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INTERPOLATION POLYNOMIALS AND INEQUALITIES FOR CONVEX FUNCTIONS OF HIGHER ORDER

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Abstract. In this paper we prove several inequalities for convex function of a higher order. Generalizations of Hadamard's inequalities and the conversion of Jensen's inequality for (n) – convex function and with conditions on the regular, real (signed) Borel measure are presented by Lidstone's and Hermite's interpolating polynomials. As a discrete form we also obtain a generalization of Perovič's inequality, i.e. Giaccardi's inequality. The Abel-Gontscharoff interpolating polynomial with two-point right focal conditions leads us to an inequality like converse of Jensen inequality for a regular, signed measure and, as a consequence, to some inequalities related with Hadamard's and Petrović's inequalities.

1. Introduction

In [4] A. M. Fink has considered Féjer generalizations [13] of the left side Hadamard inequality involving convex functions of higher order. He has also proved the generalization of the right side of this inequality:

THEOREM A 1. *Let $f \in C^2[-1,1]$ be convex, μ a regular, non-negative measure on the collection of all Borel sets in $\mathbf{R}$, and $G(s,t)$ the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[-1,1]$. If the function $s \to \int_{-1}^{1} G(s,\cdot)d\mu$ is non-positive, then*

$$\int_{-1}^{1} f\, d\mu \leqslant P_0 \frac{f(-1)+f(1)}{2} + P_1 \frac{f(1)-f(-1)}{2} \tag{1.1}$$

where $P_k = \int_{-1}^{1} x^k\, d\mu(x)$, $k \in \{0,1\}$.

The following Theorem corresponds to some conversions of the well known Jensen inequality for convex functions (see Lemma 1 in [1]):

THEOREM A 2. *Let $f : I = [m,M] \to \mathbf{R}$, $(-\infty < m < M < \infty)$ be convex and $g : [a,b] \to [m,M]$ be such that $g \in L^1(\mu)$ on $[a,b]$ for a non-negative measure μ satisfying $\int_a^b d\mu = 1$. Then*

$$\int_a^b f(g(t))d\mu(t) \leqslant \frac{M - \int_a^b g(t)d\mu(t)}{M-m} f(m) + \frac{\int_a^b g(t)d\mu(t) - m}{M-m} f(M). \tag{1.2}$$

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The following result from [1] is generalization of some complements of the well known Jensen inequality for convex functions obtained by P. R. Beesack:

THEOREM A 3. *Let the conditions of Theorem 2 be satisfied and let J be an interval such that $J \supset f(I)$. If $\Phi(u,v)$ is a real function defined on $J \times J$, non-decreasing in u, then*

$$\Phi\Big[\int_a^b f(g(t))d\mu(t), f\Big(\int_a^b g(t)d\mu(t)\Big)\Big]$$
$$\leqslant \max_{x\in[m,M]} \Phi\left[\frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M), f(x)\right]. \tag{1.3}$$

The right-hand side of (1.3) is a nondecreasing function of M and a nonincreasing function of m.

The following generalization of the well known Petrović inequality for convex functions [20] was given by F. Giaccardi [19] (see also [3], p. 153–154):

THEOREM A 4. *Let $\mathbf{p}$, $\mathbf{x} \in \mathbf{R^n}$ satisfy*

$$(x_i - x_0)\left(\sum_{k=1}^n p_k x_k - x_i\right) \geqslant 0, \qquad i = 1,..,n, \tag{1.4}$$

where $x_0, \sum_{k=1}^n p_k x_k \in [a,b]$ $\sum_{k=1}^n p_k x_k \neq x_0$. If $f : [a,b] \to \mathbf{R}$ is convex, then

$$\sum_{k=1}^n p_k f(x_k) \leqslant Af\left(\sum_{k=1}^n p_k x_k\right) + B\left(\sum_{k=1}^n p_k - 1\right)f(x_0), \tag{1.5}$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, \ B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}.$$

The notion of n-convexity goes back to Popoviciu [10]; we follow the definition given by Karlin [15]:

DEFINITION 1. A function $f : [a,b] \to \mathbf{R}$ is said to be (n)-convex on $[a,b]$, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, n^{th} order divided difference of f satisfies

$$[x_0, ..., x_n]f \geqslant 0.$$

In fact, Popoviciu proved that each continuous (n)-convex function on $[a,b]$ is the uniform limit of the sequence of (n)-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [16].

In this paper we give a proof of some inequalities for convex function of higher order ($n \geqslant 2$). Generalizations of Hadamard's inequalities and the conversion of Jensen's inequality for (n)-convex function and with conditions on the regular, real (signed) Borel measure are presented by Lidstone's and Hermite's interpolating polynomials. As a discrete form we also obtain a generalization of Perovič's inequality, i.e. Giaccardi's inequality. The Abel-Gontscharoff interpolating polynomial with two-point right focal conditions leads us to an inequality like converse of Jensen inequality for a regular, signed measure and, as a consequence, to some inequalities related with Hadamard's and Petrović's inequalities.

In Sections 2.2, 3.3 and 4.2 we give some inequality which are the generalizations or is like the converse of the Jensen inequality for regular, signed Borel measure μ. In Sections 2.3, 3.2 and 4.3 we obtained new inequalities which are the generalizations of Hadamard inequality for convex function of higher order. Also, as a discrete form, in Section 2.4, 3.4 and 4.4 we obtain some generalization of Giaccardi inequality for a n-convex function.

2. Lidstone interpolating polynomial and some inequalitiesfor convex functions of higher order

2.1. Lidstone interpolating polynomial

In the year 1929. G. J. Lidstone [5] introduced a generalization of Taylor's series. It approximates a given function in the neighbourhood of two points instead of one. D. V. Widder [6] has given the following fundamental result:

LEMMA A 1. *Let* $f(t) \in C^{(2m)}[0,1]$ *then*

$$f(t) = \sum_{k=0}^{m-1}[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t)] + \int_0^1 G_m(t,s) f^{(2m)}(s)ds \tag{2.1}$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if } s \leqslant t, \\ (s-1)t, & \text{if } t \leqslant s, \end{cases} \tag{2.2}$$

$$G_n(t,s) = \int_0^1 G_1(t,p)G_{n-1}(p,s)\,dp, n \geqslant 2 \tag{2.3}$$

and $\Lambda(t)$ *is the unique polynomial (Lidstone polynomial) of degree* $(2n+1)$ *defined by the relations*

$$\begin{aligned} \Lambda_0(t) &= t \\ \Lambda_n''(t) &= \Lambda_{n-1}(t) \\ \Lambda_n(0) &= \Lambda_n(1) = 0, n \geqslant 1 \end{aligned} \tag{2.4}$$

which can be expressed, in terms of $G_n(t,s)$ *and the Fourier series expansions, as*

$$\begin{aligned} \Lambda_n(t) &= \int_0^1 G_n(t,s)s\,ds \\ &= (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t. \end{aligned}$$

Another explicit representations of Lidstone polynomial are given by [7] and [8],

$$\begin{aligned} \Lambda_n(t) &= \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!}\right] \\ &- \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!};\ n = 1,2,.. \end{aligned}$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right); \; n = 1, 2..$$

where B_{2k+4} is the $(2k+4)$ th Bernoulli number and $B_n(t)$ is a Bernoulli polynomial.

2.2. Generalization of the conversion of the Jensen inequality

Using Widder's Lemma A1 we can get the following theorem (see [21]):

THEOREM 1. *Let $f(t) : [M_1, M_2] \to R$ be $(2m)$-convex function and let $g : [a,b] \to [M_1, M_2]$ be given function . Let μ be a regular, real (signed) Borel measure. If*

$$\int_a^b G_m\left(\frac{g(t)-M_1}{M_2-M_1}, s\right) d\mu(t) \leqslant 0, \; \forall s \in [0,1] \tag{2.5}$$

then

$$\int_a^b f(g(t))\, d\mu(t) \leqslant \sum_{k=0}^{m-1} (M_2-M_1)^{2k} [f^{(2k)}(M_1) \int_a^b \Lambda_k\left(\frac{M_2-g(t)}{M_2-M_1}\right) d\mu(t)$$
$$+ f^{(2k)}(M_2) \int_a^b \Lambda_k\left(\frac{g(t)-M_1}{M_2-M_1}\right) d\mu(t)]. \tag{2.6}$$

If the reverse inequality in (2.5) is valid then the reverse inequality in (2.6) is valid, too.

COROLLARY 1. *Let $f(t) : [M_1, M_2] \to R$ be a $(2m)$-convex function and let $g : [a,b] \to [M_1, M_2]$ be given function. For nonnegative measures $\mu \geqslant 0$, (2.6) holds for m odd and the reverse inequality holds if m is even.*

This corollary follows from the easily proved inequality $(-1)^m G_m(s,t) \geqslant 0$. For $m = 1$ in Theorem 1 we have a following corollary which is generalization of Theorem A2 for signed measure μ.

COROLLARY 2. *Let $f(t)$ be convex function on $[M_1, M_2]$ and μ be a regular, real (signed) Borel measure such that $\int_a^b d\mu(t) = 1$. Let $g : [a,b] \to [M_1, M_2]$ be a given function, integrable with respect to μ. If*

$$\int_a^b G\Big(\frac{g(t)-M_1}{M_2-M_1}, s\Big) d\mu(t) \leqslant 0, \; \forall s \in [0,1] \tag{2.7}$$

then

$$\int_a^b f(g(t))\, d\mu(t) \leqslant f(M_1)\Big(\frac{M_2 - \int_a^b g(t) d\mu(t)}{M_2-M_1}\Big)$$
$$+ f(M_2)\Big(\frac{\int_a^b g(t) d\mu(t) - M_1}{M_2-M_1}\Big) \tag{2.8}$$

If reverse inequality in (2.7) is valid then the reverse inequality in (2.8) is valid , too.

As a consequence of Theorem 1 and Corollary 2 we get a generalization of Theorem A3 for non positive measure μ:

THEOREM 2. *Let the conditions of Corollary 2 be satisfied and let J be an interval such that $J \supset f(I)$. If $\Phi(u,v)$ is real function defined on $J \times J$, non-decreasing in u, then*

$$\begin{aligned} &\Phi\Big[\int_a^b f(g(t))d\mu(t), f\Big(\int_a^b g(t)d\mu(t)\Big)\Big] \\ \leqslant \quad &\max_{x\in[M_1,M_2]} \Phi\left[\frac{M_2-x}{M_2-M_1}f(M_1)+\frac{x-M_1}{M_2-M_1}f(M_2), f(x)\right] \\ (= \quad &\max_{\theta\in[0,1]} \Phi[\theta f(M_1)+(1-\theta)f(M_2), f(\theta M_1+(1-\theta)M_2)]) \end{aligned} \tag{2.9}$$

The right-hand side of (4.9) is a nondecreasing function of M_2 and a nonincreasing function of M_1.

The following two Corollaries of Theorem 2 are generalization of Corollaries 1 and 2 from [1] and [3] for non positive measure μ (see [21]).

COROLLARY 3. *Let f be a convex function on $I=[M_1,M_2]$, $(-\infty < m < M < \infty)$, such that $f''(x) \geqslant 0$ with equality for at most isolated points of I. Suppose that either (i) $f(x)>0$ for all $x \in I$ or (ii) $f(x)<0$ for all $x\in I$. Let $g:[a,b]\to[M_1,M_2]$ be a given function such that $M_1 \leqslant \int_a^b g(t)d\mu(t) \leqslant M_2$ where μ is a regular, signed Borel measure. If (2.7) holds, then*

$$\int_a^b f(g(t))d\mu(t) \leqslant \lambda f\left(\int_a^b g(t)d\mu(t)\right) \tag{2.10}$$

holds for some $\lambda > 1$ in case (i) or $\lambda \in (0,1)$ in case (ii).
More precisely: set $r=\frac{f(M_2)-f(M_1)}{M_2-M_1}$;
if $r=0$ then $\lambda=\frac{f(M_1)}{f(x_o)}$ suffices for (2.10)
where x_o is the unique solution of the equation $f'(x)=0$;
if $r\neq 0$ then $\lambda=\frac{r}{f'(x_o)}$ suffices for (2.10)
where x_o is the unique solution of the equation $rf(x)-f'(x)(f(M_1)+r(x-M_1))=0$

COROLLARY 4. *Let f be differentiable function on $I=[M_1,M_2]$, and f' strictly increasing on I. Let $g:[a,b]\to[M_1,M_2]$ be a given function such that $M_1\leqslant\int_a^b g(t)d\mu(t)\leqslant M_2$ where μ is a regular, signed Borel measure. If (2.7) then*

$$\int_a^b f(g(t))d\mu(t) \leqslant \lambda + f\left(\int_a^b g(t)d\mu(t)\right) \tag{2.11}$$

for some $\lambda>1$ satisfying $0<\lambda<(M_2-M_1)(r-f'(M_1))$, where r is defined as in Corollary 3. More precisely :
$\lambda = f(M_1)-f(x_o)+r(x_o-M_1)$ suffices for (2.11)
where x_o is an unique solution of the equation $f'(x)=r$.

2.3. Generalization of a Hadamard inequality

For $g(t) = t$ in Theorem 1 we have the generalization of Fink's generalization of Hadamard's inequality (1.1) for $(2m)$-convex function, (see [21]):

COROLLARY 5. *Let $f : [a,b] \to R$ be a $(2m)$-convex function. If*

$$\int_a^b G_m\left(\frac{t-a}{b-a}, s\right) d\mu(t) \leqslant 0, \ \forall s \in [0,1] \tag{2.12}$$

where μ is a regular, signed Borel measure then

$$\int_a^b f(t)\, d\mu(t) \leqslant \sum_{k=0}^{m-1} (b-a)^{2k} [f^{(2k)}(a) \int_a^b \Lambda_k\left(\frac{b-t}{b-a}\right) d\mu(t) + f^{(2k)}(b) \int_a^b \Lambda_k\left(\frac{t-a}{b-a}\right) d\mu(t)]. \tag{2.13}$$

If the reverse inequality in (2.12) is valid then the reverse inequality in (2.13) is valid, too.

2.4. Generealisation of Giaccardi inequality

For the discrete case of Theorem 1 we get an inequality of Petrović and Giaccardi for convex functions of higher order, generalization of Theorem A4, (see [9]):

COROLLARY 6. *Let $\mathbf{p}$ and $\mathbf{x}$ be two given real n-tuples such that*

$$(x_i - x_0)\left(\sum_{k=1}^n p_k x_k - x_i\right) \geqslant 0, i = 1,..,n \tag{2.14}$$

$$x_0, \sum_{k=1}^n p_k x_k \in [M_1, M_2]$$

$$\sum_{k=1}^n p_k x_k \neq x_0.$$

and

$$\sum_{k=1}^n p_k G_m\left(\frac{x_k - x_0}{\sum_{j=1}^n p_j x_j - x_0}, s\right) \leqslant 0, \ \forall s \in [0,1] \tag{2.15}$$

are valid. If $f : [M_1, M_2] \to R$ is a $(2m)$-convex function then

$$\begin{aligned}\sum_{i=1}^n p_i f(x_i) \ &\leqslant \ \sum_{k=0}^{m-1} \left(\sum_{j=1}^n p_j x_j - x_0\right)^{2k} \left[f^{(2k)}(x_0) \sum_{i=1}^n p_i \Lambda_k\left(\frac{\sum_{j=1}^n p_j x_j - x_i}{\sum_{j=1}^n p_j x_j - x_0}\right)\right. \\ &+ \ \left. f^{(2k)}\left(\sum_{j=1}^n p_j x_j\right) \sum_{i=1}^n p_i \Lambda_k\left(\frac{x_i - x_0}{\sum_{j=1}^n p_j x_j - x_0}\right)\right]\end{aligned} \tag{2.16}$$

If the reverse inequality in (2.15) is valid then the reverse inequality in (2.16) is valid, too.

3. Hermite interpolating polynomial and some inequalitiesfor convex functions of higher order

3.1. Hermite interpolating polynomial

Let $-\infty < a < b < \infty$, and $a \leqslant a_1 < a_2 ... < a_r \leqslant b$, $(r \geqslant 2)$ be given. It is well known, that for $f \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$ [14], [7] exists, fulfilling one of the following conditions:

Hermite conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j);\ 0 \leqslant i \leqslant k_j,\ 1 \leqslant j \leqslant r,\ \sum_{j=1}^{r} k_j + r = n,$$

in particular:

Simple Hermite or Osculatory conditions: $(n = 2m,\ r = m,\ k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j),\ P_O'(a_j) = f'(b_j),\ 1 \leqslant j \leqslant m,$$

Lagrange conditions: $(r = n,\ k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j),\ 1 \leqslant j \leqslant n,$$

Type $(m, n-m)$ **conditions:** $(r = 2,\ 1 \leqslant m \leqslant n-1,\ k_1 = m-1,\ k_2 = n-m-1)$

$$\begin{aligned} P_{mn}^{(i)}(a) &= f^{(i)}(a_j),\ 0 \leqslant i \leqslant m-1 \\ P_{mn}^{(i)}(b) &= f^{(i)}(b_j),\ 0 \leqslant i \leqslant n-m-1, \end{aligned}$$

Two-point Taylor conditions: $(n = 2m,\ r = 2,\ k_1 = k_2 = m-1)$

$$P_{2T}^{(i)}(a) = f^{(i)}(a),\ P_{2T}^{(i)}(b) = f^{(i)}(b),\ 0 \leqslant i \leqslant m-1.$$

The associated error $|e_H(t)|$ can be represented in terms of the Green's function $G_H(t,s)$ for the multipoint boundary value problem $z^{(n)}(t) = 0$, $z^{(i)}(a_j) = 0,\ 0 \leqslant i \leqslant k_j,\ 1 \leqslant j \leqslant r$, that is, the following result holds [7]:

THEOREM A 5. *Let $F \in C^n[a,b]$, and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned} F(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j) + \int_a^b G_H(t,s)F^{(n)}(s)ds, \end{aligned} \tag{3.1}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k_j!}\left[\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right]^{(k)}_{t=a_j}(t-a_j)^k, \tag{3.2}$$

where

$$\omega(t) = \prod_{j=1}^{r}(t-a_j)^{k_j+1}, \tag{3.3}$$

and G_H is the Green's function, defined by

$$G_H(t,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t), & s \leqslant t \\ -\sum_{j=l+1}^{l}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t), & s \geqslant t. \end{cases} \tag{3.4}$$

for all $a_l \leqslant s \leqslant a_{l+1}, l = 1,...,r-1$.

We use the following lemma describing positivity properties of the Green's function (3.4), done by Levin [17] (i) and Beesack [18] (ii):

LEMMA A 2. *The Green's function $G_H(t,s)$ has the following properties:*

$$(i)\ \frac{G_H(t,s)}{\omega(t)} > 0,\ a_1 \leqslant t \leqslant a_r,\ a_1 < s < a_r, \tag{3.5}$$

$$(ii)\ G_H(t,s) \leqslant \frac{1}{(n-1)!(b-a)}|\omega(t)|, \tag{3.6}$$

$$(iii)\ \int_a^b |G_H(t,s)|ds = \frac{1}{n!}|\omega(t)|. \tag{3.7}$$

3.2. Generalization of Hadamard inequality

By using Theorem A5, Lemma A2 and a condition for the Green's function, we prove this theorem (see [22]):

THEOREM 3. *Let $f : [a,b] \to \mathbf{R}$ be (n)-convex function, $-\infty \leqslant a \leqslant a_1 < a_2 ... < a_r \leqslant b \leqslant \infty$ be given, $(r \geqslant 2)$, $k_j \in N$, $j = 1,...r$, $\sum_{j=1}^{r} k_j + r = n$, and let μ be a regular, signed measure on Borel sets.*

(I) If

$$\int_a^b G_H(t,s)d\mu(t) \leqslant 0,\ \forall s \in [a,b], \tag{3.8}$$

then

$$\int_a^b f(t)d\mu(t) \leqslant \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)\int_a^b H_{ij}(t)d\mu(t). \tag{3.9}$$

The reverse inequality in (3.8) implies the reverse inequality in (3.9).

(II) If μ is a positive measure, then

(i) if $\omega(t) > 0$, the reverse of the inequality in (3.9) holds;

(ii) if $\omega(t) < 0$, the inequality in (3.9) is always valid.

For the other cases we have the following corollaries (see [22]):

COROLLARY 7. **(I)** *Let $f : [a,b] \to \mathbf{R}$ be (n)-convex, and $1 \leqslant m \leqslant n-1$ be fixed. If*

$$\int_a^b G_{mn}(t,s)d\mu(t) \leqslant 0,\ \forall s \in [a,b] \tag{3.10}$$

then

$$\int_a^b f(t)d\mu(t) \leqslant \sum_{i=0}^{m-1} f^{(i)}(a)\int_a^b \tau_i(t)d\mu(t) + \sum_{i=0}^{n-m-1} f^{(i)}(b)\int_a^b \nu_i(t)d\mu(t), \tag{3.11}$$

where

$$\tau_i(t) = \frac{1}{i!}(t-a)^i\left(\frac{t-b}{a-b}\right)^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}\left(\frac{t-a}{b-a}\right)^k, \tag{3.12}$$

$$\nu_i(t) = \frac{1}{i!}(t-b)^i\left(\frac{t-a}{b-a}\right)^{m}\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}\left(\frac{t-b}{a-b}\right)^k. \tag{3.13}$$

The Green's function can be explicitly calculated:

$$G_{mn}(t,s)=\begin{cases} \sum_{i=0}^{m-1}\left[\sum_{p=0}^{m-1-i}\binom{n-m+p-1}{p}\left(\frac{t-a}{b-a}\right)^p\right]\frac{(t-a)(a-s)^{n-i-1}}{i!(n-i-1)!}\left(\frac{b-t}{b-a}\right)^{n-m}, & s \leqslant t\\ -\sum_{i=0}^{n-m-1}\left[\sum_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\left(\frac{b-t}{b-a}\right)^q\right]\frac{(t-b)(b-s)^{n-i-1}}{i!(n-i-1)!}\left(\frac{t-a}{b-a}\right)^{m}, & s \geqslant t.\end{cases}$$

The reverse inequality in (3.10) implies the reverse inequality in (3.11).
(II) *If μ is a positive measure then:*
- *(i) if m and n have the same parity, the reverse of the inequality in (3.11) holds;*
- *(ii) if m and n have different parity, inequality in (3.11) always holds.*

COROLLARY 8. **(I)** *Let $f \in C^{2m}[a,b]$ be $(2m)$-convex, and G_{2T} the Green's function of the two-point Taylor problem:*

$$G_{2T}(t,s) = \frac{(-1)^m}{(2m-1)!}\begin{cases} p^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(t-s)^{m-1-j}q^j(t,s), & s \leqslant t\\ q^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(s-t)^{m-1-j}p^j(t,s), & s \geqslant t,\end{cases} \tag{3.14}$$

$$p(t,s) = \frac{(s-a)(b-t)}{b-a},\quad q(t,s) = p(s,t),\ \ \forall t,s \in [a,b].$$

Suppose that the measure μ is such that for all s

$$\int_a^b G_{2T}(s,t)d\mu(t) \leqslant 0. \tag{3.15}$$

Then

$$\begin{aligned}\int_a^b f(t)d\mu(t) \leqslant & \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\Big[f^{(i)}(a)\int_a^b \frac{(t-a)^i}{i!}\left(\frac{t-b}{a-b}\right)^m\left(\frac{t-a}{b-a}\right)^k d\mu(t)\\ & + f^{(i)}(b)\int_a^b \frac{(t-b)^i}{i!}\left(\frac{t-a}{b-a}\right)^m\left(\frac{t-b}{a-b}\right)^k d\mu(t)\Big].\end{aligned} \tag{3.16}$$

The reverse inequality in (3.15) implies the reverse inequality in (3.16).
(II) *If μ is positive, then*

(i) if m is even, the reverse of (3.16) holds;
(ii) if m is odd, (3.16) holds.

COROLLARY 9. **(I)** *Let $f:[a,b]\to\mathbf{R}$ be (4)-convex, and μ such that for all s*

$$\int_a^b G_O(t,s)d\mu(t)\leqslant 0. \tag{3.17}$$

Then

$$\begin{aligned}\int_a^b f(t)d\mu(t) &\leqslant \frac{f(a)}{(b-a)^2}\int_a^b[1+\frac{2(t-a)}{b-a}](t-b)^2d\mu(t)\\ &+ \frac{f(b)}{(b-a)^2}\int_a^b[1-\frac{2(t-b)}{b-a}](t-a)^2d\mu(t)\\ &+ \frac{f'(a)}{(b-a)^2}\int_a^b(t-a)(t-b)^2d\mu(t)\\ &+ \frac{f'(b)}{(b-a)^2}\int_a^b(t-b)(t-a)^2d\mu(t),\end{aligned} \tag{3.18}$$

where

$$G_O(t,s)=\begin{cases}\frac{(a-s)^2}{3!}(\frac{b-t}{b-a})^2[(t-s)+2\frac{(t-a)(b-s)}{b-a}], & s\leqslant t\\ \frac{(b-s)^2}{3!}(\frac{t-a}{b-a})^2[(s-t)+2\frac{(s-a)(b-t)}{b-a}], & s\geqslant t.\end{cases} \tag{3.19}$$

If the reverse inequality in (3.17) is valid, then the reverse inequality in (3.18) is also valid.
(II) *If μ is positive, then the reverse inequality in (3.18) holds.*

As a consequence of Theorem 3 for the Lagrange condition $n=2$, we obtain a Fink inequality (1.1) Theorem A1, (see also [21]).

COROLLARY 10. **(I)** *Let $f\in C^2[a,b]$ be convex, and G_L special Lagrange Greens's function defined by*

$$G_L(t,s)=\begin{cases}\frac{(a-s)(t-b)}{a-b}, & s\leqslant t\\ \frac{-(b-s)(t-a)}{b-a,} & s\geqslant t.\end{cases} \tag{3.20}$$

If μ is such a measure that for all $s\in[a,b]$

$$\int_a^b G_L(t,s)d\mu(t)\leqslant 0, \tag{3.21}$$

then

$$\begin{aligned}\int_a^b f(t)d\mu(t) &\leqslant f(a)\int_a^b\frac{b-t}{b-a}d\mu(t)+f(b)\int_a^b\frac{t-a}{b-a}d\mu(t)\\ &= P_0\Big[\frac{bf(b)-af(a)}{b-a}\Big]+P_1\Big[\frac{f(b)-f(a)}{b-a}\Big],\end{aligned} \tag{3.22}$$

where $P_k=\int_a^b t^k\,d\mu(t)$, $k=0,1$. The reverse inequality in (3.21) implies the reverse inequality in (3.22).
(II) *If μ is positive, then inequality in (3.22) holds.*

3.3. Generalization of the conversion of the Jensen inequality

By using Theorem A5 we also can get this theorem (see [22]):

THEOREM 4. *Let $f : [M_1, M_2] \to \mathbf{R}$ be (n)-convex and $m, n \in \mathbf{N}$ such that $1 \leqslant m \leqslant n-1$. Let $g : [a,b] \to [M_1, M_2]$ be integrable with respect to μ. If μ is such that*

$$\int_a^b G_{mn}(g(t),s)d\mu(t) \leqslant 0, \qquad \forall s \in [M_1, M_2], \tag{3.23}$$

then

$$\int_a^b f(g(t))d\mu(t) \leqslant \sum_{i=0}^{m-1} f^{(i)}(M_1) \int_a^b \tau_i(g(t))d\mu(t) + \sum_{i=0}^{n-m-1} f^{(i)}(M_2) \int_a^b \nu_i(g(t))d\mu(t), \tag{3.24}$$

where τ_i and ν_i are defined on $[M_1, M_2]$ by:

$$\begin{aligned} \tau_i(x) &= \frac{1}{i!}(x-M_1)^i \left(\frac{x-M_2}{M_1-M_2}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-M_1}{M_2-M_1}\right)^k \\ \nu_i(x) &= \frac{1}{i!}(x-M_2)^i \left(\frac{x-M_1}{M_2-M_1}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-M_2}{M_1-M_2}\right)^k. \end{aligned}$$

If the reverse inequality in (3.23) is valid, then the reverse inequality in (3.24) holds.

The following corollary is generalization of Theorem A2 for $(2m)$-convex function and signed measure μ.

COROLLARY 11. *Let $f(t) : [M_1, M_2] \to \mathbf{R}$ be $(2m)$-convex, $g : [a,b] \to [M_1, M_2]$ integrable with respect to μ,and μ such that*

$$\int_a^b G_{2T}(g(t),s)d\mu(t) \leqslant 0, \qquad \forall s \in [M_1, M_2]. \tag{3.25}$$

Then

$$\begin{aligned} \int_a^b f(g(t))d\mu(t) \leqslant & \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \qquad (3.26) \\ & \left[\ f^{(i)}(M_1) \int_a^b \frac{(g(t)-M_1)^i}{i!} \left(\frac{g(t)-M_2}{M_1-M_2}\right)^m \left(\frac{g(t)-M_1}{M_2-M_1}\right)^k d\mu(t) \right. \\ & \left. + \ f^{(i)}(M_2) \int_a^b \frac{(g(t)-M_2)^i}{i!} \left(\frac{g(t)-M_1}{M_2-M_1}\right)^m \left(\frac{g(t)-M_2}{M_1-M_2}\right)^k d\mu(t) \right], \end{aligned}$$

where the Green's function G_{2T} is defined on $[M_1, M_2] \times [M_1, M_2]$. If the reverse inequality in (3.25) is valid, then the reverse inequality in (3.26) is also valid.

For the Lagrange condition $n = 2$, the following corollary is the same generalization of Theorem A2 for a signed measure μ as Corollary 2, (see also [21]).

COROLLARY 12. *Let $f(t)$ be convex on $[M_1, M_2]$, and $g : [a,b] \to [M_1, M_2]$ integrable with respect to μ. If μ is such that $\int_a^b d\mu(t) = 1$ and*

$$\int_a^b G_L(g(t), s)\, d\mu(t) \leqslant 0, \qquad \forall s \in [M_1, M_2], \tag{3.27}$$

then

$$\int_a^b f(g(t))d\mu(t) \leqslant f(M_1)\left(\frac{M_2 - \int_a^b g(t)d\mu(t)}{M_2 - M_1}\right) + f(M_2)\left(\frac{\int_a^b g(t)d\mu(t) - M_1}{M_2 - M_1}\right). \tag{3.28}$$

The reverse inequality in (3.27) implies the reverse inequality in (3.28).

By using Corollary 12 we can, also obtain the results which are the generalizations of Theorem A3 (see [22]).

3.4. Generalization of Giaccardi and Petrović inequality

In discrete case of Theorem 4 we also obtain some interesting inequalities for convex function of higher order, some being generalizations of the well known ones mentioned before .

COROLLARY 13. *Let $\mathbf{p}, \mathbf{x} \in \mathbf{R^n}$ be such that $x_0 \leqslant x_i \leqslant \sum_{k=1}^n p_k x_k \quad x_0$, $\sum_{k=1}^n p_k x_k \in [c,d]$ and $\sum_{k=1}^n p_k x_k \neq x_0$. If we choose $\mathbf{p} = (p_1, \ldots p_n)$ in such a way that*

$$\sum_{k=1}^n p_k G_{mN}(x_k, s) \leqslant 0, \qquad \forall s \in [c,d], \tag{3.29}$$

then for (N)-convex function $f : [c,d] \to \mathbf{R}$ and $1 \leqslant m \leqslant N-1$ we have the following inequality:

$$\sum_{i=1}^n p_i f(x_i) \leqslant \sum_{k=0}^{m-1} f^{(k)}(x_0) \sum_{i=1}^n p_i \tau_k(x_i) + \sum_{k=0}^{N-m-1} f^{(k)}(\sum_{j=1}^n p_j x_j) \sum_{i=1}^n p_i \nu_k(x_i). \tag{3.30}$$

The reverse inequality in (3.29) implies the reverse inequality in (3.30).

In case of two-point Taylor conditions, we have the following generalization of Giaccardi inequality (1.5) for $(2m)$-convex function (see also [21]):

COROLLARY 14. *Let $\mathbf{p}, \mathbf{x} \in \mathbf{R}^n$ be such that (2.14) holds, $x_0, \sum_{k=1}^n p_k x_k \in [c,d]$, $\sum_{k=1}^n p_k x_k \neq x_0$. If*

$$\sum_{k=1}^n p_k G_{2T}(x_k, s) \leqslant 0, \forall s \in [c,d], \tag{3.31}$$

then for a $(2m)$*-convex function* $f : [c,d] \to \mathbf{R}$ *the following inequality holds:*

$$\begin{aligned} \sum_{i=1}^{n} p_i f(x_i) \leqslant & \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \\ & [\; f^{(i)}(x_0) \sum_{i=1}^{n} p_i \frac{(x_i-c)^i}{i!} \left(\frac{x_i-d}{c-d}\right)^m \left(\frac{x_i-c}{d-c}\right)^k \\ & + f^{(i)}(\sum_{j=1}^{n} p_j x_j) \sum_{i=1}^{n} \frac{(x_i-d)^i}{i!} \left(\frac{x_i-c}{d-c}\right)^m \left(\frac{x_i-d}{c-d}\right)^k]. \end{aligned} \tag{3.32}$$

The reverse inequality in (3.31) also implies the reverse inequality in (3.32).

4. Abel-Gontscharoff interpolating polynomial and some inequalities for convex functions of higher order

4.1. Abel-Gontscharoff interpolating polynomial

Let $-\infty < a < b < \infty$, and $a \leqslant a_1 \leqslant a_2 ... \leqslant a_n \leqslant b$ be given. It is well known, that for $f \in C^n[a,b]$ a unique polynomial $P(t)$ of degree $(n-1)$, [7] exists, fulfilling one of the following conditions:

Abel-Gontscharoff conditions:

$$P_A^{(i)}(a_{i+1}) = f^{(i)}(a_{i+1});\; 0 \leqslant i \leqslant n-1,$$

in particular, for fixed $0 \leqslant \alpha \leqslant n-2$:

Two-point right focal conditions:

$$\begin{aligned} P_{2F}^{(i)}(a_1) &= f^{(i)}(a_1),\; 0 \leqslant i \leqslant \alpha, \\ P_{2F}^{(i)}(a_2) &= f^{(i)}(a_2),\; \alpha+1 \leqslant i \leqslant n-1, \\ a \leqslant a_1 < a_2 \leqslant b. & \end{aligned}$$

The associated error $|e_{2F}(t)|$ can be represented in terms of the Green's function $G_{2F}(t,s)$ for the boundary value problem
$z^{(n)}(t) = 0$, $z^{(i)}(a_1) = 0,\, 0 \leqslant i \leqslant \alpha,\;\; z^{(i)}(a_2) = 0,\, \alpha+1 \leqslant i \leqslant n-1$:

$$G_{2F}(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum_{i=0}^{\alpha} \binom{n-1}{i} (t-a_1)^i (a_1-s)^{n-i-1}, & a \leqslant s \leqslant t \\ -\sum_{i=\alpha+1}^{n-1} \binom{n-1}{i} (t-a_1)^i (a_1-s)^{n-i-1}, & t \leqslant s \leqslant b. \end{cases} \tag{4.1}$$

For $n=2$ Green's function for 'two-point right focal' problem is

$$G_{2F}(t,s) = \begin{cases} (a_1-s), & a \leqslant s \leqslant t \\ -(t-a_1), & t \leqslant s \leqslant b. \end{cases} \tag{4.2}$$

The following result holds [7]:

THEOREM A 6. *Let* $F \in C^n[a,b]$, *and let* P_{2F} *be its 'two-point right focal' interpolating polynomial. Then*

$$\begin{aligned} F(t) &= P_{2F}(t) + e_{2F}(t) \\ &= \sum_{i=0}^{\alpha} \frac{(t-a_1)^i}{i!} F^{(i)}(a_1) \\ &+ \sum_{j=0}^{n-\alpha-2} \Big[\sum_{i=0}^{j} \frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!} \Big] F^{(\alpha+1+j)}(a_2) \\ &+ \int_a^b G_{2F}(t,s) F^{(n)}(s) ds, \end{aligned} \tag{4.3}$$

where G_{2F} is the Green's function, defined by (4.1).

We use the following inequalities [2] which describes the positivity property of the Green's function (4.1):

LEMMA A 3. *The Green's function $G_{2F}(t,s)$ has the following properties for fixed $0 \leqslant \alpha \leqslant n-2$:*

$$(-1)^{n-1-\alpha} G_{2F}(t,s) \geqslant 0, \quad a_1 \leqslant s, t \leqslant a_2. \tag{4.4}$$

4.2. Some inequalities related with the converse of the Jensen inequality

THEOREM 5. **(I)** *Let $f : [M_1, M_2] \to \mathbf{R}$ be (n)-convex and α be fixed, $0 \leqslant \alpha \leqslant n-2$. Let μ be a regular, signed Borel measure. Let $g : [a,b] \to [M_1, M_2]$ be integrable with respect to μ. If μ is such that*

$$\int_a^b G_{2F}(g(t), s) d\mu(t) \leqslant 0, \qquad \forall s \in [M_1, M_2], \tag{4.5}$$

where G_{2F} is defined by (4.1), then

$$\begin{aligned} &\int_a^b f(g(t)) d\mu(t) \leqslant \sum_{i=0}^{\alpha} f^{(i)}(M_1) \frac{1}{i!} \int_a^b (g(t)-M_1)^i d\mu(t) \\ &+ \sum_{j=0}^{n-\alpha-2} f^{(\alpha+1+j)}(M_2) \\ &\Big[\sum_{i=0}^{j} \frac{(M_1-M_2)^{j-i}}{(\alpha+1+i)!(j-i)!} \int_a^b (g(t)-M_1)^{\alpha+1+i} d\mu(t) \Big] \end{aligned} \tag{4.6}$$

If the reverse inequality in (4.5) is valid, then the reverse inequality in (4.6) holds.
(II) *Let $f \in C^n[a,b]$ be (n)-convex. Suppose that the measure μ is positive, then*

(i) if $n-\alpha-1$ is even, the reverse of (4.6) holds;

(ii) if $n-\alpha-1$ is odd, (4.6) holds.

Proof. By using Theorem A6, for $f(g(t))$ we have

$$
\begin{aligned}
f(g(t)) &= \sum_{i=0}^{\alpha} \frac{(g(t)-M_1)^i}{i!} f^{(i)}(M_1) \\
&+ \sum_{j=0}^{n-\alpha-2} \Big[\sum_{i=0}^{j} \frac{(g(t)-M_1)^{\alpha+1+i}(M_1-M_2)^{j-i}}{(\alpha+1+i)!(j-i)!}\Big] f^{(\alpha+1+j)}(M_2) \\
&+ \int_{M_1}^{M_2} G_{2F}(g(t),s) f^{(n)}(s)ds, \quad t \in [a,b].
\end{aligned}
$$

We can then integrate and use the condition (4.5) to get the result (4.6). Statement (II) follows from the fact (4.4). □

For the condition $n=2$, the following corollary is some version of Theorem A2 for a signed measure μ, (see also [21],[22]).

COROLLARY 15. *Let $f(t)$ be convex on $[M_1,M_2]$, and $g:[a,b]\to[M_1,M_2]$ be integrable with respect to μ. If μ is such that $\int_a^b d\mu(t)=1$ and*

$$\int_a^b G_{2F}(g(t),s)\, d\mu(t) \leqslant 0, \qquad \forall s \in [M_1,M_2], \tag{4.7}$$

where G_{2F} is defined as (4.2), then

$$\int_a^b f(g(t))d\mu(t) \leqslant f(M_1)+f'(M_2)\left(\int_a^b g(t)d\mu(t)-M_1\right). \tag{4.8}$$

The reverse inequality in (4.7) implies the reverse inequality in (4.8).

By using Corollary 15 we can obtain the result which is related with Theorem A3 (see [21], [22]):

COROLLARY 16. *Let the conditions of Corollary 15 be satisfied, and let J be an interval such that $J \supset f(I)$. If $\Phi(u,v)$ is a real function defined on $J\times J$, non-decreasing in u, then*

$$
\begin{aligned}
&\Phi\Big[\int_a^b f(g(t))d\mu(t), f\Big(\int_a^b g(t)d\mu(t)\Big)\Big] \\
&\leqslant \max_{x\in[M_1,M_2]} \Phi\left[f(M_1)+(x-M_1)f'(M_2), f(x)\right]
\end{aligned}
\tag{4.9}
$$

The right-hand side of (4.9) is a non-decreasing function of M_2, and a non-increasing function of M_1.

4.3. Some inequalities related with a Hadamard inequality

COROLLARY 17. **(I)** *Let $f \in C^n[a,b]$ be (n)-convex, and G_{2F} the Green's function of the 'two-point right focal' problem. Suppose that the measure μ is such* *that (4.5)* $|_{g(t)=t}$ *holds for all* s, *then*

$$\begin{aligned}\int_a^b f(t)d\mu(t) &\leqslant \sum_{i=0}^{\alpha} f^{(i)}(a)\frac{1}{i!}\int_a^b (t-a)^i d\mu(t) \\ &+ \sum_{j=0}^{n-\alpha-2} f^{(\alpha+1+j)}(b)\Big[\sum_{i=0}^{j}\frac{(a-b)^{j-i}}{(\alpha+1+i)!(j-i)!}\int_a^b (t-a)^{\alpha+1+i}d\mu(t)\Big].\end{aligned} \tag{4.10}$$

The reverse inequality in (4.5) $|_{g(t)=t}$ *implies the reverse inequality in (4.10).*
(II) *Let* $f \in C^n[a,b]$ *be (* n *)-convex. Suppose that the measure* μ *is positive, then*
(i) if $n-\alpha-1$ *is even, the reverse of (4.10) holds;*
(ii) if $n-\alpha-1$ *is odd, (4.10) holds.*

Proof. By using Theorem 5, $g(t)=t$, we first prove statement (I). Statement (II) follows from the fact (4.4). □

For the condition $n=2$, we obtain an inequality related with Fink inequality (1.1) and Corollary 10 (see also [21], [22]).

COROLLARY 18. *(I) Let* $f \in C^2[a,b]$ *be convex, and* G_{2F} *the Greens's function defined by (4.2). If* μ *is a measure such that for all* $s \in [a,b]$

$$\int_a^b G_{2F}(s,t)d\mu(t) \leqslant 0, \tag{4.11}$$

then

$$\begin{aligned}\int_a^b f(t)d\mu(t) &\leqslant f(a)\int_a^b d\mu(t) + f'(b)\int_a^b (t-a)d\mu(t) \\ &= P_0[f(a)-af'(b)] + P_1 f'(b),\end{aligned} \tag{4.12}$$

where $P_k = \int_a^b t^k\, d\mu(t)$, $k=0,1$. *The reverse inequality in (4.11) implies the reverse inequality in (4.12).*
(II) Let $f \in C^2[a,b]$ *be convex. If* μ *is a positive measure, then inequality (4.12) holds.*

Proof. Statement (II) follows from the fact that $G_{2F}(s,t) < 0, \forall s,t \in [a,b]\times[a,b]$. □

4.4. Some inequalities related with Giaccardi and Petrović inequality

In discrete case we also obtain some interesting inequalities for convex function of higher order, wich are related with the well known ones mentioned before.

COROLLARY 19. *Let* $\mathbf{p},\mathbf{x} \in \mathbf{R^n}$ *be such that* $x_0 \leqslant x_i \leqslant \sum_{k=1}^n p_k x_k$, x_0, $\sum_{k=1}^n p_k x_k \in [c,d]$ *and* $\sum_{k=1}^n p_k x_k \neq x_0$. *If we choose* $\mathbf{p}=(p_1,\dots p_n)$ *in such a way that*

$$\sum_{k=1}^n p_k G_{2F}(x_k,s) \leqslant 0, \qquad \forall s \in [c,d], \tag{4.13}$$

where G_{2F} *is defined in (4.1) then for* (N)*-convex function* $f : [c,d] \to \mathbf{R}$ *and* $0 \leqslant \alpha \leqslant N-2$ *we have the following inequality:*

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant \sum_{i=0}^{\alpha} f^{(i)}(x_0) \frac{1}{i!} \sum_{k=1}^{n} p_k (x_k - x_0)^i + \sum_{j=0}^{N-\alpha-2} f^{(\alpha+1+j)}\Big(\sum_{l=1}^{n} p_l x_l\Big) \left[\sum_{i=1}^{j} \frac{(x_0 - \sum_{l=1}^{n} p_l x_l)^{j-i}}{(\alpha+1+i)!(j-i)!} \sum_{k=1}^{n} p_k (x_k - x_0)^{\alpha+1+i}\right]. \tag{4.14}$$

The reverse inequality in (4.13) implies the reverse inequality in (4.14).

Proof. We take a discrete measure μ in Theorem 5, and $g(t) = t$ for all $t \in [a,b]$. We can suppose that $[c,d] \supset [M_1, M_2]$, where M_1, M_2 are as in Theorem 5. For a specific choice of M_1 and M_2, take

$$M_1 = x_0,\ M_2 = \sum_{k=1}^{n} p_k x_k,$$

we have

$$x_0 \leqslant x_i \leqslant \sum_{k=1}^{n} p_k x_k. \quad \square$$

In case $N = 2$ we have the following type of Petrović inequality (1.5) for convex function also [21], [22]):

COROLLARY 20. *Let* $\mathbf{p}, \mathbf{x} \in \mathbf{R}^n$ *be as Corollary 19. If*

$$\sum_{k=1}^{n} p_k G_{2F}(x_k, s) \leqslant 0,\ \forall s \in [c,d], \tag{4.15}$$

where G_{2F} *is defined in (4.2), then for a convex function* $f : [c,d] \to \mathbf{R}$ *the following inequality holds:*

$$\sum_{k=1}^{n} p_k f(x_k) \ \leqslant\ f(x_0) \sum_{k=1}^{n} p_k + f'\Big(\sum_{l=1}^{n} p_l x_l\Big) \sum_{k=1}^{n} p_k (x_k - x_0). \tag{4.16}$$

The reverse inequality in (4.15) also implies the reverse inequality in (4.16).

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