N}}$ converges to $T\pi$ in $\mathscr{L}(\ell_q(2^{-\theta m}G_m), (B_0, B_1)_{\theta,q})$, we have that $T\pi$ is compact. This implies compactness of $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ because π is a metric surjection.

Conversely, suppose that $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ is compact. Since $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q}$ it is clear that $T : A_0 \cap A_1 \longrightarrow (B_0, B_1)_{\theta,q}$ is compact.

To check that condition (b) is also satisfied, first note that formula (1), properties (I) and (II) and the interpolation theorem yield that

$$\max\left\{\|Q_n^+\|_{\ell_q(2^{-\theta m}G_m),\ell_1(2^{-m}G_m)}\ ,\ \|Q_n^-\|_{\ell_q(2^{-\theta m}G_m),\ell_1(G_m)}\right\} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Take any $v \in \ell_q(2^{-\theta m}G_m)$ with $\|v\|_{\ell_q(2^{-\theta m}G_m)} \leqslant 1$ and put $w = (I - P_{n+1})v$. Then

$$\|w\|_{\ell_q(2^{-\theta m}G_m)} \leqslant \|v\|_{\ell_q(2^{-\theta m}G_m)} \leqslant 1.$$

It follows from

$$\|T\pi(I - P_{n+1})v\|_{\theta,q} = \|T\pi(I - P_n)(I - P_{n+1})v\|_{\theta,q} = \|T\pi(I - P_n)w\|_{\theta,q}$$

that

$$\|T\pi(I - P_{n+1})\|_{\ell_q(2^{-\theta m}G_m),(B_0,B_1)_{\theta,q}} \leqslant \|T\pi(I - P_n)\|_{\ell_q(2^{-\theta m}G_m),(B_0,B_1)_{\theta,q}}.$$

Whence, the sequence $\left\{\|T\pi(I - P_n)\|_{\ell_q(2^{-\theta m}G_m),(B_0,B_1)_{\theta,q}}\right\}_{n\in\mathbf{N}}$ is non-increasing and therefore it should be convergent, say to δ.

In order to show that $\delta = 0$, choose $\{v_n\}_{n\in\mathbf{N}}$ in the unit ball of $\ell_q(2^{-\theta m}G_m)$ so that $\delta = \lim_{n\to\infty} \|T\pi(I - P_n)v_n\|_{\theta,q}$. The sequence $\{(I - P_n)v_n\}_{n\in\mathbf{N}}$ is bounded in $\ell_q(2^{-\theta m}G_m)$. By compactness of $T\pi : \ell_q(2^{-\theta m}G_m) \longrightarrow (B_0, B_1)_{\theta,q}$ we can select from $\{T\pi(I - P_n)v_n\}_{n\in\mathbf{N}}$ a convergent subsequence $\{T\pi(I - P_{n'})v_{n'}\}$. Let b its limit. So $\delta = \|b\|_{\theta,q}$. We have

$$T\pi(I - P_{n'})v_{n'} = T\pi Q_{n'}^+ v_{n'} + T\pi Q_{n'}^- v_{n'}$$

and, according to (2), sequences $\{T\pi Q_{n'}^+ v_{n'}\}$, $\{T\pi Q_{n'}^- v_{n'}\}$ converge to 0 in $B_0 + B_1$. By compatibility $b = 0$ and so $\delta = 0$.

The proof is complete.

4. Compact operators between K-spaces

Subsequently, the real interpolation space $(B_0,B_1)_{\theta,q}$ is viewed as a K-space. We put $F_m = (B_0+B_1, K(2^m,\cdot))$ and we form the space $\ell_q(2^{-\theta m}F_m)$. Then the operator $j : (B_0,B_1)_{\theta,q} \longrightarrow l_q(2^{-\theta m}F_m)$ that associates with any $b \in B_0+B_1$ the constant sequence $jb = \{\dots,b,b,b,\dots\}$ is a metric injection.

Operators P_n, Q_n^+ and Q_n^- will be considered now acting between spaces $\ell_q(2^{-\theta m}F_m)$. They satisfy analogous properties to (I), (II) and (III) but replacing the couple $(\ell_1(G_m),\ \ell_1(2^{-m}G_m))$ by $(\ell_\infty(F_m),\ \ell_\infty(2^{-m}F_m))$, and replacing the space $\ell_q(2^{-\theta m}G_m)$ by $\ell_q(2^{-\theta m}F_m)$.

We next establish the characterization for operators between K-spaces.

THEOREM 4.1. *Let (A_0,A_1) and (B_0,B_1) be Banach couples and let T be a linear operator such that $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ are bounded. Then for any $0<\theta<1$ and $1 \leqslant q < \infty$, the interpolated operator $T : (A_0,A_1)_{\theta,q} \longrightarrow (B_0,B_1)_{\theta,q}$ is compact if and only if the following conditions hold*

(a) $T : (A_0,A_1)_{\theta,q} \longrightarrow B_0+B_1$ *is compact.*

(b) $\sup\left\{ \Big(\sum_{|m|>n} (2^{-\theta m}K(2^m,Ta))^q\Big)^{\frac{1}{q}} \ :\ \|a\|_{\theta,q} \leqslant 1 \right\} \longrightarrow 0 \quad as \quad n\to\infty.$

Proof. First we show that conditions (a) and (b) imply compactness of the interpolated operator.

Since $P_nj \in \mathscr{L}\big(B_0+B_1, \ell_\infty(F_m)\cap\ell_\infty(2^{-m}F_m)\big)$, we have that $P_njT : (A_0,A_1)_{\theta,q} \longrightarrow \ell_\infty(2^{-mi}F_m)$ is compact for $i=0,1$. Lions-Peetre Lemma /(i) and formula

$$(\ell_\infty(F_m)\,,\ \ell_\infty(2^{-m}F_m))_{\theta,q} = l_q(2^{-\theta m}F_m) \tag{3}$$

yield that $P_njT : (A_0,A_1)_{\theta,q} \longrightarrow l_q(2^{-\theta m}F_m)$ is compact for any $n\in\mathbf{N}$. But, according to (b), the sequence $\{P_njT\}_{n\in\mathbf{N}}$ converges to jT in $\mathscr{L}((A_0,A_1)_{\theta,q}, \ell_q(2^{-\theta m}F_m))$, so jT is compact. This implies that $T : (A_0,A_1)_{\theta,q} \longrightarrow (B_0,B_1)_{\theta,q}$ is compact because j is a metric injection.

Conversely, assume that $T : (A_0,A_1)_{\theta,q} \longrightarrow (B_0,B_1)_{\theta,q}$ is compact. Since $(B_0,B_1)_{\theta,q} \hookrightarrow B_0+B_1$, it is obvious that $T : (A_0,A_1)_{\theta,q} \longrightarrow B_0+B_1$ is compact.

To check condition (b), take any $\varepsilon>0$. By compactness of $T : (A_0,A_1)_{\theta,q} \longrightarrow (B_0,B_1)_{\theta,q}$, there exists a finite set $\{a_1,\dots,a_r\} \subset (A_0,A_1)_{\theta,q}$ such that

$$T\left(U_{(A_0,A_1)_{\theta,q}}\right) \subseteq \bigcup_{k=1}^{r} \left(Ta_k + \varepsilon U_{(B_0,B_1)_{\theta,q}}\right).$$

Here $U_{(A_0,A_1)_{\theta,q}}$ denotes the closed unit ball of $(A_0,A_1)_{\theta,q}$ and $U_{(B_0,B_1)_{\theta,q}}$ has a similar meaning. Since $q<\infty$, for any $b\in(B_0,B_1)_{\theta,q}$ we have that

$$\lim_{n\to\infty} \|P_njb - jb\|_{l_q(2^{-\theta m}F_m)} = 0.$$

Whence, we can find $N\in\mathbf{N}$ such that for any $n\geqslant N$ it follows that

$$\|P_njTa_k - jTa_k\|_{l_q(2^{-\theta m}F_m)} \leqslant \varepsilon \quad , \quad k=1,\dots,r.$$

Consequently, for $n \geqslant N$, given any $a \in U_{(A_0,A_1)_{\theta,q}}$ if we choose k so that $\|Ta - Ta_k\|_{\theta,q} \leqslant \varepsilon$, we obtain

$$\begin{aligned}\|P_njTa - jTa\|_{l_q(2^{-\theta m}F_m)} &\leqslant \|P_njTa - P_njTa_k\|_{l_q(2^{-\theta m}F_m)} \\ &\quad + \|P_njTa_k - jTa_k\|_{l_q(2^{-\theta m}F_m)} + \|jTa_k - jTa\|_{l_q(2^{-\theta m}F_m)} \\ &\leqslant \|Ta - Ta_k\|_{\theta,q} + \varepsilon + \|Ta_k - Ta\|_{\theta,q} \leqslant 3\varepsilon.\end{aligned}$$

In other words, if $n \geqslant N$ then

$$\sup\left\{ (\sum_{|m|>n} (2^{-\theta m} K(2^m, Ta))^q)^{\frac{1}{q}} \; : \; \|a\|_{\theta,q} \leqslant 1 \right\} \leqslant 3\varepsilon.$$

The proof is complete.

Suppose now $q = \infty$, and let $(B_0, B_1)^\circ_{\theta,\infty}$ be the closed subspace of $(B_0, B_1)_{\theta,\infty}$ generated by $B_0 \cap B_1$. It is not hard to check that

$$(B_0, B_1)^\circ_{\theta,\infty} = \left\{ b \in B_0 + B_1 \; : \; \lim_{|m|\to\infty} \{2^{-\theta m} K(2^m, b)\} = \lim_{n\to\infty} \|P_njb - jb\|_{l_\infty(2^{-\theta m}F_m)} = 0 \right\}.$$

So, techniques used in the proof of Theorem 4.2 may be modified to give

THEOREM 4.2. *Let (A_0, A_1) and (B_0, B_1) be Banach couples and let T be a linear operator such that $T : A_0 \longrightarrow B_0$ and $T : A_1 \longrightarrow B_1$ are bounded. Then for any $0 < \theta < 1$, the interpolated operator $T : (A_0, A_1)^\circ_{\theta,\infty} \longrightarrow (B_0, B_1)^\circ_{\theta,\infty}$ is compact if and only if the following conditions hold*

(a) $T : (A_0, A_1)^\circ_{\theta,\infty} \longrightarrow B_0 + B_1$ *is compact.*

(b) $\sup\left\{ \sup_{|m|>n} \{2^{-\theta m} K(2^m, Ta)\} \; : \; a \in U_{(A_0,A_1)^\circ_{\theta,\infty}} \right\} \longrightarrow 0 \quad$ *as* $n \to \infty$.

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REFERENCES

[1] C. BENNETT AND R. SHARPLEY, *Interpolation of operators*, Academic Press, New York, 1988.

[2] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces. An introduction*, Springer, Berlin, 1976.

[3] Y. BRUDNYI AND N. KRUGLJAK, *Interpolation functors and interpolation spaces*, Vol. 1, North-Holland, Amsterdam, 1991.

[4] F. COBOS, M. CWIKEL AND P. MATOS, *Best possible compactness results of Lions-Peetre type*, Proc. Edinb. Math. Soc. **44** (2001) 153–172.

[5] F. COBOS, L. M. FERNÁNDEZ-CABRERA, A. MARTÍNEZ AND E. PUSTYLNIK, *Some interpolation results that are exclusive property of compact operators*, Proc. Royal Soc. Edinburgh 132A (2002) (to appear).

[6] F. COBOS, T. KÜHN AND T. SCHONBEK, *One-sided compactness results for Aronszajn-Gagliardo functors*, J. Funct. Analysis **106** (1992) 274–313.
[7] F. COBOS AND J. PEETRE, *Interpolation of compactness using Aronszajn-Gagliardo functors*, Israel J. Math. **68** (1989) 220–240.
[8] F. COBOS, L.-E. PERSSON, *Real interpolation of compact operators between quasi-Banach spaces*, Math. Scand. **82** (1998) 138–160.
[9] M. CWIKEL, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. **65** (1992) 333–343.
[10] J. L. LIONS AND J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. **19** (1964) 5–68.
[11] J. PEETRE, *A theory of interpolation of normed spaces*, Lecture notes, Brasilia 1963 [Notas Mat. **39** (1968) 1–86].
[12] A. PERSSON, *Compact linear mappings between interpolation spaces*, Ark. Mat. **5** (1964) 215–219.
[13] T. SCHONBEK, *Interpolation of compact operators by the complex method and equicontinuity*, Indiana U. Math. J. **49** (2000) 1229–1245.
[14] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

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