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A MINIMAX PROBLEM ABOUT UNIT VECTORS IN THE PLANE

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Abstract. For $n \geqslant 2$, we obtain the extremal values of the minimax problem for exponential sums

$$\mu(n) := \min_{|x|=1} \max \left\{ \left| \sum_{k=0}^{n-1} x^k \right|, \left| \sum_{k=0}^{n-1} x^{kn} \right| \right\}.$$

Moreover, we show that the polynomial with coefficients 0 and 1 derived from $\mu(n)$ does not have zeros on the unit circle.

1. Introduction

For $n \geqslant 2$, Kim [2] obtained the extremal values of the minimax problem for exponential sums

$$f(n) := \min_{a_i \text{ real}} \max \left\{ \left| \sum_{k=1}^{n} e^{ia_k} \right|, \left| \sum_{k=1}^{n} e^{ina_k} \right| \right\}.$$

In fact, the values are

$$f(n) = \begin{cases} 1, & n = 2, \\ \sqrt{2c+3}, & n = 3, \\ 0, & n \geqslant 4, \end{cases}$$

where c is the real root of $1 + 2(1 + x + x^2 + x^3) = 0$, thus $f(3) = 0.769292\cdots$. In particular, Kim [2] reduced the two extremal problems $f(3)$ and

$$\max_{a,b,c \in [0,2\pi]} \min \left\{ \left| (e^{ia} - e^{ib})(e^{ib} - e^{ic})(e^{ic} - e^{ia}) \right|, \right.$$
$$\left. \left| (e^{i3a} - e^{i3b})(e^{i3b} - e^{i3c})(e^{i3c} - e^{i3a}) \right| \right\}$$

to the problems about the packing of certain convex sets in the plane and obtained the results. This new packing method also could be used to solve some other extremal problems (see [2]). While using packing method to obtain the extremal value

$$f(3) = \min_{0 \leqslant a,b < 2\pi} \max \left\{ \left| 1 + e^{ia} + e^{ib} \right|, \left| 1 + e^{i3a} + e^{i3b} \right| \right\},$$

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Kim [2] realized that this extremal value was attained when $b = 2a$. Hence we have another representation of $f(3)$ as follows;

$$f(3) = \min_{|x|=1} \max\left\{\left|1+x+x^2\right|, \left|1+x^3+x^6\right|\right\} = 0.769292\cdots. \tag{1}$$

In this paper, we consider following generalization of (1);

$$\mu(n) := \min_{|x|=1} \max\left\{\left|r_n^1(x)\right|, \left|r_n^n(x)\right|\right\} \quad (n \geqslant 2), \tag{2}$$

where $r_n^t(x) = \sum_{k=0}^{n-1} x^{kt}$. Note that switching min and max in (2) leads to trivial problems. On the other hand, there is another representation of $\mu(n)$;

$$\frac{1}{n}\,\mu(n)^2 = \min_{a\in[0,1)} \max\left\{\Delta_n(a),\, \Delta_{n^2}(na)\right\},$$

where

$$\Delta_n(x) = \sum_{-n}^{n}\left(1-\frac{|k|}{n}\right)e^{2\pi ikx} = \frac{1}{n}\left(\frac{\sin \pi nx}{\sin \pi x}\right)^2$$

is Fejĕr's kernel. In fact, for $x = e(a) := e^{2\pi ia}$, $a \in [0,1)$,

$$r_n^1(x) = \frac{e(na)-1}{e(a)-1} = \frac{e\left(\frac{na}{2}\right)}{e\left(\frac{a}{2}\right)}\frac{e\left(\frac{na}{2}\right)-e\left(-\frac{na}{2}\right)}{e\left(\frac{a}{2}\right)-e\left(-\frac{a}{2}\right)} = e\left(\frac{(n-1)a}{2}\right)\frac{\sin \pi na}{\sin \pi a},$$

and similarly

$$r_n^n(x) = e\left(\frac{n(n-1)a}{2}\right)\frac{\sin \pi n^2 a}{\sin \pi na}.$$

We will show in Section 2 that

$$\mu(n) = \begin{cases} 1, & n = 2,\\ 0.769292\cdots, & n = 3,\\ \csc\frac{(n-2)\pi}{(n-1)^2}\sin\frac{(n-2)n\pi}{(n-1)^2}, & n \geqslant 4. \end{cases}$$

While obtaining $\mu(n)$ in Section 2, we will encounter with the polynomial with coefficients 0 and 1

$$\begin{aligned} Q_n(x) := {} & x^{(n-2)n} + x^{(n-3)n} + x^{(n-4)n} + \cdots + x^{2n} + x^n + 1\\ & + x\left(x^{(n-3)n} + x^{(n-4)n} + x^{(n-5)n} + \cdots + x^{2n} + x^n + 1\right)\\ & + x^2\left(x^{(n-4)n} + x^{(n-5)n} + x^{(n-6)n} + \cdots + x^{2n} + x^n + 1\right)\\ & + \cdots\\ & + x^{n-3}\left(x^n + 1\right)\\ & + x^{n-2}. \end{aligned}$$

The zeros of polynomials with each coefficient either 0 or 1 are "near" the unit circle. More precisely, if we let W denote the complete set of zeros of polynomials from the set

$$P = \{f(x) = 1 + \sum_{j=1}^{n} \epsilon_j x^j : \epsilon_j \in \{0, 1\} \text{ for all } j\}$$

and let $\overline{W}$ denote the closure of W, then Odlyzko and Poonen [4] showed that (a) if $z \in W$, then $0.61803\cdots = \phi^{-1} < |z| < \phi = 1.61803\cdots$, where $\phi = (1+\sqrt{5})/2$, and (b) there is an open neighborhood of $\{z \in \mathbb{C} : |z| = 1, z \neq 1\}$ contained in $\overline{W}$. The bound in (a) was also proved independently in different contexts by Flatto, Lagarias, and Poonen [3] and by Solomyak [5]. In Section 3, we also show that the polynomial $Q_n(x)$ derived from $\mu(n)$ does not have zeros on the unit circle.

2. The extremal values of $\mu(n)$

It is obvious that $\mu(2) = 1$. By (1), we have $\mu(3) = \sqrt{2c+3}$, where c is the real root of $1 + 2(1 + x + x^2 + x^3) = 0$. Thus $\mu(3) = 0.769292\cdots$. Hence we only need to consider integers $n \geqslant 4$. Throughout the rest of this paper, we let n be an integer $\geqslant 4$. Our first assertion is the following.

PROPOSITION 2.1. *If $\mu(n)$ is extremal, then the two moduli are equal.*

Proof. For $0 \leqslant a < 2\pi$, let $R_n^h(a) := \left|r_n^h(e^{ia})\right|^2$ for $h = 1, n$. It is enough to show that, if $\mu(n)$ is extremal, then $R_n^1(a) = R_n^n(a)$ for some a. The extremal value of $\mu(n)$ is attained at a point a satisfying one of the following; (i) either $R_n^1(a)$ or $R_n^n(a)$ is a local minimum whose value is greater than and equal to the other value, (ii) $R_n^1(a) = R_n^n(a)$. We show that (i) is not the case either. We can easily compute that

$$R_n^1(a) = \frac{e^{ina}-1}{e^{ia}-1}\,\frac{e^{-ina}-1}{e^{-ia}-1} = \frac{1-\cos na}{1-\cos a} = \csc^2\frac{a}{2}\,\sin^2\frac{an}{2}$$

by double-angle formulas. We may assume that $0 < a < \pi$. By simple calculation,

$$\begin{aligned}\frac{d}{da}R_n^1(a) &= \csc^2\frac{a}{2}\,\sin\frac{an}{2}\left(n\cos\frac{an}{2} - \cot\frac{a}{2}\sin\frac{an}{2}\right)\\ &= \frac{1}{2}\csc^3\frac{a}{2}\,\sin\frac{an}{2}\,t_n^1(a),\end{aligned}$$

where $t_n^1(a) = (n-1)\sin(a(n+1)/2) - (n+1)\sin(a(n-1)/2)$. Observe that the zeros of $\csc^3(a/2)\sin(an/2) = 0$ are

$$a_k := \frac{2k\pi}{n} \quad \left(0 < k \leqslant \left\lfloor\frac{n}{2}\right\rfloor\right),$$

and $R_n^1(a_k) = 0$. Now we show that the only local minimum of R_n^1 is 0 at $a = a_k$. But $t_n^1(a)$ has at least $\lfloor\frac{n}{2}\rfloor - 1$ zeros since

$$t_n^1(a_k) = (n-1)\sin\frac{k(n+1)\pi}{n} - (n+1)\sin\frac{k(n-1)\pi}{n}$$

for $0 < k \leqslant \lfloor \frac{n}{2} \rfloor$, and

$$t_n^1(a_k) \begin{cases} < 0, & k \text{ odd}, \\ > 0, & k \text{ even}. \end{cases}$$

It follows from

$$\frac{d}{da} t_n^1(a) = -(n+1)(n-1)\sin\frac{a}{2}\,\sin\frac{an}{2}$$

that all zeros of $t_n^1(a)$ are a_k, where $(0 < k \leqslant \lfloor n/2 \rfloor)$. Since $(R_n^1)^2(a_k) = 0$, $R_n^1(a)$ has only local minimum 0 at $a = a_k$. On the other hand, $R_n^n(a_k) = 0$, which completes the proof. $\square$

We have that $r_n^1(1) = r_n^n(1) = n$. For x with $x^{n-1} = 1$ and $x \neq 1$, an easy computation yields that $r_n^1(x) = r_n^n(x) = 1$. Hence we have $\mu(n) \leqslant 1$. By Proposition 2.1, in order that $\mu(n)$ be extremal, we must have $|r_n^1(x)| = |r_n^n(x)|$. In the rest of this section, we assume that $x^n \neq \pm 1$, since, if $x^n = 1$, then $r_n^n(x) = n$, and if $x^n = -1$, then $|r_n^1(x)| = 2/|x-1| \geqslant 1$. In the following, we omit some easy trigonometric computations.

PROPOSITION 2.2. *Let* $x = e^{i\zeta}$, $0 < \zeta < 2\pi$, *and*

$$r_n^1(x) = \frac{x^n - 1}{x - 1} = se^{i\rho}, \quad r_n^n(x) = \frac{x^{n^2} - 1}{x^n - 1} = se^{i\theta}, \quad (s > 0).$$

If $\mu(n)$ *is extremal, then*

$$\sin(\theta - n\rho) = 0$$

and ζ *is a* $(n-1)^2$*-th root of unity.*

Proof. Let $0 < \zeta < 2\pi$ and

$$r_n^1(x) = \frac{x^n - 1}{x - 1} = se^{i\rho}, \quad r_n^n(x) = \frac{x^{n^2} - 1}{x^n - 1} = se^{i\theta}.$$

Substituting $e^{i\zeta}$ for x in $r_n^1(x) = se^{i\rho}$ derives

$$(s(\cos\rho - \cos(\zeta+\rho)) + \cos(n\zeta) - 1) + i\,(s(\sin\rho - \sin(\zeta+\rho)) + \sin(n\zeta)) = 0$$

and

$$s = \frac{1 - \cos(n\zeta)}{\cos\rho - \cos(\zeta+\rho)} = -\frac{\sin(n\zeta)}{\sin\rho - \sin(\zeta+\rho)}. \tag{3}$$

Here $\cos\rho \neq \cos(\zeta+\rho)$ and $\sin\rho \neq \sin(\zeta+\rho)$, since $x^n \neq \pm 1$. From the equation (3), we can compute that

$$\csc\frac{\zeta}{2}\csc\left(\frac{\zeta}{2}+\rho\right)\sec\left(\frac{\zeta}{2}+\rho\right)\sin\frac{n\zeta}{2}\sin\left(\frac{\zeta}{2}(1-n)+\rho\right) = 0.$$

Similarly, from $r_n^n(x) = se^{i\theta}$, we have

$$\csc\frac{n\zeta}{2}\csc\left(\frac{n\zeta}{2}+\theta\right)\sec\left(\frac{n\zeta}{2}+\theta\right)\sin\frac{n^2\zeta}{2}\sin\left(\frac{n\zeta}{2}(1-n)+\theta\right) = 0.$$

It follows from $\sin\left(\frac{\zeta}{2}(1-n)+\rho\right)=0$ and $\sin\left(\frac{n\zeta}{2}(1-n)+\rho\right)=0$ that

$$\sin(\theta-n\rho)=0$$

and

$$\zeta=\frac{2j\pi}{(n-1)^2}$$

for some integer j. $\square$

Now we compute $\mu(n)$ for $n\geqslant 4$.

THEOREM 2.3. *For $n\geqslant 4$, we have*

$$\mu(n)=\csc\frac{(n-2)\pi}{(n-1)^2}\sin\frac{(n-2)n\pi}{(n-1)^2}.$$

Proof. By Proposition 2.2, $\mu(n)$ is attained when x is a $(n-1)^2$-th root of unity. Note that $x\neq 1$. We can compute that, for $x=e^{i\frac{2k\pi}{(n-1)^2}}$ for some integer k, $0<k<(n-1)^2$,

$$r_n^1(x)=\csc\frac{k\pi}{(n-1)^2}\sin\frac{kn\pi}{(n-1)^2}\left(\cos\frac{k\pi}{n-1}-i\sin\frac{k\pi}{n-1}\right).$$

Hence, by Proposition 2.1, we have

$$\mu(n)=\min_{\substack{0<k<(n-1)^2\\ k\in\mathbb{Z}}}\csc\frac{k\pi}{(n-1)^2}\left|\sin\frac{kn\pi}{(n-1)^2}\right|.$$

But $\sin\frac{kn\pi}{(n-1)^2}$ vanishes when $k=(n-1)^2h/n\notin\mathbb{Z}$ for $0<h\leqslant n-1$. Moreover we may check that the function $\csc\frac{k\pi}{(n-1)^2}\sin\frac{kn\pi}{(n-1)^2}$ has only one critical point on each subinterval $((n-1)^2h/n,(n-1)^2(h+1)/n)$, where $0<h\leqslant n-2$. This implies that

$$\mu(n)=\min\left\{\min_{0<h\neq n-1}\alpha(h),\min_{0<h<n-1}\beta(h)\right\},$$

where

$$\alpha(h)=\csc\frac{\left(\left\lfloor\frac{h(n-1)^2}{n}\right\rfloor-1\right)\pi}{(n-1)^2}\sin\frac{\left(\left\lfloor\frac{h(n-1)^2}{n}\right\rfloor-1\right)n\pi}{(n-1)^2}$$

and

$$\beta(h)=\csc\frac{\left(\left\lceil\frac{h(n-1)^2}{n}\right\rceil-1\right)\pi}{(n-1)^2}\left|\sin\frac{\left(\left\lceil\frac{h(n-1)^2}{n}\right\rceil-1\right)n\pi}{(n-1)^2}\right|.$$

Since $(h(n-1)^2)/n=(n-2)h+h/n$, we have

$$\alpha(h)=\csc\frac{(h(n-2)-1)\pi}{(n-1)^2}\sin\frac{(h(n-2)-1)n\pi}{(n-1)^2},$$

$$\beta(h)=\csc\frac{h(n-2)\pi}{(n-1)^2}\left|\sin\frac{h(n-2)n\pi}{(n-1)^2}\right|.$$

But, for n even, the $\alpha(h)$ is minimal when $h=n/2-1$ or $h=n/2$. In fact,

$$\begin{aligned}\alpha(h) &= \csc\frac{(h(n-2)-1)\,\pi}{(n-1)^2}\left|\sin\left(h-\frac{n+h}{(n-1)^2}\right)\right| \\ &= \csc\frac{(h(n-2)-1)\,\pi}{(n-1)^2}\,\sin\left(\frac{n+h}{(n-1)^2}\pi\right).\end{aligned}$$

Observe that

$$\frac{n+h}{(n-1)^2}\pi < \frac{\pi}{2},$$

and, as h increases from 1 to $n-1$, $\frac{(h(n-2)-1)}{(n-1)^2\pi}$ is faster than $\frac{n+h}{(n-1)^2}\pi$. Since

$$\frac{(h(n-2)-1)\,\pi}{(n-1)^2} = \frac{\pi}{2} \quad \text{when } h = \frac{n}{2}+\frac{3}{2(n-2)},$$

the $\min_{0<h\leqslant n-1}\alpha(h)$ is attained when $h=n/2-1$ or $h=n/2$. But substituting $n/2-1$ and $n/2$ for h in $\frac{(h(n-2)-1)\pi}{(n-1)^2}$ give

$$\frac{n^2-4n+2}{2(n-1)^2}$$

and

$$\frac{(n^2-2n-2)\pi}{2(n-1)^2}\ \left(<\frac{\pi}{2}\right),$$

respectively. Since

$$\frac{(n^2-6)\pi}{2(n-1)^2}-\frac{\pi}{2}-\left(\frac{\pi}{2}-\frac{(n^2-2n-2)\pi}{2(n-1)^2}=\frac{(n-5)\pi}{(n-1)^2}\right)>0,$$

the minimum is attained when $h=n/2$. We can show in the similar way that, for n odd, the minimum is attained when $h=\lfloor n/2\rfloor=(n-1)/2$. Hence the $\min_{0<h\leqslant n-1}\alpha(h)$ equals

$$\begin{cases}\alpha\left(\frac{n}{2}-1\right)=\csc\frac{(n^2-4n+2)\pi}{2(n-1)^2}\sin\frac{(3n-2)\pi}{2(n-1)^2}, & n \text{ even},\\ \alpha\left(\frac{(n-1)}{2}\right)=\csc\frac{(n-3)n\pi}{2(n-1)^2}\sin\frac{(3n-1)\pi}{2(n-1)^2}, & n \text{ odd}.\end{cases}$$

Next we attain the extremal value $\min_{0<h<n-1}\beta(h)$. A calculation yields

$$\begin{aligned}(n-1)^2(\beta^2)'(h) &= (n-2)\pi\csc^3\frac{h(n-2)\pi}{(n-1)^2}\sin\frac{h(n-2)n\pi}{(n-1)^2} \\ &\quad -\left((n-1)\sin\frac{h(n^2-n-2)\pi}{(n-1)^2}+(n+1)\sin\frac{h(n-2)\pi}{n-1}\right)>0\end{aligned}$$

by considering for cases of h even and h odd, respectively. Hence

$$\min_{0<h<n-1}\beta(h)=\beta\,(1)=\csc\frac{(n-2)\pi}{(n-1)^2}\sin\frac{(n-2)n\pi}{(n-1)^2}.$$

In all, we have

$$\mu(n)=\begin{cases}\min\left\{\alpha\left(\frac{n}{2}-1\right),\ \beta(1)\right\}, & n \text{ even},\\ \min\left\{\alpha\left(\frac{n-1}{2}\right),\ \beta(1)\right\}, & n \text{ odd}.\end{cases}$$

On the other hand, for $n=2k$, we can compute that

$$\begin{aligned}&\alpha\left(\frac{n}{2}-1\right)-\beta(1)\\ =&\frac{1}{4}\csc\frac{(k-1)\pi}{(2k-1)^2}\csc\frac{(2k^2-4k+1)\pi}{2(2k-1)^2}\sec\frac{(k-1)\pi}{(2k-1)^2}\sec\frac{(2k^2-4k+1)\pi}{2(2k-1)^2}\\ &\left(\sin\frac{2(k-1)\pi}{(2k-1)^2}\sin\frac{(3k-1)\pi}{(2k-1)^2}-\sin\frac{4(k-1)k\pi}{(2k-1)^2}\right)>0.\end{aligned}$$

Also, for $n=2k+1$,

$$\begin{aligned}&\alpha\left(\frac{n-1}{2}\right)-\beta(1)\\ =&\frac{1}{8}\csc\frac{(2k-1)\pi}{8k^2}\csc\frac{(2k^2-k-1)\pi}{8k^2}\sec\frac{(2k-1)\pi}{8k^2}\sec\frac{(2k^2-k-1)\pi}{8k^2}\\ &\left(\cos\frac{(k+2)\pi}{4k^2}-\cos\frac{5\pi}{4k}+\sin\frac{\pi}{4k}-\sin\frac{(k+2)\pi}{4k^2}\right)>0.\end{aligned}$$

This completes the proof □

3. Polynomials with coefficients 0 and 1

By Proposition 2.1, if $\mu(n)$ is extremal, then $|r_n^1(x)|=|r_n^n(x)|$. In this section, we consider the case $r_n^1(x)=r_n^n(x)$. For x with $|x|=1$, $x^n\neq 1$, we have

$$r_n^1(x)=r_n^n(x)\Leftrightarrow(x-1)(x^{n^2}-1)-(x^n-1)^2=0.$$

Define, for $n\geqslant 4$,

$$\begin{aligned}F_n(x)&=\frac{(x-1)(x^{n^2}-1)-(x^n-1)^2}{x(x+1)(x-1)^3},\\ P_n(x)&=\begin{cases}\frac{x^n-1}{x^2-1}, & n \text{ even},\\ \frac{x^n-1}{x-1}, & n \text{ odd},\end{cases}\\ Q_n(x)&=x^{(n-2)n}+x^{(n-3)n}+x^{(n-4)n}+\cdots+x^{2n}+x^n+1\\ &\quad+x\left(x^{(n-3)n}+x^{(n-4)n}+x^{(n-5)n}+\cdots+x^{2n}+x^n+1\right)\\ &\quad+x^2\left(x^{(n-4)n}+x^{(n-5)n}+x^{(n-6)n}+\cdots+x^{2n}+x^n+1\right)\\ &\quad+\cdots\\ &\quad+x^{n-3}\left(x^n+1\right)\\ &\quad+x^{n-2}.\end{aligned}$$

Then the following can be verified with tedious calculation.

LEMMA 3.1. *For $n \geqslant 4$, we have*

$$F_n(x) = P_n(x)P_{n-1}(x)Q_n(x).$$

In this section, we investigate the zero distributions of the polynomial $Q_n(x)$. Define, for $n \geqslant 4$,

$$\widetilde{F_n}(x) = x(x+1)(x-1)^3 F_n(x) = (x-1)(x^{n^2}-1) - (x^n-1)^2.$$

LEMMA 3.2. *The polynomial $Q_n(x)$ does not have a zero x with $x^{n-1} = 1$.*

Proof. Suppose that x is a zero of $Q_n(x)$ such that $x^{n-1} = 1$ and $x \neq 1$. Since $x^n = x$, we have

$$Q_n(x) = (n-1)x^{n-2} + (n-2)x^{n-3} + (n-3)x^{n-4} + \cdots + 2x + 1. \tag{4}$$

Then the zeros of

$$(x-1)Q_n(x) = 0$$

are those of

$$x^{n-1} = \frac{x^{n-2} + x^{n-3} + \cdots + x + 1}{n-1}.$$

This contradicts to the fact that the average of points on the unit circle is strictly inside the unit circle unless all of the points are equal. □

We use Lemma 3.1 and Lemma 3.2 to prove the following.

THEOREM 3.3. *The polynomials $Q_n(x)$ does not have zeros on the unit circle.*

Proof. It is obvious that $Q_n(1) \neq 0$ and, for n even, $Q_n(-1) \neq 0$. It is easy to compute that, for n odd, $Q_n(-1) = 1 + (-1)(n-1)/2 = (3-n)/2 \neq 0$. Thus it is enough to show that all zeros of $\widetilde{F_n}(x)$ with modulus 1 except 1 and -1 are not the zeros of $Q_n(x)$. Let $x = e^{ia}$ is a zero of both $\widetilde{F_n}(x)$ and $Q_n(x)$. Then, by Proposition 2.2, $x^{(n-1)^2} = 1$. But $x^{n^2} = x^{2n-1}$ and

$$\begin{aligned}\widetilde{F_n}(x) &= x^{n^2+1} - x^{n^2} - x - x^{2n} + 2x^n \\ &= -x(x^{n-1}-1)^2.\end{aligned}$$

However, by Lemma 3.2, the polynomial $Q_n(x)$ does not have a zero with $x^{n-1} = 1$. This contradicts to the assumption. This proves the result. □

REMARK 3.4. By Eneström-Kakeya theorem (see p. 136 of [1]) and the proof of Lemma 3.2, the polynomial equation in (4)

$$(n-1)x^{n-2} + (n-2)x^{n-3} + (n-3)x^{n-4} + \cdots + 2x + 1 = 0$$

has all its zeros strictly inside the unit circle

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