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LOWER AND UPPER BOUNDS FOR THE PROBABILITY THAT AT LEAST r AND EXACTLY r OUT OF n EVENTS OCCUR

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Abstract. Lower and upper bounds are presented for the probability that at least r or exactly r out of n events occur, in terms of sums of joint probabilities of at most m events, where $m < n$.

1. Problem formulation and preliminary Lemmas

Let $A_1, \ldots, A_n$ be arbitrary events in an arbitrary probability space. Introduce the indicator variables:

$$X_i = \begin{cases} 1, \text{if } A_i \text{ occurs}, \\ 0, \text{otherwise}, \end{cases} \qquad i = 1, \ldots, n$$

and let

$$\nu = X_1 + \cdots + X_n.$$

Clearly, ν designates the number of events which occur. In this paper we present new lower and upper bounds for $P(\nu \geqslant r)$, and $P(\nu = r)$, where $1 \leqslant r \leqslant n$. The bounds for $P(\nu \geqslant r)$ generalize bounds presented in Prékopa, Vizvári, Regös and Gao [7], who dealt with the case of $r = 1$.

Our new bounds are different from those presented in Boros and Prékopa [1], for $P(\nu \geqslant r)$ and $P(\nu = r)$ and also from those in Sathe, Pradha and Shah [8], for $P(\nu \geqslant r)$. The new bounds are based on joint probabilities of at least r and at most $m + r$ events which are taken into account in more detailed forms, than in the above cited papers. In Section 6 we present numerical examples and in almost all of them our bounds outperform the just mentioned other ones.

In what follows, the sets

$$A_{i_1} \cap \cdots \cap A_{i_k} \cap \bar{A}_{i_{k+1}} \cap \cdots \cap \bar{A}_{i_n}, \tag{1}$$

which subdivide the sample space into 2^n disjoint parts, will be referred to as atoms. Our bounds are based on simple lemmas and linear programming formulations, where the

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optimum values provide us with the bounds. The first lemma enunciates a well-known fact. For a proof the reader is referred to Prékopa [5] (pp. 182-183).

LEMMA 1.1. *We have the equality*

$$\binom{\nu}{k} = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} X_{i_1}\cdots X_{i_k}. \tag{2}$$

LEMMA 1.2. *Let* $1 \leqslant j_1 < \cdots < j_r \leqslant n$. *We have the equality*

$$\binom{\nu-r}{k-r} X_{j_1}\cdots X_{j_r} = \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ \{i_1,\ldots,i_k\} \supset \{j_1,\ldots,j_r\}}} X_{i_1}\cdots X_{i_k}, \tag{3}$$

where $1 \leqslant k \leqslant r$.

Proof. Equation (3) holds trivially, if $k = r$. Otherwise, it can be written in the following equivalent form:

$$\binom{\nu-r}{k-r} X_{j_1}\cdots X_{j_r} = X_{j_1}\cdots X_{j_r} \sum_{\substack{1 \leqslant i_1 < \cdots < i_{k-r} \leqslant n \\ \{i_1,\ldots,i_{k-r}\} \subset \{1,\ldots,n\}\setminus\{j_1,\ldots,j_r\}}} X_{i_1}\cdots X_{i_{k-r}}. \tag{4}$$

If $X_{j_i} = 0$, for at least one i, then both sides are 0. If $X_{j_1} = \cdots = X_{j_r} = 1$, then (4) follows by (2), if we reduce the sample space determined by these equations.

LEMMA 1.3. *Let* $1 \leqslant j_1 < \cdots < j_r \leqslant n$, *and*

$$x_{j_1\cdots j_r h} = P(X_{j_1} = 1,\ldots,X_{j_r} = 1, \nu = h), \tag{5}$$

where $h \geqslant r$. *We define*

$$p_{i_1\cdots i_k} = P(X_{i_1} = 1,\ldots X_{i_k} = 1),$$

for any $1 \leqslant i_1 < \cdots < i_r \leqslant n$.

Then we have the equality:

$$\sum_{h=r}^{n} \binom{h-r}{k-r} x_{j_1\cdots j_r h} = \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ \{i_1,\ldots,i_k\} \supset \{j_1,\ldots,j_r\}}} p_{i_1\cdots i_k}, \tag{6}$$

where $k \geqslant r$.

Proof. If we take the expectations on both sides in (3), then we obtain (6).

LEMMA 1.4. *Introduce the notation:*

$$y_{j_1\cdots j_r h} = \frac{x_{j_1\cdots j_r h}}{\binom{h}{r}}, \quad h = r, \ldots, n. \tag{7}$$

We have the equalities

$$\sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} y_{j_1\cdots j_r h} = P(\nu = h) \tag{8}$$

and

$$\sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \sum_{h=r}^{n} y_{j_1\cdots j_r h} = P(\nu \geqslant r). \tag{9}$$

Proof. As $j_1, \ldots, j_r$ vary such that $1 \leqslant j_1 < \cdots < j_r \leqslant n$, any atom for which $\nu = h$ will came up $\binom{h}{r}$ times in the events $A_{j_1} \cap \cdots \cap A_{j_r}$. This implies that

$$\sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} x_{j_1\cdots j_r h} = \binom{h}{r} P(\nu = h). \tag{10}$$

Equation (10) is the same as (8). Finally,

$$\begin{aligned} \sum_{1 \leqslant j_1 < \cdots j_r \leqslant n} \sum_{h=r}^{n} y_{j_1\cdots j_r h} & \\ &= \sum_{h=r}^{n} \binom{h}{r}^{-1} \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} x_{j_1\cdots j_r h} \\ &= \sum_{h=r}^{n} P(\nu = h) = P(\nu \geqslant r). \end{aligned} \tag{11}$$

2. Linear programming formulations of the probability bounding problems

We rewrite equation (6) by replacing $\binom{h}{r} y_{j_1\cdots j_r h}$ for $x_{j_1\cdots j_r h}$ and introducing

$$S'_{j_1\cdots j_r k} = \frac{1}{\binom{k}{r}} \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ \{i_1, \ldots, i_k\} \supset \{j_1, \ldots, j_r\}}} p_{i_1\cdots i_k}. \tag{12}$$

Since

$$\binom{h-r}{k-r}\binom{h}{r} = \binom{h}{k}\binom{k}{r},$$

equation (6) takes the form

$$\sum_{h=r}^{n} \binom{h}{k} y_{j_1 \cdots j_r h} = S'_{j_1 \cdots j_r k}, \tag{13}$$

where $k \geqslant r$ and $1 \leqslant j_1 < \cdots < j_r \leqslant n$. If we consider (13) as a system of linear equations, then we observe that for a fixed $j_1, \ldots, j_r$ the right hand side values $S'_{j_1 \cdots j_r k}$, $k \geqslant r$ uniquely determine the values of $y_{j_1 \cdots j_r h}$, $h \geqslant r$. In fact, the matrix of the equation is

$$\begin{pmatrix} \binom{r}{r} & \binom{r+1}{r} & \cdots & \binom{n}{r} \\ & \binom{r+1}{r+1} & \cdots & \binom{n}{r+1} \\ & & \ddots & \vdots \\ & & & \binom{n}{n} \end{pmatrix}, \tag{14}$$

where in the blank positions we have zeros, and this matrix is clearly nonsingular. Thus, the sets of values $S'_{j_1 \cdots j_r k}$, $k \geqslant r$ and $y_{j_1 \cdots j_r h}$, $h \geqslant r$ uniquely determine each other for fixed $j_1, \ldots, j_r$, and also if $j_1, \ldots, j_r$ vary in all possible ways.

In practice the probabilities $x_{j_1 \cdots j_r h}$, hence also the values $y_{j_1 \cdots j_r h}$, are unknown but known are the $S'_{j_1 \cdots j_r k}$ for some k. Assume that we know

$$S'_{j_1 \cdots j_r k},\ k = r, r+1, \ldots, m_{j_1 \cdots j_r}, \tag{15}$$

for all $1 \leqslant j_1 < \cdots < j_r \leqslant n$.

Now we relax the equation (13) in such a way that we keep only those which have right hand side values (15). The obtained equations, together with the nonnegativity restrictions

$$y_{j_1 \cdots j_r h} \geqslant 0,\ \text{for } h = r, \ldots, n, \tag{16}$$

for all $1 \leqslant j_1 < \cdots < j_r \leqslant n$, determine a set of feasible solutions. For such a feasible solution the equation (11) is not necessarily valid any more. However, the optimal solutions of the LP's:

$$\min(\max) \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \sum_{h=r}^{n} y_{j_1 \cdots j_r h} \tag{17}$$

subject to

$$\begin{gathered} \sum_{h=r}^{n} \binom{h}{k} y_{j_1 \cdots j_r h} = S'_{j_1 \cdots j_r k},\ k = r, r+1, \ldots, m_{j_1 \cdots j_r} \\ y_{j_1 \cdots j_r h} \geqslant 0,\ h = r, r+1, \ldots, n \\ 1 \leqslant j_1 < \cdots < j_r \leqslant n \end{gathered} \tag{18}$$

are lower and upper bounds, respectively, for $P(\nu \geqslant r)$. Similarly, the optimal solutions of the LP's:

$$\min(\max) \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} y_{j_1 \cdots j_r h} \tag{19}$$
$$\text{subject to (18)}$$

are lower and upper bounds, respectively, for the probability $P(\nu = r)$.

The constraints (18) split into $\binom{n}{r}$ subconstraints such that they contain disjoint sets of variables. These variables are coupled only in the objective function, which is the sum of the objective functions of the subproblems. Thus, the minimization, as well as the maximization problem (17) splits into $\binom{n}{r}$ subproblems and the optimum value of the original problem is simply the sum of the optimum values of the subproblems.

Let us introduce the notations:

$P_{(r)}$ = probability that at least r out of the n events occur
$P_{[r]}$ = probability that exactly r out of the n events occur
$L_{(r)}$ = optimum value of the minimization problem (17)-(18)
$U_{(r)}$ = optimum value of the maximization problem (17)-(18)
$L_{[r]}$ = optimum value of the minimization problem (19)
$U_{[r]}$ = optimum value of the maximization problem (19)
$l_{j_1 .. j_r (r)}$ = optimum value of the minimization subproblem (17)-(18), corresponding to $j_1 \cdots j_r$
$u_{j_1 .. j_r (r)}$ = optimum value of the maximization subproblem (17)-(18), corresponding to $j_1 \cdots j_r$
$l_{j_1 \cdots j_r [r]}$ = optimum value of the minimization subproblem (19), corresponding to $j_1 \cdots j_r$
$u_{j_1, \ldots, j_r [r]}$ = optimum value of the maximization subproblem (19), corresponding to $j_1 \cdots j_r$.

Then, we have the relations

$$L_{(r)} = \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} l_{j_1 \cdots j_r (r)} \tag{20}$$

$$U_{(r)} = \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} u_{j_1 \cdots j_r (r)} \tag{21}$$

$$L_{[r]} = \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} l_{j_1 \cdots j_r [r]} \tag{22}$$

$$U_{[r]} = \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} u_{j_1 \cdots j_r [r]}. \tag{23}$$

Since the true probability distribution, that provides us with the input data $S'_{j_1 \cdots j_r k}$, $k = r, r+1, \ldots, m_{j_1 \cdots j_r}$, $1 \leqslant j_1 < \cdots j_r \leqslant n$, is among the feasible solutions of the constraints (18), it follows that $L_{(r)} \leqslant P_{(r)}$, $L_{[r]} \leqslant P_{[r]}$, and these bounds are sharp. For the same reason we also have the inequalities: $P_{(r)} \leqslant U_{(r)}$, $P_{[r]} \leqslant U_{[r]}$. However, it may happen, that $U_{(r)} > 1$ and $U_{[r]} > 1$. In these cases the sharp upper bounds are equal to 1. Summarizing, the sharp bounds for $P_{(r)}$ and $P_{[r]}$ are given by:

$$L_{(r)} \leqslant P_{(r)} \leqslant \mathrm{Min}(U_{(r)}, 1) \tag{24}$$

$$L_{[r]} \leqslant P_{[r]} \leqslant \mathrm{Min}(U_{[r]}, 1). \tag{25}$$

Luckily, we are able to provide simple solutions to the subproblems, which, in turn, provide us with the sharp bounds (24) and (25).

3. Solutions to the subproblems of problems (17)-(18)

Let us pick one subproblem from problem (17)-(18), and, for the sake of simplicity, suppress the subscripts $j_1, \ldots, j_r$ in the variables as well as the right hand side values. Assume that $r + m \leqslant n$ and introduce the notations:

$$A = \begin{pmatrix} \binom{r}{r} & \binom{r+1}{r} & \binom{r+2}{r} & \cdots & \binom{r+m}{r} & \cdots & \binom{n}{r} \\ & \binom{r+1}{r+1} & \binom{r+2}{r+1} & \cdots & \binom{r+m}{r+1} & \cdots & \binom{n}{r+1} \\ & & & & & & \vdots \\ & & & & \binom{r+m}{r+m} & \cdots & \binom{n}{r+m} \end{pmatrix}$$

$$y = (y_r, y_{r+1}, \cdots, y_n)^T$$

$$b = (S'_r, S'_{r+1}, \cdots, S'_{r+m})^T$$

$$c = (1, 1, \cdots, 1)^T,$$

where c has $n - r + 1$ components. Then the subproblem takes the form

$$\begin{aligned} \min(\max)\ & c^T y \\ \text{subject to} & \\ Ay &= b \\ y &\geqslant 0. \end{aligned} \tag{26}$$

The LP (26) falls into the category of totally positive linear programs (see Prékopa [4]), meaning that all minors of order $m+1$ from A and all minors of order $m+2$ from $\begin{pmatrix} c^T \\ A \end{pmatrix}$ are positive. This fact is ensured by a theorem on binomial determinants proved by Gessel and Viennot [2] and Prékopa [3], stating that any minor of the matrix $\binom{i}{k}_{i,k=0}^{n}$, that has all positive entries above its main diagonal, is positive.

In what follows we use some notions, notations and facts in linear programming. For a simple and short presentation of the basic ideas, methods and theorems of linear programming the reader is referred to Prékopa [6].

Let $a_r, \ldots, a_n$ designate the columns of the matrix A and take $m+1$ of them to form a basis B. The basis is said to be primal feasible, if $B^{-1}b \geqslant 0$, and dual feasible in the minimization (maximization) problem, if $c_B^T B^{-1} a_k \leqslant c_k$, $k = r, \ldots, n$ ($c_B^T B^{-1} a_k \geqslant c_k$, $k = r, \ldots, n$). If k is the subscript of a basic vector, then $c_B^T B^{-1} a_k = c_k$. If for all other k subscripts the inequalities are strict, then we call the basis dual non-degenerate.

The following theorem specializes the assertions of theorem 12.1, 12.2 and 12.3 in Prékopa [4] to the present case. Since the proof is short, we present it for the reader's convenience. We remark, however, that the main ideas of the proof are the same as those in the proofs of Theorems 8, 9, 10 in Prékopa [3].

THEOREM 3.1. *All bases in problem (26) are dual non-degenerate and the dual feasible bases have the following structure, described in terms of the subscripts of the basic vectors:*

$$\begin{array}{lll} & m+1 \textit{ even} & m+1 \textit{ odd} \\ \text{min } \textit{problem} & i, i+1, \ldots, j, j+1 & i, i+1, \ldots, j, j+1, n \\ \text{max } \textit{problem} & r, i, i+1, \ldots, j, j+1, n & r, i, i+1, \ldots, j, j+1. \end{array} \tag{27}$$

Proof. Let B designate the basis. Since we have the equalities

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -c_B^T B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix}^{-1} \begin{pmatrix} c_p \\ a_p \end{pmatrix} = \begin{pmatrix} 1 & -c_B^T B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} c_p \\ a_p \end{pmatrix} = \begin{pmatrix} c_p - c_B^T B^{-1} a_p \\ B^{-1} a_p \end{pmatrix},$$

it follows that

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix} \begin{pmatrix} c_p - z_p \\ d_p \end{pmatrix} = \begin{pmatrix} c_p \\ a_p \end{pmatrix}, \tag{28}$$

where $r \leqslant p \leqslant n$, $d_p = B^{-1} a_p$, $z_p = c_B^T B^{-1} a_p$. Equation (28) is a system of linear equations for the unknown components in $\left(c_p - z_p, d_p^T\right)^T$, but at this point we are interested only in the first component of it. By Cramer's rule we obtain

$$c_p - z_p = \frac{\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}}{|B|}, \quad r \leqslant p \leqslant n. \tag{29}$$

If p is a nonbasic subscript, then the determinant in the numerator is different from 0. In fact, the determinant is either a minor of the matrix A or can be made one by changing the order of the columns. Thus, all bases are dual non-degenerate.

If we use the above result, then we can state that the basis B is dual feasible in the minimization problem (26) iff $c_p - z_p > 0$ for every nonbasic p, i.e., the determinant in the numerator of (29) is positive for every nonbasic p.

This is, however,equivalent to the requirement that the basis has the structure presented in the first line of (27).

Similarly, if B is a basis in the maximization problem (26), it is dual feasible iff $c_p - z_p < 0$ for every nonbasic p, i.e., the numerator in (29) is negative for every nonbasic p. This takes us to the structure in the second line of (27).

Based on this theorem we can derive formulas for the optimum values of problem (26), if m is small. Otherwise we can give simple dual type algorithm that solves the problem for the general case. As the formulas are simple and elegant only if $m \leqslant 2$, we restrict ourselves to these cases. We have the formulas for $m = 3$, but we disregard their presentations because they are too complicated.

Case $m = 0$, S_r' is known. We mention it for the sake of completeness, because the bounds are trivial. The optimum values of the minimization and maximization problems are $S_r'/\binom{n}{r}$, and S_r', respectively, hence

$$l_{(r)} = \frac{S_r'}{\binom{n}{r}}, \quad u_{(r)} = S_r' \tag{30}$$

and the sharp bounds can be expressed by the relations

$$\frac{S_r}{\binom{n}{r}} \leqslant P_{(r)} \leqslant \min\,(S_r, 1). \tag{31}$$

Case $m = 1$, S_r', S_{r+1}' are known. Minimization problem. Any dual feasible basis is of the form $B = (a_{r+i-1}, a_{r+i})$, where $1 \leqslant i \leqslant n - r$. This is also primal feasible, i.e., optimal, if for the solutions of the equations

$$\begin{aligned} \binom{r+i-1}{r} y_{r+i-1} + \binom{r+i}{r} y_{r+i} &= S_r' \\ \binom{r+i-1}{r+1} y_{r+i-1} + \binom{r+i}{r+1} y_{r+i} &= S_{r+1}' \end{aligned} \tag{32}$$

we obtain $y_{r+i-1} \geqslant 0$, $y_{r+i} \geqslant 0$. The optimum value is $l_{(r)} = y_{r+i-1} + y_{r+i}$. The result is

$$l_{(r)} = \frac{1}{\binom{r+i-1}{r-1}} \left[\frac{r(r+1)}{r+i} S_r' - \frac{r^2(r+1)}{i(r+i)} S_{r+1}' \right], \tag{33}$$

where

$$i = 1 + \left\lfloor (r+1)\frac{S_{r+1}'}{S_r'} \right\rfloor .$$

Case $m = 1$, S_r', S_{r+1}' are known. Maximization problem. The only dual feasible basis is $B = (a_r, a_n)$. Since problem (26) has feasible solution, (we assume that the components of b have been computed exactly by the use of the real data) and finite optimum, it follows that there exists a basis which is both primal and dual feasible.

This implies that B has that property. The optimum value is $u_r = y_r + y_n$, where y_r and y_n are obtained from the equations

$$\begin{aligned} y_r \; + \; \binom{n}{r} y_n &= S_r' \\ \binom{n}{r+1} y_n &= S_{r+1}'. \end{aligned} \tag{34}$$

Simple calculation shows that

$$u_{(r)} \;=\; S_r' \;-\; \frac{r+1}{n-r}\left(1 \;-\; \frac{1}{\binom{n}{r}}\right) S_{r+1}'. \tag{35}$$

Case $m = 2$**,** S_r'**,** S_{r+1}'**,** S_{r+2}' **are known. Minimization problem.** Any dual feasible basis has the form: $B = (a_{r+i-1}, a_{r+i}, a_n)$, where $1 \leqslant i \leqslant n-r$. The equations for the basic variables are the following:

$$\begin{aligned} \binom{r+i-1}{r} y_{r+i-1} \;+\; \binom{r+i}{r} y_{r+i} \;+\; \binom{n}{r} y_n \;&=\; S_r' \\ \binom{r+i-1}{r+1} y_{r+i-1} \;+\; \binom{r+i}{r+1} y_{r+i} \;+\; \binom{n}{r+1} y_n \;&=\; S_{r+1}' \\ \binom{r+i-1}{r+2} y_{r+i-1} \;+\; \binom{r+i}{r+2} y_{r+i} \;+\; \binom{n}{r+2} y_n \;&=\; S_{r+2}'. \end{aligned} \tag{36}$$

If we introduce the new variables

$$\binom{r+i-1}{r-1} y_{r+i-1} \;=\; z_{r+i-1}, \; \binom{r+i}{r-1} y_{r+i} \;=\; z_{r+i}, \; \binom{n}{r-1} y_n \;=\; z_n, \tag{37}$$

then (36) takes the form

$$\begin{aligned} i z_{r+i-1} \;+\; (i+1) z_{r+i} \;+\; (n-r+1) z_n \;&=\; r S_r' \\ i(i-1) z_{r+i-1} \;+\; (i+1) i z_{r+i} \;+\; (n-r+1)(n-r) z_n \;&=\; r(r+1) S_{r+1}' \\ i(i-1)(i-2) z_{r+i-1} \;+\; (i+1) i (i-1) z_{r+i} & \\ +\; (n-r+1)(n-r)(n-r-1) z_n \;&=\; r(r+1)(r+2) S_{r+2}'. \end{aligned} \tag{38}$$

The condition on i that produces $z_{r+i-1} \geqslant 0$, $z_{r+i} \geqslant 0$, $z_n \geqslant 0$ can be obtained by simple calculations. After simplification the result is:

$$i \;=\; 1 \;+\; \left\lfloor \frac{-(r+1)(r+2)S_{r+2}' + (n-r-1)(r+1)S_{r+1}'}{-(r+1)S_{r+1}' + (n-r)S_r'} \right\rfloor. \tag{39}$$

It remains to solve equations (36) and obtain the optimum value $l_{(r)} = y_{r+i-1} + y_{r+i} + y_n$. The result is:

$$l_{(r)} \;=\; y_{r+i-1} \;+\; y_{r+i} \;+\; y_n$$

$$= \frac{1}{D}\frac{1}{i}\frac{1}{\binom{r+i-1}{r-1}}\begin{vmatrix} rS'_r & 1 & 1 \\ r(r+1)S'_{r+1} & i & n-r \\ r(r+1)(r+2)S'_{r+2} & i(i-1) & (n-r)(n-r-1) \end{vmatrix}$$

$$+ \frac{1}{D}\frac{1}{i+1}\frac{1}{\binom{r+i}{r-1}}\begin{vmatrix} 1 & rS'_r & 1 \\ i-1 & r(r+1)S'_{r+1} & n-r \\ (i-1)(i-2) & r(r+1)(r+2)S'_{r+2} & (n-r)(n-r-1) \end{vmatrix} \quad (40)$$

$$+ \frac{1}{D}\frac{1}{n-r+1}\frac{1}{\binom{n}{r-1}}\begin{vmatrix} 1 & 1 & rS'_r \\ i-1 & i & r(r+1)S'_{r+1} \\ (i-1)(i-2) & i(i-1) & r(r+1)(r+2)S'_{r+2} \end{vmatrix}.$$

That is,

$$l_{(r)} = \frac{1}{D}\frac{1}{i}\frac{1}{\binom{r+i}{r}}\Big[i(n-r)(rn+n+1-r^2-ri-2i-r)S'_r \quad (41)$$

$$+ (r+1)(2r^2n+nr+ri^2+2i^2-ir-n^2r-r^3-r^2-2i)S'_{r+1}$$

$$+ (r+1)(r+2)(rn-r^2-ri-i)S'_{r+2}\Big]$$

$$+ \frac{1}{D}\frac{1}{n-r+1}\frac{1}{\binom{n}{r-1}}\Big[ri(i-1)S'_r$$

$$-2r(r+1)(i-1)S'_{r+1} + r(r+1)(r+2)S'_{r+2}\Big],$$

where

$$D = \begin{vmatrix} 1 & 1 & 1 \\ i-1 & i & n-r \\ (i-1)(i-2) & i(i-1) & (n-r)(n-r-1) \end{vmatrix} = (n-r-i+1)(n-r-i). \quad (42)$$

Case $m=2$, S'_r, S'_{r+1}, S'_{r+2} are known. Maximization problem. Any dual feasible basis has the form: $B = (a_r, a_{r+i-1}, a_{r+i})$, the equations for the basic components are:

$$\begin{aligned} y_r + \binom{r+i-1}{r} y_{r+i-1} + \binom{r+i}{r} y_{r+i} &= S'_r \\ \binom{r+i-1}{r+1} y_{r+i-1} + \binom{r+i}{r+1} y_{r+i} &= S'_{r+1} \\ \binom{r+i-1}{r+2} y_{r+i-1} + \binom{r+i}{r+2} y_{r+i} &= S'_{r+2}. \end{aligned} \quad (43)$$

If we introduce the new varibales $y_r = z_r$, $\binom{r+i-1}{r} y_{r+i-1} = z_{r+i-1}$, $\binom{r+i}{r} y_{r+i} = z_{r+i}$, then equations (43) take the form:

$$\begin{aligned} z_r \; + \; z_{r+i-1} \; + \; z_{r+i} &= S_r' \\ (i-1)z_{r+i-1} \; + \; iz_{r+i} &= (r+1)S_{r+1}' \\ (i-1)(i-2)z_{r+i-1} \; + \; i(i-1)z_{r+i} &= (r+1)(r+2)S_{r+2}'. \end{aligned} \tag{44}$$

The value of i that ensures $z_r \geqslant 0$, $z_{r+i-1} \geqslant 0$, $z_{r+i} \geqslant 0$ can be obtained from the last two equations. Simple calculation shows that

$$i \;=\; 2 \;+\; \left\lfloor (r+2)\frac{S_{r+2}'}{S_{r+1}'} \right\rfloor . \tag{45}$$

For the optimum value $u_{(r)} = y_r + y_{r+i-1} + y_{r+i}$ we obtain

$$\begin{aligned} u_{(r)} \;=\; S_r' \;-\; \frac{r+1}{\binom{r+i-1}{r}} \left(2\binom{r+i-1}{r}\frac{1}{i} \;-\; \frac{r+2}{r+i} \right) S_{r+1}' \\ +\; \frac{(r+1)(r+2)}{\binom{r+i-1}{r}} \left(\binom{r+i-1}{r}\frac{1}{i(i-1)} \;-\; \frac{r+1}{(i-1)(r+i)} \right) S_{r+2}'. \end{aligned} \tag{46}$$

For the case of $r=1$ and $m=1$ or $m=2$ the above bounds coincide with known lower and upper bounds for the probability that at least one out of n events occur. For these known formulas and the references see Prékopa [5].

ALGORITHMIC BOUNDS. We have not presented formulas for $l_{(r)}$ and $u_{(r)}$ if $S_r', \ldots, S_{r+m}'$ are known and $m \geqslant 3$. However, a simple dual type algorithm for the solution of any of the (minimization or maximization) subproblems can be given. It consists of the following steps.

Step 1. Find an initial dual feasible basis B to the problem in agreement with Theorem 3.1.

Step 2. Check for $B^{-1}b \geqslant 0$. If it holds, then go to Step 4. Otherwise, go to Step 3.

Step 3. Pick any k for which $(B^{-1}b)_k < 0$. Remove the kth column from B and replace it by the uniquely determined vector which restores the dual feasible basis structure in Theorem 3.1. Go to Step 2.

Step 4. Stop, the basis is optimal and the corresponding basic solution is an optimal solution to the problem.

4. Solutions to the subproblems of problem (19)

We keep the notations introduced in Section 3 but now $c = (1, 0, \ldots, 0)^T$. We pick a subproblem, suppress the subscripts $j_1, \ldots, j_s$ and cast it in the form (26).

Below we state a theorem that can be proved by the use of similar ideas what we have used in the proof of Theorem 3.1.

THEOREM 4.1. *Any basis that does not contain a_r is dual feasible in the minimization problem (19) (and it is also dual-degenerate). All other bases in problem (19) are dual non-degenerate and have the following structure, described in terms of the basic subscripts:*

	$m+1$ *even*	$m+1$ *odd*
min *problem*	$r, r+1, i, i+1, \ldots, j, j+1$	$r, r+1, i, i+1, \ldots, j, j+1, n$
max *problem*	$r, i, i+1, \ldots, j, j+1, n$	$r, i, i+1, \ldots, j, j+1.$

Below we derive formulas for the optimum values of the problem, when $m = 0,\ 1,\ 2$.

Case $m = 0$**,** S_r' **is known.** The optimum values for the minimization and maximization problems are 0 and S_r', respectively. Thus,

$$l_{[r]} = 0, u_{[r]} = S_r'.$$

Case $m = 1$**,** S_r', S_{r+1}' **are known. Minimization problem.** The only dual feasible basis that contains a_r is (a_r, a_{r+1}). The equations for the basic components of the optimal solution are:

$$\begin{aligned} y_r + (r+1)y_{r+1} &= S_r', \\ y_{r+1} &= S_{r+1}'. \end{aligned}$$

The basis is primal feasible iff $y_r = S_r' - (r+1)S_{r+1}' \geqslant 0$. Otherwise the optimal basis does not contain a_r, and the optimum value is 0. Thus, we have the optimum value:

$$l_{[r]} = \mathrm{Max}(S_r' - (r+1)S_{r+1}',\ 0). \tag{47}$$

Case $m = 1$**,** S_r'**,** S_{r+1}' **are known. Maximization problem.** The only dual feasible basis is (a_r, a_n). The equations for the basic components of the optimal solution are the same as in (34). This time, however, y_r alone gives the optimum value:

$$u_{[r]} = S_r' - \frac{r+1}{n-r}S_{r+1}'. \tag{48}$$

Case $m = 2$**,** S_r'**,** S_{r+1}'**,** S_{r+2}' **are known. Minimization problem.** The only dual feasible basis that contains a_r is: (a_r, a_{r+1}, a_n). The equations for y_r, y_{r+1}, y_n are:

$$\begin{aligned} y_r + (r+1)y_{r+1} + \binom{n}{r} y_n &= S_r' \\ y_{r+1} + \binom{n}{r+1} y_n &= S_{r+1}' \\ \binom{n}{r+2} y_n &= S_{r+2}'. \end{aligned}$$

Since $y_n \geqslant 0$, the basis is primal feasible iff

$$y_{r+1} \;=\; S'_{r+1} \;-\; \frac{r+2}{n-r-1} S'_{r+2} \geqslant 0 \tag{49}$$

$$y_r \;=\; S'_r \;-\; (r+1)S'_{r+1} \;+\; \frac{(r+1)(r+2)}{n-r} S'_{r+2} \geqslant 0. \tag{50}$$

If both (49) and (50) hold, then $l_{[r]} = y_r$. Otherwise, $l_{[r]} = 0$.

Case $m = 2$, S'_r, S'_{r+1}, S'_{r+2} are known. Maximization problem. Any dual feasible basis has the form $B = (a_r, a_{r+i-1}, a_{r+i})$, $1 \leqslant i \leqslant n-r$. The equations for the basic components of the optimal solution are identical to those in (43). Thus, i is given by (45) but now the optimum value is equal to y_r. Simple calculation shows that

$$u_{[r]} \;=\; S'_r \;-\; 2\frac{r+1}{i} S'_{r+1} \;+\; \frac{(r+1)(r+2)}{(i-1)i} S'_{r+2}, \tag{51}$$

where, as we have mentioned above,

$$i \;=\; 2 \;+\; \left\lfloor (r+2)\frac{S'_{r+2}}{S'_{r+1}} \right\rfloor. \tag{52}$$

For the case of $r = 1$ and $m = 1$ or $m = 2$ the above bounds coincide with the known lower and upper bounds for the probability that exactly one out of n events occur. For these known formulas and the references see Prékopa [5].

For the case of a general m an algorithmic solution of the optimization problem, that produces the bound, can be given here too. It follows the same scheme as the algorithm that we have presented at the end of Section 3.

5. Summary of bounding formulas

We present the complete bounding formulas for the cases of $m = 1, 2$.

Lower and upper bounds for $P_{(r)}$. Case $m = 1$. By (20), (21), (33), (34) and (35) we obtain the formulas

$$P_{(r)} \;\geqslant\; \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \binom{r + i_{j_1 \ldots j_r} - 1}{r-1}^{-1} \left[\frac{r(r+1)}{r + i_{j_1 \ldots j_r}} S'_{j_1 \ldots j_r, r} - \frac{r^2(r+1)}{i_{j_1 \ldots j_r}(r + i_{j_1 \ldots j_r})} S'_{j_1 \ldots j_r, r+1} \right], \tag{53}$$

where

$$i_{j_1 \ldots j_r} \;=\; 1 \;+\; \left\lfloor (r+1)\frac{S'_{j_1 \ldots j_r, r+1}}{S'_{j_1 \ldots j_r, r}} \right\rfloor \tag{54}$$

and

$$P_{(r)}$$

$$\leqslant \text{ Min}\left[\sum_{1\leqslant j_1<\cdots<j_r\leqslant n}\left(S'_{j_1..j_r,r} - \frac{r+1}{n-r}\left(1 - \frac{1}{\binom{n}{r}}\right)S'_{j_1..j_r,r+1}\right), 1\right]. \quad (55)$$

Case $m = 2$**.** By (20), (21), (39) and (41), we obtain the lower bound:

$$P_{(r)} \geqslant \sum_{1\leqslant j_1<\cdots<j_r\leqslant n} l_{j_1..j_r(r)} \quad (56)$$

where

$$l_{j_1..j_r(r)} = \frac{1}{D_{j_1..j_r}}\frac{1}{i_{j_1..j_r}}\frac{1}{\binom{r+i_{j_1..j_r}}{r}}$$

$$\times\Big[i_{j_1..j_r}(n-r)(rn+n+1-r^2-ri_{j_1..j_r}-2i_{j_1..j_r}-r)S'_{j_1..j_r,r} \quad (57)$$

$$+ (r+1)(2r^2n+nr+ri^2_{j_1..j_r}-i_{j_1..j_r}r-n^2r-r^3-r^2-2i_{j_1..j_r})S'_{j_1..j_r,r+1}$$

$$+ (r+1)(r+2)(rn-r^2-ri_{j_1..j_r}-i_{j_1..j_r})S'_{j_1..j_r,r+2}\Big]$$

$$+ \frac{1}{D_{j_1..j_r}}\frac{1}{n-r+1}\frac{1}{\binom{n}{r-1}}\Big[i_{j_1..j_r}r(i_{j_1..j_r}-1)S'_{j_1..j_r,r}$$

$$-2r(r+1)(i_{j_1..j_r}-1)S'_{j_1..j_r,r+1}+r(r+1)(r+2)S'_{j_1..j_r,r+2}\Big],$$

where

$$D_{j_1..j_r} = (n-r-i_{j_1..j_r}+1)(n-r-i_{j_1..j_r}), \quad (58)$$

and

$$i_{j_1..j_r} = 1+\left\lfloor\frac{-r(r+1)(r+2)S'_{j_1..j_r,r+2}+(n-r-1)r(r+1)S'_{j_1..j_r,r+1}}{-r(r+1)S'_{j_1..j_r,r+1}+(n-r)rS'_{j_1..j_r,r}}\right\rfloor. \quad (59)$$

In case of the upper bound we use (20), (21), (45) and (46). The result is:

$$P_{(r)} \leqslant \text{ Min}\left(\sum_{1\leqslant j_1<\cdots<j_r\leqslant n} u_{j_1..j_r(r)}, 1\right), \quad (60)$$

where

$$u_{j_1..j_r(r)} = S'_{j_1..j_r,r}$$

$$-\frac{r+1}{\binom{r+i_{j_1..j_r}-1}{r}}\left(2\binom{r+i_{j_1\dots j_r}-1}{r}\frac{1}{i_{j_1\dots j_r}} - \frac{r+2}{r+i_{j_1\dots j_r}}\right)S'_{j_1\dots j_r,r+1} \qquad (61)$$

$$+\frac{(r+1)(r+2)}{\binom{r+i_{j_1\dots j_r}-1}{r}}\left(\binom{r+i_{j_1\dots j_r}-1}{r}\frac{1}{2\binom{i_{j_1\dots j_r}-1}{2}} - \frac{r+1}{(i_{j_1\dots j_r}-1)(r+i_{j_1\dots j_r})}\right)S'_{j_1\dots j_r,r+2}$$

and

$$i_{j_1\dots j_r} = 2+\left\lfloor (r+2)\frac{S'_{j_1\dots j_r,r+2}}{S'_{j_1\dots j_r,r+1}}\right\rfloor . \qquad (62)$$

Lower and upper bounds for $P_{[r]}$. The bounds given by (53)-(61) reduce to those in Prékopa, Vizvári, Regös and Gao [7] if $r=1$. Those, in turn, are more general than the well-known binomial moment bounds for $P_{(1)}$, based on S_1, S_2 and S_1, S_2, S_3, respectively.

Case $m=1$. By (22), (23), (47) and (48) we obtain the bounds:

$$P_{[r]} \geqslant \sum_{1\leqslant j_1<\dots<j_r\leqslant n} \text{Max}\left(S'_{j_1\dots j_r,r} - (r+1)S'_{j_1\dots j_r,r+1},\ 0\right) \qquad (63)$$

and

$$P_{[r]} \leqslant \text{Min}\left[\sum_{1\leqslant j_1<\dots<j_r\leqslant n}\left(S'_{j_1\dots j_r,r} - \frac{r+1}{n-r}S'_{j_1\dots j_r,r+1}\right),\ 1\right]. \qquad (64)$$

Case $m=2$. First we look at the lower bound. If

$$S'_{j_1\dots j_r,r+1} - \frac{r+2}{n-r-1}S'_{j_1\dots j_r,r+2} \geqslant 0$$

$$S'_{j_1\dots j_r,r} - (r+1)S'_{j_1\dots j_r,r+1} + \frac{(r+1)(r+2)}{n-r}S'_{j_1\dots j_r,r+2} \geqslant 0, \qquad (65)$$

then

$$l_{j_1\dots j_r[r]} = S'_{j_1\dots j_r,r} - (r+1)S'_{j_1\dots j_r,r+1} + \frac{(r+1)(r+2)}{n-r}S'_{j_1\dots j_r,r+2}, \qquad (66)$$

otherwise $l_{j_1\dots j_r[r]}=0$. By (22) the bound is given by

$$P_{[r]} \geqslant \sum_{1\leqslant j_1<\dots<j_r\leqslant n} l_{j_1\dots j_r[r]}. \qquad (67)$$

The upper bound can be obtained by the use of (23) and (51). It is given by

$$P_{[r]} \leqslant \operatorname{Min}\left[\sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \left(S'_{j_1 \ldots j_r, r} - 2 \frac{r+1}{i_{j_1 \ldots j_r}} S'_{j_1 \ldots j_r, r+1} + \frac{(r+1)(r+2)}{2 \binom{i_{j_1 \ldots j_r}}{2}} S'_{j_1 \ldots j_r, r+2} \right),\ 1 \right], \tag{68}$$

where

$$i_{j_1 \ldots j_r} = 2 + \left\lfloor (r+2) \frac{S'_{j_1 \ldots j_r, r+2}}{S'_{j_1 \ldots j_r, r+1}} \right\rfloor . \tag{69}$$

6. Numerical example

We present one example to illustrate how the method mentioned in this paper works. Let the elementary events be $\omega_1, \ldots, \omega_{15}$ and $x_1, \ldots, x_{15}$ the corresponding probabilities, respectively. We define three event sequences $A_j^{(k)},\ j = 1, \ldots, 20,\ k = 1,\ 2,\ 3$ and the matrices $R^{(k)} = (r_{ij}^{(k)})$ where $r_{ij}^{(k)} = 1$ if $\omega_i \in A_j^{(k)}$ and $r_{ij}^{(k)} = 0$ if $\omega_i \notin A_j^{(k)},\ k = 1, 2, 3$. We present the matrices $R^{(1)}$, $R^{(2)}$, $R^{(3)}$ in detailed forms.

$$R^{(1)} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$R^{(2)} = \begin{pmatrix}
0&1&0&1&0&1&0&0&1&0&1&0&1&0&0&1&0&0&1&0\\
1&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&0&1&0&1\\
1&0&1&1&1&0&0&1&0&1&0&0&1&0&0&0&0&0&1&0\\
0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&0&0&1&0&0\\
0&1&1&1&1&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0\\
1&0&0&0&0&0&1&0&1&1&1&1&1&0&1&0&0&0&0&1\\
1&1&0&0&0&0&1&1&0&1&0&0&1&0&0&1&0&0&0&0\\
1&0&0&1&0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1\\
0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&1\\
1&1&0&1&0&0&0&1&0&1&0&0&0&1&0&0&0&0&1&0\\
0&1&0&1&1&0&0&0&1&1&0&1&0&0&1&1&0&0&0&1\\
1&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&1\\
0&1&0&0&1&1&0&0&0&1&0&0&1&0&0&0&0&1&0&0\\
0&1&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\
0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}$$

$$R^{(3)} = \begin{pmatrix}
1&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&1\\
0&1&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0\\
1&0&1&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0\\
0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
1&0&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&1&0\\
0&1&0&0&1&0&1&0&1&0&1&1&0&0&1&0&1&0&0&1\\
1&0&0&1&0&1&0&1&0&0&1&0&1&0&0&1&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&1&0&0&0&0&1&0&1&0&1\\
1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&0&0&1&0&1\\
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1\\
0&0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
1&0&0&1&0&1&0&0&0&1&0&0&1&0&1&0&0&1&0&1\\
0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&0\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0
\end{pmatrix}$$

For each event sequence, we compute the bounds for $P(\nu \geqslant 2)$ for three different probability distributions.

Distribution 1: $x_1 = 0.012$, $x_2 = 0.022$, $x_3 = 0.023$, $x_4 = 0.033$, $x_5 = 0.034$, $x_6 = 0.044$, $x_7 = 0.045$, $x_8 = 0.055$, $x_9 = 0.056$, $x_{10} = 0.066$, $x_{11} = 0.011$, $x_{12} = 0.077$, $x_{13} = 0.078$, $x_{14} = 0.088$, $x_{15} = 0.356$.

Distribution 2: $x_1 = 0.023$, $x_2 = 0.034$, $x_3 = 0.045$, $x_4 = 0.056$, $x_5 = 0.067$, $x_6 = 0.078$, $x_7 = 0.067$, $x_8 = 0.056$, $x_9 = 0.045$, $x_{10} = 0.038$, $x_{11} = 0.067$, $x_{12} = 0.022$, $x_{13} = 0.033$, $x_{14} = 0.044$, $x_{15} = 0.315$.

Distribution 3: $x_1 = 0.0329$, $x_2 = 0.1076$, $x_3 = 0.0599$, $x_4 = 0.1108$, $x_5 = 0.042$, $x_6 = 0.055$, $x_7 = 0.0508$, $x_8 = 0.1142$, $x_9 = 0.048$, $x_{10} = 0.0235$, $x_{11} = 0.0676$, $x_{12} = 0.0295$, $x_{13} = 0.0441$, $x_{14} = 0.1265$, $x_{15} = 0.1371$.

All the upper bounds are 1. The lower bounds are presented in Table 1.

Table 1: Lower Bounds for the Numerical Example

Distribution	Event Sequence	Our method	SPS	BP	$S_1,\ S_2,\ S_3$
Distribution1	1	0.9061	0.8228	0.8228	0.8509
Distribution1	2	0.9022	0.8244	0.8244	0.8530
Distribution1	3	0.9821	0.8707	0.8707	0.8988
Distribution2	1	0.8777	0.8239	0.8239	0.8563
Distribution2	2	0.8900	0.8272	0.8272	0.8499
Distribution2	3	0.9784	0.8398	0.8398	0.8796
Distribution3	1	0.9379	0.9132	0.9132	0.9317
Distribution3	2	0.9097	0.9089	0.9089	0.9247
Distribution3	3	0.9896	0.9358	0.9358	0.9459

In Table 1, we compare our results with the results obtained by the formulae in Sathe, Pradhan and Shah [8], Boros and Prékopa [1] and optimal solutions obtained by using S_1, S_2 and S_3. In Sathe, Pradhan and Shah [8], for any integer k, they defined

$$U_k = S_1 - k$$
$$V_k = S_2 - \frac{k(k-1)}{2},$$

where

$$S_m = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} P(A_{i_1} \ldots A_{i_m}),\ m = 1,\ 2,\ 3. \tag{70}$$

Hence, if $2V_{r-1} < (n+r-2)U_{r-1}$, then

$$P(\nu \geqslant r) \geqslant 2\frac{(i-1)U_{r-1} - V_{r-1}}{(i-r)(i-r+1)}, \tag{71}$$

where

$$i = 1 + \left\lfloor \frac{2V_{r-1}}{U_{r-1}} \right\rfloor. \tag{72}$$

In Boros and Prékopa [1], the bounds for $P(\nu \geqslant r)$ in case of $m = 2$ is given as follows:

$$P(\nu \geqslant r) \geqslant \frac{(r-1)(r-2i-2) + 2iS_1 - 2S_2}{(i-r+2)(i-r+1)}, \tag{73}$$

where

$$i = \left\lfloor \frac{2S_2 - (r-2)S_1}{S_1 - (r-1)} \right\rfloor. \tag{74}$$

In Table 1, column "Our method" corresponds to the results obtained by the use of (53) and (54), column "SPS ", column "BP" correspond to the results obtained by (71) and (73) respectively, column "$S_1,\ S_2,\ S_3$" corresponds to the optimal solution of the following linear programming problem:

$$\min \sum_{i=r}^{n} v_i$$

subject to

$$\sum_{i=m}^{n} \binom{i}{m} v_i = S_m,\ m = 0,1,2,3 \tag{75}$$

$$v_i \geqslant 0,\ i = 0,1,\ldots,n,$$

where $S_0 = 1$ and $S_m,\ m = 1,2,3$ are defined as (70). From Table 1, we observe that the bounds proposed in this paper outperforms the other three bounds in most cases. Meanwhile, we observe that when $r = 2$, the bounds from Sathe, Pradhan and Shah [8] are exactly the same as those from Boros and Prékopa [1]. This fact can be easily shown by replacing r by 2 in (71) and (73).

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