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A NEW EXTRAGRADIENT METHOD FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we consider and analyze a new extragradient method for solving pseudomonotone variational inequalities. The new method converges for pseudomonotone Lipschitz continuous operators, which is a weaker condition than monotonicity. The new iterative method includes the extragradient method as a special case. Our proof of convergence is very simple as compared with other methods.

1. Introduction

Variational inequalities have had a great impact and influence in the development of almost all branches of pure and applied sciences. There are a substantial number of numerical methods including projection, the Wiener-Hopf equations , auxiliary principle techniques for solving variational inequalities, see, for example, [1-12]. It is well known that the convergence of the projection method requires the operator T to be strongly monotone and Lipschitz continuous. These strict conditions rule out many applications of the projection method for a wide class of problems. These facts motivated to modify the projection method and its variant forms. The extragradient method [2] overcomes this difficulty by the technique of updating the solution, which modified the projection method by performing additional step and projection at each step according to double projection formula. Recently much attention has been given to modify the extragradient type methods, see, for example, [9, 10, 12] and the reference therein. Inspired and motivated by the research going in this areas, we suggest and analyze a new iterative method for solving variational inequalities. The convergence of our new method, that is, Algorithm 3.3, requires only the pseudomonotonicity of the Lipschitz continuous operator. In fact, our results represent a significant refinement and improvement of the previously known methods.

2. Formulation

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a closed convex set in H and $T : H \longrightarrow H$ be a

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nonlinear operator. We now consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geqslant 0, \quad \text{for all} \quad v \in K. \tag{2.1}$$

Problem (2.1) is called the variational inequality , which was introduced and studied by Stampacchia [11] in 1964. It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the variational inequalities (2.1), see [1-12] and the references therein.

If $K^* = \{u \in H : \langle u, v \rangle \geqslant 0$, for all $v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$Tu \in K^* \quad \text{and} \quad \langle Tu, u \rangle = 0, \tag{2.2}$$

which are known as the generalized complementarity problems. Such problems have been studied extensively in the literature, see, for example, [2-7].

LEMMA 2.1. *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \textit{for all} \quad v \in K, \tag{2.3}$$

if and only if $u = P_K(z)$, where P_K is the projection of H onto K. Also, the projection operator P_K is nonexpansive.

DEFINITION 2.1. For all $u, v \in H$, the operator $T : H \longrightarrow H$ is said to be
(i) *monotone*, if

$$\langle Tu - Tv, u - v \rangle \geqslant 0.$$

(ii) *pseudomonotone*, if

$$\langle Tu, v - u \rangle \geqslant 0 \quad \text{implies} \quad \langle Tv, v - u \rangle \geqslant 0.$$

(iii) *Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\|Tu - Tv\| \leqslant \delta ||u - v||^2.$$

Note that monotonicity implies pseudomonotonicity but the converse is not true, see [4].

3. Main Results

In this section, we use the projection technique to suggest some iterative methods for solving the variational inequalities. For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

LEMMA 3.1. *The function $u \in K$ is a solution of (2.1) if and only if $u \in K$ satisfies the relation*

$$u = P_K[u - \rho Tu], \tag{3.1}$$

where $\rho > 0$ *is a constant.*

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative formulation is very important from the numerical analysis point of view. This fixed-point formulation was used to suggest and analyze the following iterative method.

ALGORITHM 3.1([2]). For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T u_n], \quad n = 0, 1, 2, \dots$$

where $0 < \rho < \frac{2\alpha}{\beta^2}$, α is the strongly monotonicity constant and $\beta > 0$ is the Lipschitz continuity constant of the nonlinear operator T. We note that the projection method requires the restrictive assumption that T must be strongly monotone or co-coercive for convergence. To overcome this difficulty, a technique of updating u was used to suggest the double formula. The equation (3.1) can be written as

$$u = P_K[u - \rho T P_K[u - \rho T u]]. \tag{3.2}$$

This fixed-point formulation enables us to suggest the following iterative method, which is known as the extragradient method, see [2].

ALGORITHM 3.2 ([2]). For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T P_K[u_n - \rho T u_n)], \quad n = 0, 1, 2, \dots .$$

It is well known [2] that the convergence of the extragradient algorithm 3.2 requires the monotonicity and Lipschitz continuity of the operator. In this paper, we suggest and analyze a new iterative method for solving the variational inequalities, the convergence of which requires only the pseudomonotonicity of the Lipschitz continuity of the operator.

We now define the residue vector by the relation

$$R(u) = u - P_K[u - \rho T P_K[u - \rho T u]]. \tag{3.3}$$

From Lemma 3.1, it is clear that $u \in K$ is a solution of (2.1) if and only if $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{3.4}$$

For stepsize $\gamma \in (0,2)$, and $\eta \in (0,1)$, we rewrite the equation (3.3) in the form

$$u = P_K[u - \rho\gamma T(u - \eta R(u))]. \tag{3.5}$$

We use this fixed-point formulation to suggest and analyze the following extragradient type method for solving the variational inequalities (2.1).

ALGORITHM 3.3. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = P_K[u_n - \rho\gamma T(u_n - \eta_n R(u_n))], \quad n = 0, 1, 2, \dots \tag{3.6}$$

where $\eta_n = a^{m_k}$, and m_k is the smallest positive integer m such that

$$\|T(u_n - a^m R(u_n))\|^2 \leqslant (1 - \delta a^m) a^m \|R(u_n)\|, \tag{3.7}$$

where $\delta > 0$ is a constant constants. We remark that the search line (3.7) is well defined, see [12].

For $\eta_n = 0$, Algorithm 3.3 is exactly the extragradient Algorithm 3.2. This clearly shows that new Algorithm 3.3 includes the extragradient method as a special case. It is worth mentioning that Algorithm 3.3 is different from the methods of [9,12].

For the convergence analysis of Algorithm 3.3, we need the following results, which are proved by using the technique of Noor [9]. For the sake of simplicity, we take $\rho = 1$.

LEMMA 3.2. *Let $\overline{u} \in K$ be a solution of (2.1). If $T : H \longrightarrow H$ is a pseudomonotone Lipschitz continuous operator with constant $\delta > 0$, then*

$$\langle u - \overline{u}, T(u - \eta R(u)) \rangle \geqslant (1 - \eta\delta)\eta \|R(u)\|^2, \quad \textit{for all} \quad u \in K. \tag{3.8}$$

Proof. Since T is a pseudomonotone operator, for all $u, \bar{u} \in K$,

$$\langle T\overline{u}, u - \eta R(u) - \overline{u} \rangle \geqslant 0,$$

implies

$$\langle T(u - \eta R(u)), u - \eta R(u) - \bar{u} \rangle \geqslant 0. \tag{3.9}$$

Now consider

$$\begin{aligned}
\langle u - \bar{u}, T(u - \eta R(u)) \rangle &= \langle u - (u - \eta R(u)), T(u - \eta R(u)) \rangle \\
&+ \langle u - \eta R(u) - \bar{u}, T(u - \eta R(u)) \rangle \\
&\geqslant \eta \langle R(u), T(u - \eta R(u)) \rangle \quad \text{using (3.9) and (3.3).} \\
&= -\eta \langle R(u), Tu - T(u - \eta R(u)) \rangle + \eta \langle Tu, R(u) \rangle \\
&\geqslant -\delta\eta^2 ||R(u)||^2 + \eta \langle Tu, R(u) \rangle, \qquad (3.10)
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality, Lipschitz continuity of the operator T and (3.5).

Letting $z = u - Tu, \quad u = P_K[u - TP_K[u - Tu]], \quad v = u$ in (2.3), we obtain

$$\langle P_K[u - TP_K[u - Tu]] - u + Tu, u - P_K[u - TP_K[u - Tu]] \rangle \geqslant 0,$$

which can be written, using (3.5), as

$$\langle -R(u) + Tu, R(u) \rangle \geqslant 0,$$

that is,

$$\langle Tu, R(u) \rangle \geqslant \langle R(u), R(u) \rangle = \|R(u)\|^2. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\langle u - \bar{u}, T(u - \eta R(u))] \rangle \geqslant (1 - \eta\delta)\eta ||R(u)||^2,$$

the required result. $\square$

LEMMA 3.3. *Let $\overline{u} \in K$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.3. If (3.7) holds, then*

$$\|u_{n+1} - \overline{u}\|^2 \leqslant \|u_n - \overline{u}\|^2 - \gamma(1 - \eta_n\delta)\eta_n(2 - \gamma)\|R(u_n)\|^2. \tag{3.12}$$

Proof. From (3.6) and (3.8), we have

$$\begin{aligned}
\|u_{n+1} - \overline{u}\|^2 &= \|u_n - \overline{u} - \gamma T(u_n - \eta_n R(u_n))\|^2 \\
&\leqslant \|u_n - \overline{u}\|^2 - 2\gamma\langle u_n - \bar{u}, T(u_n - \eta_n R(u_n))\rangle + \gamma^2||T(u_n - \eta_n R(u_n))||^2 \\
&\leqslant ||u_n - \bar{u}||^2 - \gamma(1 - \delta)\eta_n(2 - \gamma)||R(u_n)||^2, \quad \text{using}(3.7). \quad \square
\end{aligned}$$

THEOREM 3.1. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 3.3 and $\bar{u} \in K$ be the solution of (2.1). Then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let u^* be a solution of (2.1). Then from (3.12), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2 - \gamma)(1 - \eta_n\delta)\eta_n||R(u_n)||^2 \leqslant ||u_0 - u^*||^2,$$

which implies that either

$$\lim_{n\to\infty} R(u_n) = 0. \tag{3.13}$$

or

$$\lim_{n\to\infty} \eta_n = 0. \tag{3.14}$$

Assume that (3.13) holds. Let $\bar{u}$ be the cluster point of $\{u_n\}$ and let the subsequence $\{u_{n_i}\}$ of the sequence $\{u_n\}$ converge to $\bar{u}$. Since R is continuous, it follows that

$$R(\bar{u}) = \lim_{i\to\infty} R(u_{n_i}) = 0,$$

which implies that $\bar{u}$ is a solution of (2.1) by invoking Lemma 3.1 and

$$||u_{n+1} - \bar{u}||^2 \leqslant ||u_n - \bar{u}||^2. \tag{3.15}$$

Thus the sequence $\{u_n\}$ has exactly one cluster point and consequently $\lim_{n\to\infty} u_n = \bar{u} \in K$, satisfying the variational inequality (2.1).

Now assume that (3.14) holds, that is, $\lim_{n\to\infty} \eta_n = 0$. Using the technique of Wang et al [12], one can easily show that $R(\bar{u}) = 0$, that is, $\bar{u} \in K$ is a solution of (2.1) by invoking Lemma 2.2 and satisfies the inequality (3.15). Repeating the above arguments, we conclude that $\lim_{n\to\infty} u_n = \bar{u} \in K$, the required result. $\square$

REMARK 3.1. Since the nonlinear complementarity problem (2.2) is a special case of the problem (2.1), Theorem 3.1 continues to hold true for the problem (2.2). In this paper, we have suggested and analyzed some new extragradient methods for pseudomonotone variational inequalities and complementarity problems. The convergence of the new method requires only the pseudomonotonicity of the operator, which is a weaker condition than monotonicity. In this respect, our results represent a significant improvement. The comparison of this new method with the other methods is an intertesting problem for further research.

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