Mathematical
Inequalities
& Applications
Volume 5, Number 3 (2002), 349–359

INEQUALITITES FOR CAUCHY MEAN VALUES

LÁSZLÓ LOSONCZI

Abstract. The Cauchy Mean Value Theorem for divided differences (see e.g. [16]) states the following:

Suppose that $x_1 \leqslant \cdots \leqslant x_n$ *and* $f^{(n-1)}, g^{(n-1)}$ *exist, with* $g^{(n-1)} \neq 0$, *on* $[x_1, x_n]$. *Then there is a* $t \in [x_1, x_n]$ *(moreover* $t \in (x_1, x_n)$ *if* $x_1 < x_n$ *) such that*

$$\frac{[x_1, \ldots, x_n]_f}{[x_1, \ldots, x_n]_g} = \frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}$$

where $[x_1, \ldots, x_n]_f$ *denotes the divided difference of* f *at the points* $x_1, \ldots, x_n$.

If the function $\frac{f^{(n-1)}}{g^{(n-1)}}$ is invertible then

$$t = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1} \left(\frac{[x_1, \ldots, x_n]_f}{[x_1, \ldots, x_n]_g}\right)$$

is a mean value of $x_1, \ldots, x_n$. It is called the *Cauchy mean of the numbers* $x_1, \ldots, x_n$ and will be denoted by $\mathcal{D}_{f,g}(x_1, \ldots, x_n)$.

In this survey paper we discuss the equality, homogeneity of Cauchy means and inequalities of general nature: comparison, Minkowski's inequality of (homogeneous) Stolarsky's means and also the comparison and general comparison of Cauchy means.

1. The definition of Cauchy means

As it is well known, the Cauchy mean value theorem of the differential calculus states the following.

If f, g *are continuous real functions on* $[x_1, x_2]$ *which are differentiable in* (x_1, x_2), *and* $g'(u) \neq 0$ *for* $u \in (x_1, x_2)$ *then there is a point* $t \in (x_1, x_2)$ *such that*

$$\frac{f'(t)}{g'(t)} = \frac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}.$$

Assuming now that $\frac{f'}{g'}$ is invertible we get

$$t = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}\right).$$

Mathematics subject classification (2000): 26A15.

Key words and phrases: Cauchy mean value, divided differences, inequalities.

This research has been supported by OTKA Grant T 030082.

© ELEMENT, Zagreb
Paper MIA-05-37

This number t is called the *Cauchy mean value of the numbers* x_1, x_2 and will be denoted by $t = \mathcal{D}_{f,g}(x_1, x_2)$.

It is possible to define the Cauchy mean value for several variables. To do so we need a mean value theorem for divided differences.

For a function $f : I \to \mathbb{R}$, I being a real interval, the divided differences of f on *distinct* points $x_i \in I$ are usually defined inductively by

$$[x_1]_f := f(x_1),$$

$$[x_1, \dots, x_n]_f := \frac{[x_1, \dots, x_{n-1}]_f - [x_2, \dots, x_n]_f}{x_1 - x_n} \quad (n = 2, 3, \dots)$$

(see e.g. Aumann and Haupt [4] §3.17, their expression contains an extra factor $n-1$ on the right).

This definition must be modified if two or more points of $[x_1, \dots, x_n]_f$ coincide: if at most r points x_i coincide, the definition is then framed on the assumption that f is $(r-1)$-times differentiable on I. In the case $n = 2$ for example we obtain

$$[x_1, x_2]_f := \begin{cases} \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} & (x_1 \neq x_2), \\ f'(x_1) & (x_1 = x_2). \end{cases}$$

A full definition, as the ratio of two determinants, can be found in Schumaker [41].

Some basic properties of the divided differences are as follows:

1. A divided difference $[x_1, \dots, x_n]_f$ is independent of the order of its arguments $x_1, \dots, x_n$.

2. The second line of the above inductive definition remains valid provided only that $x_1 \neq x_n$.

3. A divided difference is a linear functional, i.e. we have

$$[x_1, \dots, x_n]_{af+bg} = a[x_1, \dots, x_n]_f + b[x_1, \dots, x_n]_g$$

for arbitrary constants a, b and arbitrary (suitably differentiable) functions f, g.

4. (Mean value theorem) If f is $(n-1)$-times differentiable on I and $x_i \in I \, (i = 1, \dots, n)$, then there is a t between the smallest and largest x_i (strictly between if the x_i are not all the same) such that

$$[x_1, \dots, x_n]_f = \frac{f^{(n-1)}(t)}{(n-1)!}.$$

5. The "Leibniz rule" for divided differences

$$[x_1, \dots, x_n]_{fg} = \sum_{i=1}^{n} [x_1, \dots, x_i]_f \cdot [x_i, \dots, x_n]_g.$$

6. The rule of adding an extra point to a divided difference:

$$[x_2, \dots, x_n]_g = [x_1, \dots, x_n]_h, \ \ h(x) := (x - x_1) g(x).$$

7. Differentiation with respect to a singly-occurring entry results in a repetition of that entry

$$\frac{d}{dx_k}[x_1,\dots,x_n]_f = [x_1,\dots,x_n,x_k]_f \quad (k=1,\dots,n).$$

8. If $f^{(n-1)}$ is continuous then $[x_1,\dots,x_n]_f$ is a continuous function of $(x_1,\dots,x_n)$.
9. If f is analytic then $[x_1,\dots,x_n]_f$ is analytic in $(x_1,\dots,x_n)$.
10. If $f^{(n-1)}$ is continuous then

$$[x_1,\dots,x_n]_f = \int\limits_{S_{n-1}} f^{(n-1)}(t)\,d\mu$$

where

$$S_{n-1} := \{\mu = (\mu_1,\dots,\mu_{n-1}) \ : \ \mu_k \geqslant 0, k=1,\dots,n-1 \text{ and } \sum_{k=1}^{n-1}\mu_k \leqslant 1\}$$

is a simplex in $\mathbb{R}^{n-1}$ and

$$t = x_n + \sum_{k=1}^{n-1}\mu_k(x_k - x_n) = \sum_{k=1}^{n-1}\mu_k x_k + \left(1 - \sum_{k=1}^{n-1}\mu_k\right)x_n.$$

This formula is equivalent to the one given by Steffenson [42], p.17 and it is valid even if some (or all) of the points $x_1,\dots,x_n$ coalesce.

The following mean value theorem (the Cauchy mean value theorem for divided differences) is due to Leach and Sholander [16] (see also Rätz and Russell [35], Páles [29]).

THEOREM LS. *Let $x_1 \leqslant \cdots \leqslant x_n$ and assume that $f^{(n-1)}, g^{(n-1)}$ exist, with $g^{(n-1)}(u) \neq 0$, on $[x_1,x_n]$. Then there is a $t \in [x_1,x_n]$ (moreover $t \in (x_1,x_n)$ if $x_1 < x_n$) such that*

$$\frac{[x_1,\dots,x_n]_f}{[x_1,\dots,x_n]_g} = \frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}.$$

Supposing that the function $u \to \dfrac{f^{(n-1)}(u)}{g^{(n-1)}(u)}$ is invertible we get that

$$t = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1}\left(\frac{[x_1,\dots,x_n]_f}{[x_1,\dots,x_n]_g}\right)$$

is a mean value of $x_1,\dots,x_n$ which, by property 1., is symmetric in its variables. It is called the *Cauchy* (*or difference*) *mean of the numbers* $x_1,\dots,x_n$ and will be denoted by $\mathcal{D}_{f,g}(x_1,\dots,x_n)$. This mean value was first defined and examined by Leach and Sholander [16] (they called it *extended* (f,g) mean of $x_1,\dots,x_n$).

To formulate our results easier we introduce some notations. For a real interval I and for a fixed integer $n \geqslant 2$ let $\mathcal{E}_n(I)$ denote the set of all pairs (f,g) of functions $f,g : I \to \mathbb{R}$ satisfying the following conditions:

- (i) f, g are n-times differentiable on I,
- (ii) $g^{(n-1)}(u) \neq 0$ for $u \in I$,
- (iii) the (first) derivative of $\dfrac{f^{(n-1)}}{g^{(n-1)}} = \dfrac{f^{(n)}g^{(n-1)} - f^{(n-1)}g^{(n)}}{\left(g^{(n-1)}\right)^2}$ is not zero on I.

In the sequel we use the notations

$$\bar{f} = f^{(n-1)}, \ \bar{g} = g^{(n-1)}, \ h = \frac{\bar{f}}{\bar{g}}$$

for any pair $(f, g) \in \mathcal{E}_n(I)$ where, for the sake of simplicity, we suppressed the dependence of the functions on n.

We remark that $(f, g) \in \mathcal{E}_n(I)$ implies that the functions $\bar{g}$ (and h') have constant sign on I. Namely if $\bar{g}$ (and h') assumed both positive and negative values then by the intermediate value property of the derivative they assume also zero value somewhere which contradicts the assumption (ii) (and (iii)). This also implies that h is strictly monotonic thus invertible. If $(f, g) \in \mathcal{E}_n(I)$ then the representation (coming from property 10.)

$$\mathcal{D}_{f,g}(x_1, \ldots, x_n) = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1} \left(\frac{\int_{S_{n-1}} f^{(n-1)}(t)\, d\mu}{\int_{S_{n-1}} g^{(n-1)}(t)\, d\mu}\right)$$

is valid which shows that $\mathcal{D}_{f,g}(x_1, x_2, \ldots, x_n)$ exists for every possible choice of $x_1, x_2, \ldots, x_n \in I$.

As usual $\mathscr{C}_n(I)$ denotes the set of all functions $f : I \to \mathbb{R}$ which have continuous nth derivative on the interval I.

2. Equality and homogeneity of Cauchy means

The equality problem of Cauchy means is the following: find necessary and sufficient conditions for the functional equation

$$\mathcal{D}_{f,g}(x_1, x_2, \ldots, x_n) = \mathcal{D}_{F,G}(x_1, x_2, \ldots, x_n) \qquad (x_1, x_2, \ldots, x_n \in I) \tag{1}$$

to hold where $n \geqslant 2$ is a fixed integer.

The solution for $n \geqslant 3$ is given by

THEOREM 1. (Losonczi [20]) *Suppose that I is a real interval, $n \geqslant 3$ is a fixed natural number and $(f, g), (F, G) \in \mathcal{E}_n(I), f, g, F, G \in \mathscr{C}_{n+2}(I)$.*

The functional equation (1) holds if and only if there exist constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma \neq 0$ such that for all $x \in I$

$$\begin{cases} f^{(n-1)}(x) &= \alpha F^{(n-1)}(x) + \beta G^{(n-1)}(x) \\ \\ g^{(n-1)}(x) &= \gamma F^{(n-1)}(x) + \delta G^{(n-1)}(x) \end{cases} \tag{2}$$

is satisfied.

The assumption $(f,g)\in\mathcal{E}_n(I)$ may impose further conditions on the constants $\alpha,\beta,\gamma,\delta$ which we do not specify here.

If we allow $n=2$ only in (1) then we need to assume even stronger differentiability conditions than in theorem 1 and in addition to (2) we get 32 new families of solutions (see Losonczi [22]).

The *homogeneity equation* on $\mathbb{R}_+=(0,\infty)$ is the functional equation

$$\mathcal{D}_{f,g}(t\mathbf{x})=t\mathcal{D}_{f,g}(\mathbf{x})\quad(\mathbf{x}\in\mathbb{R}_+^n,\ t\in\mathbb{R}_+)\tag{3}$$

expressing the fact the $\mathcal{D}_{f,g}$ is a homogeneous function of first degree.

The solution of (3) is known only for the case $n\geqslant 3$.

THEOREM 2. (Losonczi [23]) *Suppose that $n\geqslant 3$ is a fixed natural number, $(f,g)\in\mathcal{E}_n(\mathbb{R}_+)$ and $f,g\in\mathscr{C}_{n+2}(\mathbb{R}_+)$.*

Then all Cauchy mean values $\mathcal{D}_{f,g}$ satisfying the homogeneity equation (3) are generated by the functions f,g for which

$$f^{(n-1)}(t)=t^{\alpha},\qquad g^{(n-1)}=t^{\beta},\qquad(t\in\mathbb{R}_+),\tag{4}$$

$$f^{(n-1)}(t)=t^{\beta}\ln t,\qquad g^{(n-1)}=t^{\beta},\qquad(t\in\mathbb{R}_+)\tag{5}$$

where $\alpha,\beta\in\mathbb{R}$ are arbitrary constants apart from the restrictions $\alpha\neq\beta$.

The functions (4), (5) generate homogeneous Cauchy means in the two variable case too (however *there may exist other two variable* homogeneous Cauchy means). The two variable means have a very simple form and it is worth to calculate them. First we find f,g by integration from (4), (5) distinguishing the cases $\alpha,\beta=-1$ and $\alpha,\beta\neq-1$ and introducing new constants a,b. They are

$$\mathcal{D}_{f,g}(x,y)=D_{a,b}(x,y)=\left(\frac{(x^a-y^a)b}{(x^b-y^b)a}\right)^{1/(a-b)}\qquad(ab(a-b)\neq0)$$

$$\mathcal{D}_{f,g}(x,y)=D_{a,b}(x,y)=\left(\frac{x^a-y^a}{(\ln x-\ln y)a}\right)^{1/(a)}\qquad(a\neq0,b=0)$$

$$\mathcal{D}_{f,g}(x,y)=D_{a,b}(x,y)=\left(\frac{(\ln x-\ln y)b}{x^b-y^b}\right)^{1/(-b)}\qquad(a=0,b\neq0)$$

$$\mathcal{D}_{f,g}(x,y)=D_{a,b}(x,y)=e^{\left(\frac{x^a\ln(x^a/e)-y^a\ln(y^a/e)}{(x^a-y^a)a}\right)}\qquad(a\neq0,a=b)$$

$$\mathcal{D}_{f,g}(x,y)=D_{a,b}(x,y)=\sqrt{xy}\qquad(a=b=0)$$

where $a,b\in\mathbb{R}$ are constants with the restrictions indicated, $x\neq y$ and $x,y\in\mathbb{R}_+$.

The Cauchy mean

$$\mathcal{D}_{f,g}(x,y)=D_{a,c}(x,y)=e^{\frac{1}{c}\arctan\frac{p(x)-p(y)}{q(x)-q(y)}}\quad(c\neq0)$$

for

$$x, y \in I_c = (e^{-\frac{\pi}{2|c|}}, e^{\frac{\pi}{2|c|}})$$

where

$$\begin{aligned} p(x) &= x^a(-c\cos(\ln x^c) + a\sin(\ln x^c)) \qquad (a, c \in \mathbb{R}, c \neq 0) \\ q(x) &= x^a(a\cos(\ln x^c) + c\sin(\ln x^c)) \qquad (a, c \in \mathbb{R}, c \neq 0). \end{aligned}$$

is also of interest. Although it is not homogeneous in the sense of (3) it satisfies a *generalized homogeneity* equation:

$$D_{a,c}(tx, ty) = tD_{a,c}(x, y)$$

for all $x, y \in I_c$ and for those t' s for which $tx, ty \in I_c$ holds.

3. Inequalities for homogeneous Cauchy means

The Cauchy means $D_{a,b}$ were introduced by Stolarsky [43].
The power mean

$$M_a(x, y) := \begin{cases} \left(\dfrac{x^a + y^a}{2}\right)^{1/a} & \text{if } a \neq 0 \\ \sqrt{xy} & \text{if } a = 0 \end{cases}$$

with exponent a can be obtained as $D_{2a,a}$ thus $D_{2,1}$, $D_{0,0}$, and $D_{-2,-1}$ are the arithmetic, geometric, and harmonic means respectively. $D_{0,1}$ and $D_{1,1}$ are called logarithmic and identric means, respectively.

Several particular inequalities involving $D_{a,b}$ have been studied, among others, by Allasia, Giordano and Pečarić [2], Alzer [3], Brenner [5], Brenner and Carlson [6], Burk [7], Carlson [8], Dodd [11], Leach and Sholander [14], Lin [17], Pittinger [32], [33], Sándor [36], [37], [38], Seiffert [39], [40], Stolarsky [43], [44], Székely [45]. Neuman [25] studied multivariable weighted logarithmic means, Pečarić and Simić [31] introduced n-dimensional (homogeneous) weighted means (called Stolarsky-Tobey means), Pearce, Pečarić and Šunde [30] generalized Pólya's inequality to Stolarsky and Gini means.

Some of these means have applications in electrostatics [1], [34], in heat conductions, chemical problems [46] and particular recent interest is the occurrence of these means in signal processing theory in connection with time-frequency distributions [13].

There are only a few inequalities of general nature for the homogeneous Cauchy means: the comparison problem and Minkowski's inequality.

The comparison problem

$$D_{a,b}(x, y) \leqslant D_{c,d}(x, y) \quad (x, y \in \mathbb{R}_+)$$

was solved by Leach and Sholander [15]. Páles [27] gave a new proof for this result. In [28] Páles solved the comparison problem on any subinterval $[\alpha, \beta]$ of $\mathbb{R}_+$.

THEOREM 3. (Páles, [28]) *Let* $0 < \alpha < \beta < \infty$. *The inequality*

$$D_{a,b}(x,y) \leqslant D_{c,d}(x,y) \quad (x,y \in [\alpha,\beta])$$

holds if and only if the inequalities

$$a+b \leqslant c+d \quad \text{and} \quad D_{a,b}(\alpha,\beta) \leqslant D_{c,d}(\alpha,\beta)$$

are satisfied.

On the interval $\mathbb{R}_+$ the necessary and sufficient conditions can be formulated in terms of the constants a,b,c,d. Here we formulate the result of Páles [27] in the more general form of Czinder and Páles [9].

THEOREM 4. *Let* $a,b,c,d \in \mathbb{R}$. *The inequality*

$$D_{a,b}(x,y) \leqslant D_{c,d}(x,y) \quad (x,y \in \mathbb{R}_+)$$

holds if and only if the conditions

$$a+b \leqslant c+d$$

and

$$l(a,b) \leqslant l(c,d), \qquad e(a,b) \leqslant e(c,d)$$

are satisfied where the functions $l,e : \mathbb{R}^2 \to \mathbb{R}$ *are defined by*

$$l(x,y) := \begin{cases} \dfrac{x-y}{\ln x/y} & \text{if } 0 < xy,\ x \neq y, \\ x & \text{if } 0 < xy,\ x = y, \\ 0 & \text{otherwise,} \end{cases}$$

$$e(x,y) := \begin{cases} \dfrac{|x|-|y|}{x-y} & \text{if } x \neq y, \\ \operatorname{sgn} x & \text{if } x = y. \end{cases}$$

Concerning Minkowski's inequality we have

THEOREM 5. (Losonczi-Páles [24]) *The Minkowski's inequality*

$$D_{a,b}(x_1+x_2, y_1+y_2) \leqslant D_{a,b}(x_1,y_1) + D_{a,b}(x_2,y_2) \qquad (x_1,x_2,y_1,y_2 \in \mathbb{R}_+)$$

holds if and only if

$$a+b \geqslant 3, \qquad \min\{a,b\} \geqslant 1.$$

THEOREM 6. (Losonczi-Páles [24]) *The inequality*

$$D_{a,b}(x_1+x_2, y_1+y_2) \geqslant D_{a,b}(x_1,y_1) + D_{a,b}(x_2,y_2) \qquad (x_1,x_2,y_1,y_2 \in \mathbb{R}_+)$$

holds if and only if

$$a+b \leqslant 3, \qquad \min\{a,b\} \leqslant 1.$$

Páles and Czinder [9] studied the Minkowski-type inequality

$$D_{a_1,b_1}(x_1+x_2, y_1+y_2) \leqslant D_{a_2,b_2}(x_1,y_1) + D_{a_3,b_3}(x_2,y_2) \qquad (x_1,x_2,y_1,y_2 \in \mathbb{R}_+)$$

inequality and its reverse.

4. Comparison of Cauchy means

The *comparison problem* for the Cauchy means is the following. Find necessary and sufficient conditions for the functions f, g, F, G such that the inequality

$$\mathcal{D}_{f,g}(x_1, x_2, \ldots, x_n) \leqslant \mathcal{D}_{F,G}(x_1, x_2, \ldots, x_n) \qquad (x_1, x_2, \ldots, x_n \in I). \tag{6}$$

holds where $n \geqslant 2$ is a *fixed integer*.

The solution of this problem is not yet known. There are however necessary conditions (which are not sufficient) and also sufficient conditions (which are not necessary). These conditions in the special cases $g = G$, $f = F$ and $h = H$ coincide, giving necessary and sufficient conditions.

THEOREM 7. (Losonczi [21]) *Suppose that* $(f, g), (F, G) \in \mathcal{E}_n(I)$ *and* $f, g, F, G \in \mathscr{C}_{n+1}(I)$. *Then the inequality*

$$\frac{h''(x)}{h'(x)} + 2\frac{\bar{g}'(x)}{\bar{g}(x)} \leqslant \frac{H''(x)}{H'(x)} + 2\frac{\bar{G}'(x)}{\bar{G}(x)} \qquad (x \in I) \tag{7}$$

is necessary for (6) to hold (where the notations introduced at the end of Section 1 were used).

To conclude (7) it is enough to assume (6) only for the values $x_1 \in [x - \epsilon, x + \epsilon] \cap I, x_2 = \cdots = x_n = x$ for all $x \in I$ where ϵ is a positive number. Due to this we cannot expect (7) to be sufficient for (6). If however in (7) the inequality is strict (for a fixed value $x \in I$) then it can be proved that (6) holds if $(x_1, x_2 \ldots, x_n)$ is near to $(x, x, \ldots, x)$.

THEOREM 8. (Losonczi [21]) *Suppose that* $(f, g), (F, G) \in \mathcal{E}_n(I)$. *Then the inequality*

$$\frac{h(u) - h(v)}{h'(v)} \frac{\bar{g}(u)}{\bar{g}(v)} \leqslant \frac{H(u) - H(v)}{H'(v)} \frac{\bar{G}(u)}{\bar{G}(v)} \quad (u, v \in I) \tag{8}$$

is sufficient for (6) to hold.

In the special case $I = \mathbb{R}_+, \mathcal{D}_{f,g} = D_{a,b}, \mathcal{D}_{F,G} = D_{c,d}$ where a, b, c, d are positive constants with $(a - b)(c - d) \neq 0$ by theorem 4 (6) holds if and only if

$$a + b \leqslant c + d \quad \text{and} \quad \frac{a - b}{\ln a - \ln b} \leqslant \frac{c - d}{\ln c - \ln d}. \tag{9}$$

The necessary condition (7) is equivalent to

$$a + b \leqslant c + d. \tag{10}$$

The sufficient condition (8) can easily be written in the form

$$\frac{z^a - z^b}{a - b} \leqslant \frac{z^c - z^d}{c - d} \qquad (z \in \mathbb{R}_+).$$

This inequality holds if and only if (see e.g. Páles [26], Daróczy-Losonczi [10], Losonczi [19])

$$\min(a,b) \leqslant \min(c,d) \quad \text{and} \quad \max(a,b) \leqslant \max(c,d). \tag{11}$$

It is clear that (10) in general is not sufficient for (9), proving that (7) *is, in general, not sufficient for* (6).

One can easily check that for a fixed (a,b) the set of points (c,d) satisfying (9) is a proper subset of the points satisfying (11) (e.g. if $a=e, b=1$ and $c=0.9e^2, d=0.9$ then (9) is satisfied but (11) is not). Therefore (8) *is, in general, not necessary for* (6).

The next three theorems give *necessary and sufficient conditions* for (6) to hold in the special cases $g=G$, $f=F$ and $h=H$. (see [21]).

THEOREM 9. *Suppose that* $(f,g),(F,G)\in\mathcal{E}_n(I), f,g,F,G\in\mathscr{C}_{n+1}(I)$ *and*

$$g=G.$$

Then the inequality (6) holds if and only if either the function h *is (necessarily strictly) increasing on* I *and the function* χ *defined by*

$$\chi(u) := h\left(H^{-1}(u)\right) \qquad (u\in J := \{\ H(x)\mid x\in I\ \})$$

is concave on J *or the function* h *is (strictly) decreasing on* I *and* χ *is convex on* J.

This result can be applied to sharpen Theorems 2 and 3 of Elezović and Pečarić [12] (where, with our notations, the case $g(x)=G(x)=1$ was studied) to necessary and sufficient conditions for the comparison.

THEOREM 10. *Suppose that* $(f,g),(F,G)\in\mathcal{E}_n(I), f,g,F,G\in\mathscr{C}_{n+1}(I)$ *further* $\bar{f}(x)\neq 0$ $(x\in I)$ *and*

$$f=F.$$

Then the inequality (6) holds if and only if either the function k *defined by*

$$k(x) := 1/h(x) = \frac{\bar{g}(x)}{\bar{f}(x)} \qquad (x\in I)$$

is (strictly) increasing on I *and the function* ψ *defined by*

$$\psi(u) := k(K^{-1}(u)) \quad (u\in J_1 := \{K(x)\,|\,x\in I\})\ \text{where}\ K(x) := \frac{\bar{G}(x)}{\bar{f}(x)},\ (x\in I)$$

is concave in J_1 *or* k *is (strictly) decreasing on I and* ψ *is convex on* J_1.

THEOREM 11. *Suppose that* $(f,g),(F,G)\in\mathcal{E}_n(I), f,g,F,G\in\mathscr{C}_{n+1}(I)$ *and*

$$h=H.$$

Then the inequality (6) holds if and only if the function defined by

$$x\to\ln\left|\frac{\bar{g}(x)}{\bar{G}(x)}\right| \qquad (x\in I)$$

is decreasing on I.

5. The general comparison problem

The *general comparison* of three Cauchy means is the inequality

$$\mathcal{D}_{F_1,G_1}\left(k(x_1,y_1),\dots,k(x_n,y_n)\right) \leqslant k\left(\mathcal{D}_{F_2,G_2}(x_1,\dots,x_n),\mathcal{D}_{F_3,G_3}(y_1,\dots,y_n)\right) \qquad (12)$$

where $x_i \in I_2, y_i \in I_3 \ \ (i=1,\dots,n),\ I_1,I_2,I_3$ are intervals, $k : I_2\times I_3 \to I_1$ is a given (comparison) function and $(F_i,G_i)\in\mathcal{E}_n(I_i) \ \ (i=1,2,3)$.

From (12) we obtain with $k(u,v)=u, k(u,v)=u+v$ and $k(u,v)=uv$ as special cases the comparison, subadditivity and Hölder type inequality for Cauchy means.

We can find necessary conditions for (12) in the following way. Let $x\in I_2$, and $y\in I_3$ be fixed. Denoting the difference of the right and left side of (12) taken at $x_1=u, x_2=\d{o}ts=x_n=x, y_1=v, y_2=\cdots=y_n=y$ by $\Phi(u,v)$ it follows that $\Phi(u,v)\geqslant 0$ and $\Phi(x,y)=0$ thus Φ has a minimum at (x,y). Therefore the inequalities

$$(\partial_1^2\Phi)(x,y)(\partial_2^2\Phi)(x,y)-(\partial_1\partial_2\Phi)(x,y)^2\geqslant 0, \quad (\partial_1^2\Phi)(x,y)\geqslant 0, \quad (\partial_2^2\Phi)(x,y)\geqslant 0$$

are necessary for the generalized comparison (provided that the derivatives here are continuous).

REFERENCES

[1] W. H. ADAMS, *Heat transmission*, McGraw-Hill, New York, 1954.
[2] G. ALLASIA, C. GIORDANO AND J. PEČARIĆ, *On the arithmetic and logarithmic means with applications to Stirling's formula*, Atti Sem. Mat. Fiz. Univ. Modena, **47** (1999), 441–455.
[3] H. ALZER, *Bestmögliche Abschätzungen für spezielle Mittelwerte*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., **23(1)** (1993), 331–346.
[4] G. AUMANN AND O. HAUPT, *Einführung in die reelle Analysis*, Band II, W. de Gruyter, Berlin-New York, 1979.
[5] J. BRENNER, *A unified treatment and extension of some means of classical analysis I. Comparison theorems*, J. Combin. Inform System Sci., **3** (1978), 175–199.
[6] J. BRENNER AND B. C. CARLSON, *Homogeneous mean values: Weights and asymptotics*, J. Math. Anal. Appl., **13** (1987), 265–280.
[7] F. BURK, *By all means*, Amer. Math. Monthly, **92**, 1985, 50.
[8] B. C. CARLSON, *The logarithmic mean*, Amer. Math. Monthly, **79** (1972), 615–618.
[9] P. CZINDER AND ZS. PÁLES, *A general Minkowski-type inequality for two variable Stolarsy means*, (manuscript).
[10] Z. DARÓCZY AND L. LOSONCZI, *Über den Vergleich von Mittelwerten*, Publ. Math. Debrecen, **17**, (1970), 289–297.
[11] E. L. DODD, *Some generalization of the logarithmic mean and of similar means of two variates which become indeterminate when the two variates are equal*, Ann. Math. Statist., **12** (1971), 422–428.
[12] N. ELEZOVIĆ AND J. PEČARIĆ, *Differential and integral f -means and applications to digamma function*, Math. Ineq. Appl., **3** (2000), 189–196.
[13] P. FLANDRIN AND P. GONÇALVÈS, *Geometry of affine time-frequency distributions*, Applied and Computational Harmonic Anal., **3** (1996), 10–39.
[14] E. LEACH AND M. SHOLANDER, *Extended mean values*, Amer. Math. Monthly, **85** (1978), 84–90.
[15] E. LEACH AND M. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl., **92** (1983), 207–223.
[16] E. LEACH AND M. SHOLANDER, *Multi-variable extended mean values*, J. Math. Anal. Appl., **104** (1984), 390–407.
[17] T. P. LIN, *The power mean and the logarithmic mean*, Amer. Math. Monthly, **81** (1974), 879–883.
[18] L. LOSONCZI, *Equality of two variable weighted means: reduction to differential equations*, Aequationes Math. **58** (1999), 223–241.
[19] L. LOSONCZI, *Ineqwualities for integral mean values*, J. Math. Anal. Appl. **61** (1977), 586–606.

[20] L. LOSONCZI, *Equality of Cauchy means*, Publ. Math. Debrecen **57** (1999), 223–241.
[21] L. LOSONCZI, *On the comparison of Cauchy mean values*, J. of Ineq. and Appl. **7** (2002), 11–24.
[22] L. LOSONCZI, *Equality of two variable Cauchy means*, Aequationes Math. (to appear).
[23] L. LOSONCZI, *Homogeneous Cauchy Means*, in: *Functional Equations – Results and Advances* (ed. by Z. Daróczy and Zs. Páles), Kluwer Acad. Publ., Dordrecht-Boston-London (2002), 209–218.
[24] L. LOSONCZI AND ZS. PÁLES, *Minkowski's inequality for two variable difference means*, Proc. Amer. Math. Soc. **126** (1998), 779–789.
[25] E. NEUMAN, *The weighted logarithmic mean*, J. Math. Anal. Appl., **188** (1994), 885–900.
[26] ZS. PÁLES, *Inequalities for sums of powers*, J. Math. Anal. Appl., **131** (1988), 265–270.
[27] ZS. PÁLES, *Inequalities for differences of powers*, J. Math. Anal. Appl., **131** (1988), 271–281.
[28] ZS. PÁLES, *Comparison of two variable homogeneous means*, *General Inequalities 6. Proc. 6th Internat. Conf. Math. Res. Inst. Oberwolfach*, Birkhäuser Verlag Basel, 1992, pp. 59–69.
[29] ZS. PÁLES, *A unified form of the classical mean value theorems*, in: *Inequalities and Applications* (ed. by R. P. Agarwal), World Scientific Publ., Singapore-New Jersey-London-Hong Kong, 1994, 493–500.
[30] C. E. PEARCE, J. PEČARIĆ AND J. ŠUNDE, *A generalization of Pólya's inequality to Stolarsky and Gini means*, Math. Ineq. Appl., **1** (1998), 211–222.
[31] J. PEČARIĆ AND V. SIMIĆ, *Stolarsky-Tobey mean in n variables*, Math. Ineq. Appl., **2** (1999), 325–341.
[32] A. O. PITTENGER, *The symmetric, logarithmic, and power means*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. **No. 681** (1980), 19–23.
[33] A. O. PITTENGER, *The logarithmic mean in n variables*, Amer. Math. Monthly, **92** (1985), 99–104.
[34] G. PÓLYA AND G. SZEGŐ, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, N. J., 1951.
[35] J. RÄTZ AND D. RUSSELL, *An extremal problem related to probability*, Aequationes Math., **34** (1987), 316–324.
[36] J. SÁNDOR, *On certain inequalities for means*, J. Math. Anal. Appl., **189** (1995), 602–606.
[37] J. SÁNDOR, *On certain inequalities for means II*, J. Math. Anal. Appl., **199** (1996), 629–635.
[38] J. SÁNDOR, *On refinements of certain inequalities for means*, Archivum Mathematicum (Brno), **31** (1995), 279–282.
[39] H. SEIFFERT, *Ungleichungen für einen bestimmten Mittelwert*, Nieuw Arch. Wisk., **13** (1995), 195–198.
[40] H. SEIFFERT, *Ungleichungen für elementare Mittelwerte*, Arch. Math. (Basel), **63** (1995), 129–131.
[41] L. SCHUMAKER, *Spline functions*, Wiley, New York-Toronto, 1981.
[42] J. F. STEFFENSON, *Interpolation*, 2nd ed., Chelsea, New York, 1950.
[43] K. B. STOLARSKY, *Generalizations of the logarithmic mean*, Math. Mag., **48** (1975), 87–92.
[44] K. B. STOLARSKY, *The power and generalized logarithmic means*, Amer. Math. Monthly, **87** (1980), 545–548.
[45] G. SZÉKELY, *A classification of means*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **18** (1975), 129–133.
[46] K. TETTAMANTI, G. SÁRKÁNY, D. KRÁLIK AND R. SOMFAI, *Über die Annäherung logarithmischer Funktionen durch algebraische Funktionen*, Period. Polytech. Chem. Engrg., **14** (1970), 99–111.
[47] M. D. TOBEY, *Two-parameter homogeneous mean value*, Proc. Amer. Math. Soc., **18** (1967), 9–14.

László Losonczi
Institute of Mathematics and Informatics
Debrecen University
4010 Debrecen, pf. 12, Hungary
e-mail: losi@math.klte.hu