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ON THE RIESZ IDEMPOTENT OF CLASS A OPERATORS

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(*communicated by T. Furuta*)

Abstract. In this paper, we show that if T is a class A operator and λ is a non–zero isolated eigenvalue of $\sigma(T)$, then $E\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$, where E is the Riesz idempotent with respect to λ. In this case, E is self–adjoint, i.e, it is an orthogonal projection.

1. Introduction

A bounded linear operator T on a separable Hilbert space $\mathscr{H}$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. Here $\sigma(T)$ denotes the spectrum of T and we also denote the point spectrum of T by $\sigma_p(T)$. It is important and useful for studying some classes of operators to investigate whether every operator in such classes is isoloid or not. Stampfli's result that every hyponormal operator is isoloid is one of the most famous results in this subject. See [17]. The proof of this result is very excellent and simple by using the Riesz idempotent. This result was extended to several classes of operators, e.g., p-hyponormal, log-hyponormal and paranormal. See [5], [6], [7], [19], [21], [22]]. Here an operator T is called p-hyponormal for $p > 0$ if $(T^*T)^p \geqslant (TT^*)^p$, log-hyponormal if T is invertible and $\log(T^*T) \geqslant \log(TT^*)$, p-quasihyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geqslant 0$. A 1-hyponormal operator is called hyponormal and a 1-quasihyponormal operator is called quasihyponormal. An operator T is called class A if $|T^2| \geqslant |T|^2$, and paranormal if $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$ for all $x \in \mathscr{H}$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T. See Furuta [9] for properties of paranormal operators. We remark that every hyponormal, p-hyponormal or log-hyponormal operator is paranormal [3] (see also [1], [8], [10], [11], [18]). We also remark that p-quasihyponormal is paranormal if $0 < p \leqslant 1$ [15]. Class A is an interesting class of bounded linear operators on a Hilbert space, which was defined by T. Furuta, M. Ito and T. Yamazaki [10]. They showed that class A is a subclass of paranormal and contains every invertible log-hyponormal operator.

Let $T \in B(\mathscr{H})$ and let $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$. Then there exists a closed disk D_λ centered λ which satisfies $D_\lambda \cap \sigma(T) = \{\lambda\}$. The operator $E = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_\lambda} (\lambda - T)^{-1} d\lambda$ is called the Riesz idempotent with respect to λ which

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has properties that $E^2 = E, ET = TE, \ker(T-\lambda) \subset E\mathscr{H}$ and $\sigma(T|_{E\mathscr{H}}) = \{\lambda\}$. In [17], Stampfli also proved that if T is hyponormal and $\lambda \in \sigma(T)$ is isolated, then the Riesz idempotent E with respect to λ is self-adjoint and satisfies

$$E\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

Recently, M. Chō and K. Tanahashi [7] proved that if T is a p-hyponormal or log-hyponormal operator and λ is an isolated point of $\sigma(T)$, then the Riesz idempotent E with respect to λ is self-adjoint and $E\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$. This result is an extension of Stampfli's result for hyponormal operators. It is interesting to study whether Stampfli's result hold for the other classes of operators. In this paper, we extend these results to the case of class A operators and $\lambda \neq 0$. However, it is necessarily not true for the case $\lambda = 0$, we give a counter-example after. See [19] for p-quasihyponormal operators.

2. Preliminaries

The following two inequalities are important.

PROPOSITION 1. (Löwner-Heinz inequality [13], [16]) *If $A,\ B \in B(\mathscr{H})$ satisfy $0 \leqslant A \leqslant B$ and $\alpha \in (0,1]$, then $A^\alpha \leqslant B^\alpha$.*

PROPOSITION 2. (Hansen's inequality [12]) *If $A,\ B \in B(\mathscr{H})$ satisfy $A \geqslant 0$ and $\|B\| \leqslant 1$, then $(B^*AB)^\delta \geqslant B^*A^\delta B$ for all $\delta \in (0,1]$.*

DEFINITION 3. *An operator $T \in B(\mathscr{H})$ is called normaloid if*

$$\|T\| = \sup\{|z| \mid z \in \sigma(T)\}.$$

LEMMA 4. ([9], [14]) *If T is paranormal, then T is normaloid.*

LEMMA 5. *If T is paranormal, then the restriction $T|_{\mathscr{M}}$ to its invariant subspace $\mathscr{M}$ is also paranormal.*

The following Lemmas 6 and 8 have been proved in [22]. For the completeness, we contain proofs.

LEMMA 6. ([21], [22]) *If T is paranormal, then T is isoloid.*

Proof. Let $\lambda \in \sigma(T)$ be an isolated point. Then the range of Riesz idempotent $E = \frac{1}{2\pi i}\int_{\partial D_\lambda}(zI-T)^{-1}\,dz$ is an invariant closed subspace of T and $\sigma(T|_{E\mathscr{H}}) = \{\lambda\}$. Here D_λ is a closed disk with its center λ such that $\sigma(T) \cap D_\lambda = \{\lambda\}$.

If $\lambda = 0$, then $\sigma(T|_{E\mathscr{H}}) = \{0\}$. Since $T|_{E\mathscr{H}}$ is paranormal by Lemma 5, $T|_{E\mathscr{H}} = 0$ by Lemma 4. Therefore 0 is an eigenvalue of T.

If $\lambda \neq 0$, then $T|_{E\mathscr{H}}$ is an invertible paranormal operator and hence $(T|_{E\mathscr{H}})^{-1}$ is also paranormal [14]. We see $\|T|_{E\mathscr{H}}\| = |\lambda|$ and $\|(T|_{E\mathscr{H}})^{-1}\| = \frac{1}{|\lambda|}$. Let $x \in E\mathscr{H}$ be an arbitrary vector. Then $\|x\| \leqslant \|(T|_{E\mathscr{H}})^{-1}\|\|T|_{E\mathscr{H}}x\| = \frac{1}{|\lambda|}\|T|_{E\mathscr{H}}x\| \leqslant \frac{1}{|\lambda|}|\lambda|\|x\| = \|x\|$. This implies that $\frac{1}{\lambda}T|_{E\mathscr{H}}$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T|_{E\mathscr{H}}) = \{1\}$. Hence $T|_{E\mathscr{H}} = \lambda$ and λ is an eigenvalue of T. This completes the proof. $\square$

PROPOSITION 7. ([4], [23]) *If T is p-hyponormal, then $(T-\lambda)x=0$ implies that $(T-\lambda)^*x=0$.*

LEMMA 8. ([22]) *If T belongs to the class A and $\lambda \in \mathbb{C}\setminus\{0\}$, then $(T-\lambda)x=0$ implies that $(T-\lambda)^*x=0$.*

Proof. We may assume $x \neq 0$. Since $\||T^2|x\| = \|T^2x\| = |\lambda|^2\|x\|$ and

$$\begin{aligned}|\lambda|^2\|x\|^2 &= \langle\, |T|^2x, x\rangle \\ &\leqslant \langle\, |T^2|x, x\rangle \quad \text{(since } T \text{ belongs to the class } A) \\ &\leqslant \||T^2|x\|\|x\| = |\lambda|^2\|x\|^2,\end{aligned}$$

we have $|T^2|x = |\lambda|^2x$. Since

$$\|(|T^2|-|T|^2)^{\frac{1}{2}}x\|^2 = \langle\, |T^2|x,x\rangle\, - \langle\, |T|^2x,x\rangle\, = 0,$$

we also have $|T|^2x = |\lambda|^2x$. This implies that $T^*x = \overline{\lambda}x$. $\square$

3. The main theorem

THEOREM 9. *Let $T \in B(\mathscr{H})$ be a class A operator and λ be a non-zero isolated point of $\sigma(T)$. Let D_λ denote the closed disk which centered λ such that $D_\lambda \cap \sigma(T) = \{\lambda\}$. Then the Riesz idempotent $E = \frac{1}{2\pi i}\int_{\partial D_\lambda}(z-T)^{-1}\,dz$ satisfies that $E\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$. In particular, E is self-adjoint.*

Proof. Since class A operators are paranormal, λ is an eigenvalue of T by Lemma 6. As we see in the proof of Lemma 6, $E\mathscr{H} = \ker(T-\lambda)$. Since $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ by Lemma 8, it suffices to show that $\ker(T-\lambda)^* \subset \ker(T-\lambda)$. Since $\ker(T-\lambda)$ is a reducing subspace of T by Lemma 8 and the restriction of a class A operator to its reducing subspace is also a class A operator, we see that T is of the form $T = \lambda \oplus T'$ on $\mathscr{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$, where T' is a class A operator with $\ker(T'-\lambda) = \{0\}$. Since $\lambda \in \sigma(T) = \{\lambda\} \cup \sigma(T')$ is isolated, the only two cases occur. One is $\lambda \notin \sigma(T')$ and the other is that λ is an isolated point of $\sigma(T')$. The latter case, however, does not occur otherwise we have $\lambda \in \sigma_p(T')$ and this contradicts the fact that $\ker(T'-\lambda) = \{0\}$. $\ker(T-\lambda) = \ker(T-\lambda)^*$ is immediate from the invertivity of $T'-\lambda$ as an operator on $(\ker(T-\lambda))^\perp$.

Next, we show that E is self-adjoint. Since $E\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$, we have $((z-T)^*)^{-1}E = \overline{(z-\lambda)^{-1}}E$. Hence

$$\begin{aligned}E^*E &= -\frac{1}{2\pi i}\int_{\partial D_\lambda}((z-T)^*)^{-1}E\,d\overline{z} \\ &= -\frac{1}{2\pi i}\int_{\partial D_\lambda}\overline{(z-\lambda)^{-1}}E\,d\overline{z} \\ &= \overline{(\frac{1}{2\pi i}\int_{\partial D_\lambda}\frac{1}{z-\lambda}\,dz)}E = E,\end{aligned}$$

this completes the proof. □

REMARK 10. In the above theorem, to prove $E\mathscr{H} = \ker(T-\lambda)$ we only use the fact that T is paranormal, so it also holds for the case of $\lambda = 0$. The following example, however, tells us that $\ker(T-\lambda) = \ker(T-\lambda)^*$ is necessarily not true for the case of $\lambda = 0$.

We show that if T is a p-quasihyponormal operator for some $p \in (0,1]$ such that T is not p-hyponormal and 0 is an isolated point of $\sigma(T)$, then $\ker T \neq \ker T^*$ and the Riesz idempotent with respect to 0 is not self-adjoint. Let $T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix}$ be 2×2 matrix representation of T with respect to $\mathscr{H} = \overline{\operatorname{ran} T} \oplus \ker T^*$. Then $(AA^*)^p \leqslant (AA^* + SS^*)^p \leqslant (A^*A)^p$ and $\sigma(A) \subset \sigma(T) \subset \sigma(A) \cup \{0\}$, in particular, A is p-hyponormal. See [20]. By the assumption $S \neq 0$, otherwise $T = A \oplus 0$ is also p-hyponormal. If $x \in \ker A = \{u \in \overline{\operatorname{ran} T} \mid Au = 0\}$, then $A^*x = 0$ and $S^*x = 0$ by the above inequality and hence $T^*x = A^*x \oplus S^*x = 0$. This shows that $\ker A \subset \overline{\operatorname{ran} T} \cap \ker T^* = \{0\}$. So A is invertible, otherwise 0 is an isolated point of $\sigma(A)$ and hence $0 \in \sigma_p(A)$ by Lemma 6. This contradicts the fact that $\ker A = \{0\}$. It is easy to check $\ker T = \{-A^{-1}Su \oplus u \mid u \in \ker T^*\}$ and hence $\ker T \neq \ker T^*$. In this case,

$$\begin{aligned}
E &= \frac{1}{2\pi i}\int_{\partial D_0} (z-T)^{-1}\,dz \\
&= \frac{1}{2\pi i}\int_{\partial D_0} \begin{pmatrix} (z-A)^{-1} & z^{-1}(z-A)^{-1}S \\ 0 & z^{-1} \end{pmatrix} dz \\
&= \frac{1}{2\pi i}\int_{\partial D_0} \begin{pmatrix} (z-A)^{-1} & -(z^{-1}-(z-A)^{-1})A^{-1}S \\ 0 & z^{-1} \end{pmatrix} dz \\
&= \begin{pmatrix} \frac{1}{2\pi i}\int_{\partial D_0}(z-A)^{-1}\,dz & \left(\frac{1}{2\pi i}\int_{\partial D_0}(z-A)^{-1}\,dz - \frac{1}{2\pi i}\int_{\partial D_0} z^{-1}\,dz\right)A^{-1}S \\ 0 & \frac{1}{2\pi i}\int_{\partial D_0} z^{-1}\,dz \end{pmatrix} \\
&= \begin{pmatrix} 0 & -A^{-1}S \\ 0 & 1 \end{pmatrix} \\
&\qquad \left(\text{because } \frac{1}{2\pi i}\int_{\partial D_0}(z-A)^{-1}\,dz = 0 \text{ and } \frac{1}{2\pi i}\int_{\partial D_0} z^{-1}\,dz = 1\right),
\end{aligned}$$

where E is the Riesz idempotent with respect to 0. Hence $E\mathscr{H} = \ker T$ and E is not self-adjoint.

EXAMPLE 11. Let $\{e_n\}_{n=-\infty}^{\infty}$ be a completely orthonormal basis on $\mathscr{H}$ and U be a weighted bilateral shift such as $Ue_n = \begin{cases} e_{n+1} & (\text{if } n \leqslant 0) \\ \sqrt{2}e_{n+1} & (\text{if } n \geqslant 1) \end{cases}$. Let $\mathscr{K} = \mathbb{C}e_1$ (one-dimensional subspace of $\mathscr{H}$ generated by e_1) and $S : \mathscr{K} \to \mathscr{H}$ be the operator defined by $S(\alpha e_1) = \alpha e_1$ for $\alpha \in \mathbb{C}$. Then the operator $T = \begin{pmatrix} U & S \\ 0 & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{K}$

is quasihyponormal because

$$\begin{aligned}
&T^*(T^*T-TT^*)T\\
&=\begin{pmatrix} U^* & 0\\ S^* & 0\end{pmatrix}\left\{\begin{pmatrix} U^* & 0\\ S^* & 0\end{pmatrix}\begin{pmatrix} U & S\\ 0 & 0\end{pmatrix}-\begin{pmatrix} U & S\\ 0 & 0\end{pmatrix}\begin{pmatrix} U^* & 0\\ S^* & 0\end{pmatrix}\right\}\begin{pmatrix} U & S\\ 0 & 0\end{pmatrix}\\
&=\begin{pmatrix} U^* & 0\\ S^* & 0\end{pmatrix}\begin{pmatrix} U^*U-UU^*-SS^* & U^*S\\ S^*U & S^*S\end{pmatrix}\begin{pmatrix} U & S\\ 0 & 0\end{pmatrix}\\
&=\begin{pmatrix} U^* & 0\\ S^* & 0\end{pmatrix}\begin{pmatrix} 0 & U^*S\\ S^*U & S^*S\end{pmatrix}\begin{pmatrix} U & S\\ 0 & 0\end{pmatrix}=0.
\end{aligned}$$

Hence T is also class A operator, U is invertible and 0 is an isolated point of $\sigma(T)=\{0\}\cup\sigma(U)$.

In [19], we give an example of p-quasihyponormal operator for $0<p<1$ such that it is not p-hyponormal and 0 is an isolated point of its spectrum, so we do not give such example in this paper.

An operator T is called w-hyponormal [2] if its Aluthge transform $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ satisfies

$$|\widetilde{T}|\geqslant|T|\geqslant|(\widetilde{T})^*|,$$

where $T=U|T|$ is the polar decomposition of T. w-hyponormal is a subclass of class A. See [1]. As above example, in general, the Riesz idempotent of class A operator with respect to 0 is not self-adjoint and $\ker T\neq\ker T^*$. In case of w-hyponormal operators, there is also an example T of w-hyponormal operator, Example 13, which has properties that 0 is an isolated point of $\sigma(T)$, the Riesz idempotent with respect to 0 is not self adjoint and $\ker T\neq\ker T^*$.

LEMMA 12. *Let $\mathscr{H}=\oplus_{n=0}^{\infty}\mathbb{C}^2\ni x=x_0\oplus x_1\oplus\cdots$ and define an operator T on $\mathscr{H}$ by $Tx=0\oplus Bx_0\oplus Bx_1\oplus\cdots$, where $B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then for $|\lambda|<1$, $(T-\lambda)\mathscr{H}$ is closed.*

Proof. Let V be the unilateral shift on ℓ^2. It is easy to see that $\|(V-\lambda)x\|\geqslant(1-|\lambda|)\|x\|$ for $|\lambda|<1$ and $x\in\ell^2$. Using a decomposition $\mathscr{H}=\ell^2\oplus\ell^2$, we have a representation $T=V\oplus 0$. Since $T\mathscr{H}=\operatorname{ran}V\oplus\{0\}$, $T\mathscr{H}$ is closed. For $0<|\lambda|<1$, $\|(T-\lambda)x\|=\|(V-\lambda)x_1\oplus(-\lambda x_2)\|\geqslant\min\{1-|\lambda|,|\lambda|\}\|x\|$ for $x=x_1\oplus x_2$ $(x_1,x_2\in\ell^2)$, so that $(T-\lambda)\mathscr{H}$ is also closed. □

EXAMPLE 13. Let $\mathscr{H}=\oplus_{n=0}^{\infty}\mathbb{C}^2$ and define an operator T on $\mathscr{H}$ by $T(\cdots\oplus x_{-2}\oplus x_{-1}\oplus\overset{(0)}{x_0}\oplus x_1\oplus\cdots)=\cdots\oplus Ax_{-2}\oplus\overset{(0)}{Ax_{-1}}\oplus Bx_0\oplus Bx_1\oplus\cdots$, where $A=\frac{1}{4}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}$ and $B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then T is w-hyponormal and $\sigma(T)=\{0\}\cup\{z\mid\frac{1}{4}\leqslant|z|\leqslant 1\}$. Moreover $E\mathscr{H}=\ker T$, E is not self-adjoint and $\ker T\neq\ker T^*$, where E is the Riesz idempotent with respect to 0.

Proof. For $x = \cdots \oplus x_{-2} \oplus x_{-1} \oplus \overset{(0)}{x_0} \oplus x_1 \oplus \cdots$, we have

$$\begin{aligned} T^*x &= (\cdots \oplus Ax_{-0} \oplus \overset{(0)}{B}x_1 \oplus Bx_2 \oplus \cdots), \\ |T|x &= (\oplus_{n<0} Ax_n) \oplus (\oplus_{n\geqslant 0} Bx_n), \\ |\widetilde{T}|x &= (\oplus_{n<-1} Ax_n) \oplus (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}x_{-1} \oplus (\oplus_{n\geqslant 0} Bx_n), \\ |(\widetilde{T})^*|x &= (\oplus_{n<0} Ax_n) \oplus (BAB)^{\frac{1}{2}}x_0 \oplus (\oplus_{n\geqslant 1} Bx_n). \end{aligned}$$

Since $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{2}A$ and $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} = (BAB)^{\frac{1}{2}} = \frac{1}{2\sqrt{2}}B$,

$$\begin{aligned} \langle (|\widetilde{T}| - |T|)x, x \rangle &= \langle \big((A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} - A\big)x_{-1}, x_{-1} \rangle \geqslant 0 \\ \langle (|T| - |(\widetilde{T})^*|)x, x \rangle &= \langle \big(B - (BAB)^{\frac{1}{2}}\big)x_0, x_0 \rangle \geqslant 0. \end{aligned}$$

Hence T is w-hyponormal.

(i) Let $\mathscr{H}_+ = \{(T-\lambda)x \mid x \in \mathscr{H}, x = \cdots \oplus 0 \oplus x_0 \oplus x_1 \oplus x_2 \oplus \cdots\}$, $\mathscr{H}_- = \{(T-\lambda)x \mid x \in \mathscr{H}, x = \cdots \oplus x_{-4} \oplus x_{-3} \oplus 0 \oplus \cdots\}$, and $\mathscr{H}_0 = \{(T-\lambda)x \mid x \in \mathscr{H}, x = \cdots \oplus 0 \oplus x_{-2} \oplus x_{-1} \oplus 0 \oplus \cdots\}$. Then $\mathscr{H}_+ \perp \mathscr{H}_-$. We remark that $4A$ is unitary equivalent to B. By Lemma 12, $\mathscr{H}_+$ and $\mathscr{H}_-$ are closed for $|\lambda| < \frac{1}{4}$. Since $\mathscr{H}_0$ is finite dimensional, $(T-\lambda)\mathscr{H} = (\mathscr{H}_+ \oplus \mathscr{H}_-) + \mathscr{H}_0$ is closed.

(ii) It is easy to check that

$$\begin{aligned} \ker T &= \left\{ \Big[\oplus_{n\leqslant -1} c_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Big] \oplus \Big[\oplus_{n\geqslant 0} c_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] \mid \{c_n\} \in \ell^2(\mathbb{Z}) \right\}, \\ \ker T^* &= \left\{ \Big[\oplus_{n\leqslant 0} c_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Big] \oplus \Big[\oplus_{n\geqslant 1} c_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Big] \mid \{c_n\} \in \ell^2(\mathbb{Z}) \right\}. \end{aligned}$$

Hence $\ker T \neq \ker T^*$.

(iii) If $0 < |\lambda| < \frac{1}{4}$, it is easy to check that $\ker(T-\lambda) = \ker(T-\lambda)^* = \{0\}$. Since $(T-\lambda)\mathscr{H}$ is closed by Lemma 12, we have $(T-\lambda)\mathscr{H} = \overline{(T-\lambda)\mathscr{H}} = \{\ker(T-\lambda)^*\}^\perp = \mathscr{H}$ and therefore $\lambda \notin \sigma(T)$.

(iv) If $\frac{1}{4} < |\lambda| < 1$, we have

$$\ker(T-\lambda)^* = \mathbb{C}\left(\Big[\oplus_{n<0} \frac{1}{2(4\lambda)^{|n|}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big] \oplus \Big[\oplus_{n\geqslant 0} \lambda^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Big] \right).$$

(v) It follows from (iii) and (iv) that $\sigma(T) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \frac{1}{4} \leqslant |\lambda| \leqslant 1\}$.

(vi) We finally show that the Riesz idempotent E is not self-adjoint. Since T is paranormal, we have $E\mathscr{H} = \ker T$ by the proof of Lemma 6. Suppose that E is self-adjoint. Then $E\mathscr{H} \perp (1-E)\mathscr{H}$, so that $T = 0 \oplus T_2$ for some paranormal operator on $(1-E)\mathscr{H}$ with $\ker T_2 = \{0\}$. Since T_2 is isoloid, $0 \notin \sigma(T_2)$. Hence $\ker T = E\mathscr{H} = \ker T^*$. This contradicts (ii). □

REMARK 14. An operator T is called class $A(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2$. Class $A(k)$ operators were defined by T. Furuta, M. Ito and T. Yamazaki [10]. In that paper, they proved that every invertible log-hyponormal operator is a class $A(k)$

operator for $k > 0$ and every invertible class A operator is a class $A(k)$ operator for $k \geqslant 1$.

Now we assert that the operator T of Example 13 is class $A(k)$ for $k \geqslant \frac{1}{4}$. In fact, for each $k \geqslant \frac{1}{4}$,

$$\left\langle \left(\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T|^2 \right) x, x \right\rangle = \left\langle \left((ABA)^{\frac{1}{k+1}} - A^2 \right) x_{-1}, x_{-1} \right\rangle$$
$$= \left\{ \left(\frac{1}{32}\right)^{\frac{1}{k+1}} - \frac{1}{16} \right\} \left\| \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} x_{-1} \right\|^2 \geqslant 0$$
$$\text{for all } x \in \mathscr{H}.$$

Let $0 < \alpha < 1$ and $A = \alpha \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ in Example 13. Then T is class $A(k)$ with

$$k \geqslant \frac{-\log 2}{2 \log \alpha}$$

and 0 is an isolated point of $\sigma(T)$. Also $\ker T \neq \ker T^*$ and the Riesz idempotent E with respect to 0 is not self-adjoint. Since $\dfrac{-\log 2}{2\log \alpha} \to 0$ (as $\alpha \to +0$), for any $k > 0$, there exists a class $A(k)$ operator T such that 0 is an isolated point of $\sigma(T)$, $\ker T \neq \ker T^*$ and the Riesz idempotent E with respect to 0 is not self-adjoint.

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