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A SHARP NORM INEQUALITY FOR NON–ISOTROPIC DISTANCE FUNCTIONS ON $\mathbf{R}^n$

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Abstract. We obtain best possible upper and lower bounds for spherical means of distance functions associated with non-isotropic dilation groups on $\mathbf{R}^n$.

1. Introduction

Consider the function $\rho : \mathbf{R}^n \to \mathbf{R}$ defined by

$$\rho(x_1, x_2, \dots, x_n) = \left(x_1^{2p/a_1} + x_2^{2p/a_2} + \dots + x_n^{2p/a_n}\right)^{1/(2p)}$$

where, $a_1, a_2, \dots, a_n$ are positive real numbers, $p = a_1 a_2 \dots a_n$, and $n \geqslant 2$.

If $r > 0$, $\alpha \geqslant a_1 + a_2 + \dots + a_n$, S^{n-1} is the unit sphere in $\mathbf{R}^n$, σ_{n-1} is the Lebesgue surface area measure, and $I(r;\rho,\alpha)$ is the function of the positive real variable r defined by the equation

$$I(r;\rho,\alpha) = \int_{S^{n-1}} \rho(r\omega)^{-\alpha}\, d\sigma_{n-1}(\omega), \tag{1}$$

then our purpose here is to obtain sharp upper and lower bounds for $I(r;\rho,\alpha)$.

The interest in these norms arises in connection with the study of mapping properties of certain singular integral operators [1], where knowledge of the precise order of growth of the integral in (1), as a function of r, is indispensible.

We shall show that, if

$$b_i = \frac{a_i}{a_1}, \qquad i = 1, 2, \dots, n,$$

and

$$K = \int_{R^{n-1}} \frac{d\xi_2 d\xi_3 \dots d\xi_n}{\left(1 + \xi_2^{2p/a_2} + \dots + \xi_n^{2p/a_n}\right)^{\alpha/(2p)}}$$

then

$$\frac{2^n K}{(\max_i b_i)\, r^{\alpha/a_1 - b_2 - \dots - b_n + n - 1}} \leqslant I(r;\rho,\alpha) \leqslant \frac{2^n K}{(\min_i b_i)\, r^{\alpha/a_1 - b_2 - \dots - b_n + n - 1}},$$

the left-hand inequality holding for $\alpha \geqslant a_1 + a_2 + \dots + a_n$ and the right -hand one for $\alpha = a_1 + a_2 + \dots + a_n$.

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2. A more general inequality

For a point $\omega \in S^{n-1}$ write $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ and retain all notation introduced earlier. The inequalities stated above are special cases of the following two more general inequalities:

THEOREM 1. *If* $r_1, r_2, \dots, r_n$ *are positive real numbers and*

$$I = I(r_1, r_2, \dots, r_n; \rho, \alpha) = \int_{S^{n-1}} \rho(r_1\omega_1, r_2\omega_2, \dots, r_n\omega_n)^{-\alpha}\, d\sigma_{n-1}(\omega),$$

then

$$\frac{2^n K}{(\max_i b_i)\, r_1^{\alpha/a_1 - b_2 - \cdots - b_n} r_2 \dots r_n} \leqslant I \leqslant \frac{2^n K}{(\min_i b_i)\, r_1^{\alpha/a_1 - b_2 - \cdots - b_n} r_2 \dots r_n},$$

the left-hand inequality holding for $\alpha \geqslant a_1 + a_2 + \cdots + a_n$ *and the right-hand one for* $\alpha = a_1 + a_2 + \cdots + a_n$. *Furthermore, these inequalities are sharp in the case when* $\alpha = a_1 + a_2 + \cdots + a_n$.

First of all it suffices to estimate the integral over the set

$$S^+ = S^{n-1} \cap \{(x_1, x_2, \dots, x_n) : x_i > 0, i = 1, 2, \dots, n\},$$

since, by symmetry, the integral over S^{n-1} is 2^n times the integral over S^+. The proof of the Theorem will employ a certain change of variables, and it will be necessary to obtain sharp bounds for the Jacobian determinant of the transformation. The calculations will be easier to follow in case $n = 2$ and we present this first.

If $n = 2$ and $(\cos\phi_1, \sin\phi_1)$ is the usual parametrization of S^1, we introduce a new variable λ_1 defined by

$$r_2 \sin\phi_1 = (r_1 \cos\phi_1)^{b_2} \lambda_1, \qquad 0 < \phi_1 < \frac{\pi}{2}.$$

We have

$$\frac{1}{\lambda_1}\frac{d\lambda_1}{d\phi_1} = \cot\phi_1 + b_2 \tan\phi_1, \qquad 0 < \phi_1 < \frac{\pi}{2},$$

so that λ_1 is a positive, strictly increasing function of ϕ_1. Then

$$\begin{aligned} I(r_1, r_2; \rho, \alpha) &= 4\int_0^{\pi/2} \frac{d\phi_1}{[(r_1\cos\phi_1)^{2a_2} + (r_2 \sin\phi_1)^{2a_1}]^{\alpha/(2a_1a_2)}} \\ &= 4\int_0^{\infty} \frac{(d\phi_1/d\lambda_1)d\lambda_1}{(r_1\cos\phi_1)^{\alpha/a_1}(1 + \lambda_1^{2a_1})^{\alpha/(2a_1a_2)}}. \end{aligned}$$

The simple inequalities

$$\frac{\min(1, b_2)\lambda_1}{\sin\phi_1 \cos\phi_1} \leqslant \frac{d\lambda_1}{d\phi_1} \leqslant \frac{\max(1, b_2)\lambda_1}{\sin\phi_1\cos\phi_1}$$

imply that

$$\frac{\max^{-1}(1, b_2)}{(\cos\phi_1)^{\alpha/a_1 - b_2 - 1} r_1^{\alpha/a_1 - b_2} r_2} \leqslant \frac{d\phi_1/d\lambda_1}{(r_1\cos\phi_1)^{\alpha/a_1}} \leqslant \frac{\min^{-1}(1, b_2)}{(\cos\phi_1)^{\alpha/a_1 - b_2 - 1} r_1^{\alpha/a_1 - b_2} r_2}.$$

If $\alpha \geqslant a_1 + a_2$, then $0 < (\cos\phi_1)^{\alpha/a_1 - b_2 - 1} < 1$ for $0 < \phi_1 < \pi/2$, and the left-hand inequality implies that

$$\frac{\max^{-1}(1, b_2)}{r_1^{\alpha/a_1 - b_2} r_2} \leqslant \frac{d\phi_1/d\lambda_1}{(r_1 \cos\phi_1)^{\alpha/a_1}}.$$

If $\alpha = a_1 + a_2$, the right-hand inequality gives

$$\frac{d\phi_1/d\lambda_1}{(r_1 \cos\phi_1)^{\alpha/a_1}} \leqslant \frac{\min^{-1}(1, b_2)}{r_1^{\alpha/a_1 - b_2} r_2} = \frac{\min^{-1}(1, b_2)}{r_1 r_2}.$$

Returning to the original integral we conclude that

$$\begin{aligned} 4\frac{\max^{-1}(1, b_2)}{r_1^{\alpha/a_1 - b_2} r_2} \int_0^\infty \frac{d\lambda_1}{(1 + \lambda_1^{2a_1})^{\alpha/(2a_1a_2)}} &\leqslant I(r_1, r_2; \rho, \alpha) \\ \leqslant \; 4\frac{\min^{-1}(1, b_2)}{r_1^{\alpha/a_1 - b_2} r_2} \int_0^\infty \frac{d\lambda_1}{(1 + \lambda_1^{2a_1})^{\alpha/(2a_1a_2)}}, \end{aligned}$$

the left-hand inequality holding for $\alpha \geqslant a_1 + a_2$ and the right-hand one for $\alpha = a_1 + a_2$.

If $n \geqslant 3$ we let $\phi_1, \phi_2, \ldots, \phi_{n-1}$ be the usual spherical coordinates and write

$$\begin{aligned} \omega_1 &= \cos\phi_1 \\ \omega_2 &= \cos\phi_2 \sin\phi_1 \\ &\vdots \\ \omega_{n-1} &= \cos\phi_{n-1} \sin\phi_{n-2} \ldots \sin\phi_2 \sin\phi_1 \\ \omega_n &= \sin\phi_{n-1} \sin\phi_{n-2} \ldots \sin\phi_2 \sin\phi_1. \end{aligned}$$

Next we introduce positive variables $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ defined by

$$\lambda_j = \frac{r_{j+1}\omega_{j+1}}{(r_1\omega_1)^{b_{j+1}}}, \qquad j = 1, 2, \ldots, n-1, \tag{2}$$

with

$$0 < \phi_k < \frac{\pi}{2}, \qquad k = 1, 2, \ldots, n-1.$$

LEMMA 2. *The transformation defined by the equations in (2) is injective and its Jacobian determinant J_{n-1} satisfies the inequalities*

$$\frac{\min(b_1, b_2, \ldots, b_n)\, \lambda_1\lambda_2 \ldots \lambda_{n-1}}{\prod_{k=1}^{n-1} \sin\phi_k \cos\phi_k} \leqslant J_{n-1} \leqslant \frac{\max(b_1, b_2, \ldots, b_n)\, \lambda_1\lambda_2 \ldots \lambda_{n-1}}{\prod_{k=1}^{n-1} \sin\phi_k \cos\phi_k}$$

where $J_{n-1} = \frac{\partial(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})}{\partial(\phi_1, \phi_2, \ldots, \phi_{n-1})}$.

Proof. We first show that the mapping

$$(\phi_1, \phi_2, \ldots, \phi_{n-1}) \longmapsto (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$$

is injective on $0 < \phi_1, \phi_2, \ldots, \phi_{n-1} < \pi/2$. Suppose

$$\left(\frac{r_2\omega_2}{(r_1\omega_1)^{b_2}}, \frac{r_3\omega_3}{(r_1\omega_1)^{b_3}}, \ldots, \frac{r_n\omega_n}{(r_1\omega_1)^{b_n}}\right) = \left(\frac{r_2\omega_2'}{(r_1\omega_1')^{b_2}}, \frac{r_3\omega_3'}{(r_1\omega_1')^{b_3}}, \ldots, \frac{r_n\omega_n'}{(r_1\omega_1')^{b_n}}\right),$$

where ω_j' is defined as ω_j but with ϕ_j replaced by θ_j. Then

$$\left(\frac{\omega_1'}{\omega_1}\right)^{b_k} = \frac{\omega_k'}{\omega_k}, \qquad k = 2, 3, \ldots, n.$$

This gives

$$\left(\frac{\cos\theta_1}{\cos\phi_1}\right)^{b_k} = \frac{\cos\theta_k \sin\theta_{k-1} \ldots \sin\theta_1}{\cos\phi_k \sin\phi_{k-1} \ldots \sin\phi_1}, \qquad k = 2, \ldots, n-1,$$

and

$$\left(\frac{\cos\theta_1}{\cos\phi_1}\right)^{b_n} = \frac{\sin\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_1}{\sin\phi_{n-1} \sin\phi_{n-2} \ldots \sin\phi_1}. \tag{3}$$

If $\theta_1 \neq \phi_1$, assume, without loss of generality, that $0 < \theta_1 < \phi_1 < \pi/2$. Then $\cos\theta_1 > \cos\phi_1$, $\sin\theta_1 < \sin\phi_1$, and

$$1 < \left(\frac{\cos\theta_1}{\cos\phi_1}\right)^{b_2} = \frac{\cos\theta_2 \sin\theta_1}{\cos\phi_2 \sin\phi_1} < \frac{\cos\theta_2}{\cos\phi_2},$$

so that $0 < \theta_2 < \phi_2 < \pi/2$. Having

$$0 < \theta_j < \phi_j < \pi/2 \quad \text{for } j = 1, 2, \ldots, k-1,$$

we deduce that

$$1 < \left(\frac{\cos\theta_1}{\cos\phi_1}\right)^{b_k} = \frac{\cos\theta_k \sin\theta_{k-1} \ldots \sin\theta_1}{\cos\phi_k \sin\phi_{k-1} \ldots \sin\phi_1} < \frac{\cos\theta_k}{\cos\phi_k}.$$

Thus $0 < \theta_k < \phi_k < \pi/2$ and so we have

$$0 < \theta_j < \phi_j < \pi/2 \quad \text{for } j = 1, 2, \ldots, n-1.$$

It follows from (3) that

$$1 < \left(\frac{\cos\theta_1}{\cos\phi_1}\right)^{b_n} = \frac{\sin\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_1}{\sin\phi_{n-1} \sin\phi_{n-2} \ldots \sin\phi_1} < 1$$

and we have a contradiction. Therefore $\theta_1 = \phi_1$, and then $\omega_1 = \omega_1'$ from which follows that $\omega_k = \omega_k'$ for $k = 2, 3, \ldots, n$. These last equalities imply that $\theta_k = \phi_k$ for $k = 1, 2, \ldots, n-1$. Hence the defined mapping is injective.

The next step is to obtain upper and lower bounds for the Jacobian determinant of the transformation.

Since

$$\begin{aligned}\frac{1}{\lambda_1}\frac{\partial\lambda_1}{\partial\phi_1} &= \cot\phi_1 + b_2\tan\phi_1,\\ \frac{1}{\lambda_1}\frac{\partial\lambda_1}{\partial\phi_2} &= -\tan\phi_2,\\ \frac{1}{\lambda_1}\frac{\partial\lambda_1}{\partial\phi_k} &= 0 \qquad (k \geqslant 3),\end{aligned}$$

and, for $j \geqslant 2$,

$$\begin{aligned}\frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\phi_1} &= \cot\phi_1 + b_{j+1}\tan\phi_1,\\ \frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\phi_k} &= \cot\phi_k \qquad (2 \leqslant k \leqslant j),\\ \frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\phi_{j+1}} &= -\tan\phi_{j+1},\\ \frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\phi_k} &= 0 \qquad (j+1 < k \leqslant n-1),\end{aligned}$$

we may display $J_{n-1}/(\lambda_1\lambda_2\ldots\lambda_{n-1})$ as follows

$$\begin{vmatrix} \cot\phi_1 + b_2\tan\phi_1 & -\tan\phi_2 & 0 & \cdots & 0\\ \cot\phi_1 + b_3\tan\phi_1 & \cot\phi_2 & \ddots & & \vdots\\ \vdots & \vdots & \ddots & -\tan\phi_{n-2} & 0\\ \cot\phi_1 + b_{n-1}\tan\phi_1 & \cot\phi_2 & \cdots & \cot\phi_{n-2} & -\tan\phi_{n-1}\\ \cot\phi_1 + b_n\tan\phi_1 & \cot\phi_2 & \cdots & \cot\phi_{n-2} & \cot\phi_{n-1} \end{vmatrix}.$$

We also denote by L_{n-2} the determinant of the minor corresponding to $-\tan\phi_{n-1}$ in $J_{n-1}/(\lambda_1\lambda_2\ldots\lambda_{n-1})$. Thus L_{n-2} is the $(n-2)\times(n-2)$ determinant

$$\begin{vmatrix} \cot\phi_1 + b_2\tan\phi_1 & -\tan\phi_2 & 0 & \cdots & 0\\ \cot\phi_1 + b_3\tan\phi_1 & \cot\phi_2 & \ddots & & \vdots\\ \vdots & \vdots & \ddots & -\tan\phi_{n-3} & 0\\ \cot\phi_1 + b_{n-2}\tan\phi_1 & \cot\phi_2 & \cdots & \cot\phi_{n-3} & -\tan\phi_{n-2}\\ \cot\phi_1 + b_n\tan\phi_1 & \cot\phi_2 & \cdots & \cot\phi_{n-3} & \cot\phi_{n-2} \end{vmatrix}.$$

We now show that

$$\frac{\min(b_1,b_2,\ldots,b_n)\,\lambda_1\lambda_2\ldots\lambda_{n-1}}{\prod_{k=1}^{n-1}\sin\phi_k\cos\phi_k} \leqslant J_{n-1} \leqslant \frac{\max(b_1,b_2,\ldots,b_n)\,\lambda_1\lambda_2\ldots\lambda_{n-1}}{\prod_{k=1}^{n-1}\sin\phi_k\cos\phi_k}$$

for $n \geqslant 3$.

If $n = 3$, we have

$$r_2\sin\phi_1\cos\phi_2 = (r_1\cos\phi_1)^{b_2}\lambda_1, \qquad r_3\sin\phi_1\sin\phi_2 = (r_1\cos\phi_1)^{b_3}\lambda_2,$$

and

$$\begin{aligned} J_2 &= \frac{\partial(\lambda_1,\lambda_2)}{\partial(\phi_1,\phi_2)} \\ &= \lambda_1\lambda_2 \begin{vmatrix} \cot\phi_1 + b_2\tan\phi_1 & -\tan\phi_2 \\ \cot\phi_1 + b_3\tan\phi_1 & \cot\phi_2 \end{vmatrix} \\ &= \lambda_1\lambda_2\cot\phi_2(\cot\phi_1 + b_2\tan\phi_1) + \lambda_1\lambda_2\tan\phi_2(\cot\phi_1 + b_3\tan\phi_1) \\ &\leqslant \lambda_1\lambda_2\cot\phi_2\left(\frac{\max(b_1,b_2)}{\sin\phi_1\cos\phi_1}\right) + \lambda_1\lambda_2\tan\phi_2\left(\frac{\max(b_1,b_3)}{\sin\phi_1\cos\phi_1}\right) \\ &\leqslant \lambda_1\lambda_2\left(\frac{\max(b_1,b_2,b_3)}{\sin\phi_1\cos\phi_1\sin\phi_2\cos\phi_2}\right). \end{aligned}$$

A lower bound is obtained in a similar fashion. Also

$$L_2 = \begin{vmatrix} \cot\phi_1 + b_2\tan\phi_1 & -\tan\phi_2 \\ \cot\phi_1 + b_4\tan\phi_1 & \cot\phi_2 \end{vmatrix}$$

so that

$$\frac{\min(b_1,b_2,b_4)}{\sin\phi_1\cos\phi_1\sin\phi_2\cos\phi_2} \leqslant L_2 \leqslant \frac{\max(b_1,b_2,b_4)}{\sin\phi_1\cos\phi_1\sin\phi_2\cos\phi_2}.$$

Suppose now that J_{n-2} ($n \geqslant 4$) satisfies the inequalities in Lemma 2 for all $b_1, b_2, \ldots, b_{n-1}$. Explicitly, suppose that J_{n-2} satisfies

$$\frac{\min(b_1,b_2,\ldots,b_{n-1})}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k} \leqslant \frac{J_{n-2}}{\lambda_1\lambda_2\ldots\lambda_{n-2}} \leqslant \frac{\max(b_1,b_2,\ldots,b_{n-1})}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k}.$$

Then L_{n-2}, as defined above, satisfies

$$\frac{\min(b_1,b_2,\ldots,\hat{b}_{n-1},b_n)}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k} \leqslant L_{n-2} \leqslant \frac{\max(b_1,b_2,\ldots,\hat{b}_{n-1},b_n)}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k},$$

where the ˆ above a term indicates that the term should be omitted. If now we use the expansion of $J_{n-1}/(\lambda_1\lambda_2\ldots\lambda_{n-1})$ along its last column, that is

$$\frac{J_{n-1}}{\lambda_1\lambda_2\ldots\lambda_{n-1}} = (\cot\phi_{n-1})\frac{J_{n-2}}{\lambda_1\lambda_2\ldots\lambda_{n-2}} + (\tan\phi_{n-1})L_{n-2},$$

we obtain

$$\begin{aligned} \frac{J_{n-1}}{\lambda_1\lambda_2\ldots\lambda_{n-1}} &\leqslant \frac{\max(b_1,b_2,\ldots,b_{n-1})}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k}(\cot\phi_{n-1}) \\ &\quad + \frac{\max(b_1,b_2,\ldots,\hat{b}_{n-1},b_n)}{\prod_{k=1}^{n-2}\sin\phi_k\cos\phi_k}(\tan\phi_{n-1}) \\ &\leqslant \frac{\max(b_1,b_2,\ldots,b_n)}{\prod_{k=1}^{n-1}\sin\phi_k\cos\phi_k}. \end{aligned}$$

This finishes the proof of the right-hand inequality in Lemma 2. The left-hand inequality follows in a similar fashion. □

To continue the proof of the Theorem, we first express the integral over S^+ in spherical coordinates and then apply the change of variables defined in (2).

$$I = \int_0^\infty \cdots \int_0^\infty \frac{\left(\sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \ldots \sin\phi_{n-2}\right)\left(\frac{\partial(\phi_1,\ldots,\phi_{n-1})}{\partial(\lambda_1,\ldots,\lambda_{n-1})}\right)}{(r_1\cos\phi_1)^{\alpha/a_1}\left(1+\sum_{j=2}^n \lambda_{j-1}^{2p/a_j}\right)^{\alpha/(2p)}} d\lambda_1 \ldots d\lambda_{n-1}.$$

It remains to apply the inequalities in Lemma 2 for the Jacobian

$$\frac{\partial(\phi_1,\ldots,\phi_{n-1})}{\partial(\lambda_1,\ldots,\lambda_{n-1})},$$

and to note that, since $\alpha \geqslant a_1 + a_2 + \cdots + a_n$, we have

$$\frac{\alpha}{a_1} - 1 - b_2 - b_3 - \cdots - b_n \geqslant 0$$

so that

$$(\cos\phi_1)^{\alpha/a_1 - 1 - b_2 - b_3 - \cdots - b_n} \leqslant 1$$

and the left-hand inequality in Theorem 1 follows. If $\alpha = a_1 + a_2 + \cdots + a_n$, then

$$(\cos\phi_1)^{\alpha/a_1 - 1 - b_2 - b_3 - \cdots - b_n} = 1$$

and the right-hand inequality in Theorem 1 follows.

3. An inequality for distance functions associated with non-isotropic dilation groups on $\mathbf{R}^n$

Let $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ be given constants and, for $t > 0$, define $\delta_t : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\delta_t x = (t^{a_1}x_1, t^{a_2}x_2, \ldots, t^{a_n}x_n), \qquad x = (x_1, x_2, \ldots, x_n).$$

DEFINITION 3. *A function* $\rho : \mathbf{R}^n \to [0,\infty)$ *satisfying*

$$\rho(\delta_t x) = t\rho(x)$$

for all $x \in \mathbf{R}^n$ *and vanishing only at 0 is called a* δ*-homogeneous distance function.*

For elementary properties of these functions we refer the reader to [2]. The primary example of such distance functions is the function defined in the introduction, which has the additional properties of infinite differentiability on $\mathbf{R}^n - \{0\}$, and symmetry with respect to each of the hyperplanes $x_i = 0$. There are, of course other δ-homogeneous distance functions not necessarily in possession of these two properties.

It turns out that the inequalities in Theorem 1 extend to general δ-homogeneous distance functions which are C^2 in $\mathbf{R}^n - \{0\}$. We indicate briefly how these inequalities may be obtained.

Let ρ be a δ-homogeneous distance function as defined above. If, say, $x_1 > 0$, use

$$\rho(x_1, x_2, \ldots, x_n) = \left(x_1^{1/a_1}\right)\rho(1, \lambda_1, \lambda_2, \ldots, \lambda_{n-1})$$

where

$$\lambda_1 = \frac{x_2}{x_1^{a_2/a_1}},\ \lambda_2 = \frac{x_3}{x_1^{a_3/a_1}},\ \ldots,\ \lambda_{n-1} = \frac{x_n}{x_1^{a_n/a_1}}.$$

If the coordinates are expressed in spherical coordinates

$$\begin{aligned} x_1 &= r\cos\phi_1, \\ x_j &= r\left(\prod_{k=1}^{j-1}\sin\phi_k\right)\cos\phi_j \qquad (2 \leqslant j \leqslant n-1), \\ x_n &= r\left(\prod_{k=1}^{n-2}\sin\phi_k\right)\sin\phi_{n-1}, \end{aligned}$$

it becomes possible to proceed as in the proof of Theorem 1 and we arrive at two sharp inequalities for the integral $\int_{S^{n-1}\cap\{x_1>0\}}\rho(x)^{-\alpha}d\sigma_{n-1}(\omega)$. For example, one such inequality is

$$\int_{S^{n-1}\cap\{x_1>0\}} \frac{d\sigma_{n-1}(\omega)}{\rho(r\omega)^\alpha} \geqslant r^n K'$$

for an appropriate constant K' and with the same restrictions on α. Similar inequalities may be obtained for the integrals over the other parts of the sphere and addition gives an analogue of our inequalities in Theorem 1. We omit the details.

Finally it is perhaps worthwhile to remark that the right-hand inequality in Theorem 1 extends, but with a different constant, to the case where $\alpha \geqslant a_1 + a_2 + \cdots + a_n$. Since this requires a different method of proof we do not include it here.

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