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A STRENGTHENED CAUCHY–SCHWARZ INEQUALITY FOR BIDIMENSIONAL SPLINE–WAVELETS

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Abstract. Multilevel methods have been widely applied for the approximation of the solutions of the elliptic partial differential equations in the frame of finite element spaces and, recently, owing to the development of the wavelet theory, in wavelet spaces. It has been pointed out that the strengthened Cauchy-Buniakowski-Schwarz inequality is the main tool in the analysis of multilevel methods. In this paper, results on the strengthened Cauchy-Buniakowski-Schwarz inequality are reviewed for one dimensional biorthogonal wavelets and, as original contribution, a theorem is proposed for the bidimensional case, concerning order one spline-wavelets.

1. Introduction

In this paper we present some results about the so-called *strengthened Cauchy-Schwarz inequality* for biorthogonal wavelet spaces. The usual Cauchy-Schwarz inequality is refined by the strengthened one in the sense that it states the existence of a constant $\gamma \in [0,1)$ such that

$$|(v,w)| \leqslant \gamma ||v|| \, ||w||$$

for $v \in V$, $w \in W$, where V, W are linear subspaces of a Hilbert space, with $V \cap W = \{0\}$, and γ depends only on V and W, and not on the choice of the functions v and w. The smallest such quantity γ may be called the cosine of the angle between the spaces V and W.

Such inequality is a tool in the resolution of elliptic partial differential equations, because it has an important role in a posteriori error estimation and, therefore, in the formulation of iterative and adaptive wavelet-based methods (see for example [2], [7], [3]). The case of multilevel discretization in wavelet spaces has been largely studied in last years, and the development of wavelet theory has permit to make several numerical works in this field. We point-out that, in the context of approximation of partial differential equations, the more suitable wavelet multilevel decomposition is the biorthogonal one and, in particular, the biorthogonal spline-wavelet decomposition.

So in particular, to obtain results about a posteriori error estimates for adaptive wavelet-based methods, we have to check in this frame the strengthened Cauchy-Schwarz inequality. We observe that some proofs of it already exist for finite element spaces, but the case of wavelet spaces is not yet completely investigated. Since now,

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as far as we know, the only proof existing is a strengthened Cauchy-Schwarz inequality for general biorthogonal wavelets defined on the real line and on an interval (that is, for the one-dimensional case) [8]. Some new results for two-dimensional biorthogonal wavelets will be presented in this paper and in successive works.

The contents are, in short, the following. In section 2, first we recall the concept of biorthogonal wavelet multilevel decomposition, then the important example of the biorthogonal spline-wavelet decomposition is briefly examined. In section 3, results are presented of strengthened Cauchy-Schwarz inequalities for biorthogonal wavelet spaces. A special emphasis is dedicated to biorthogonal spline-wavelets.

2. Wavelet multilevel decomposition

We first present in short a definition of biorthogonal wavelets, addressed to non experts.

Let V be a separable Hilbert space equipped with inner product (u,v) and norm $||v|| = (v,v)^{\frac{1}{2}}$. Take as basic example $V = L^2(\mathbf{R})$, or some other function space on the real line. We assume that we are given two families $\{V_j\}_{j\in\mathbf{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbf{Z}}$ of closed and nested subspaces of V. Let φ and $\tilde{\varphi}$ be two functions in V such that

$$\begin{aligned} V_j =& \mathrm{span}_V\{\varphi_{jk},\ k\in\mathbf{Z}\} \qquad \forall j\in\mathbf{Z} \\ \tilde{V}_j =& \mathrm{span}_V\{\tilde{\varphi}_{jk},\ k\in\mathbf{Z}\} \qquad \forall j\in\mathbf{Z} \end{aligned}$$

where

$$\begin{aligned} \varphi_{jk}(x) =& 2^{\frac{j}{2}}\varphi(2^j x - k) \qquad j,\, k\in\mathbf{Z} \\ \tilde{\varphi}_{jk}(x) =& 2^{\frac{j}{2}}\tilde{\varphi}(2^j x - k) \qquad j,\, k\in\mathbf{Z} \end{aligned}$$

and

$$(\varphi_{jk},\tilde{\varphi}_{jk'}) = \delta_{kk'} \qquad j\in\mathbf{Z},\ k,\, k'\in\mathbf{Z}.$$

The subpaces family $\{V_j\}_{j\in\mathbf{Z}}$ must satisfy the properties

$$\overline{\bigcup_{j\in\mathbf{Z}} V_j}^{V} = V \qquad \bigcap_{j\in\mathbf{Z}} V_j = \{0\}; \tag{2.1}$$

similar properties must be satisfied by $\{\tilde{V}_j\}_{j\in\mathbf{Z}}$. The functions φ and $\tilde{\varphi}$ are named *scaling functions*, while the bases $\{\varphi_{jk}\}_{j,k\in\mathbf{Z}}$ and $\{\tilde{\varphi}_{jk}\}_{j,k\in\mathbf{Z}}$ are called *canonical bases* of the multilevel decomposition of V. In fact a function $v\in V$ could be approximated by a function $v_j\in V_j$ or $\tilde{v}_j\in\tilde{V}_j$ for (2.1); v_j and $\tilde{v}_j$ are named *j-ths approximations* of v.

Now let us introduce spaces W_j and $\tilde{W}_j$ for each $j\in\mathbf{Z}$ such that

$$V_{j+1} = V_j\oplus W_j \qquad \tilde{V}_{j+1} = \tilde{V}_j\oplus\tilde{W}_j \tag{2.2}$$

with the *biorthogonality conditions*

$$V_j\perp\tilde{W}_j \qquad \tilde{V}_j\perp W_j;$$

moreover, let ψ, $\tilde{\psi}$ be functions such that

$$W_j = \text{span}_V\{\psi_{jk},\ k \in \mathbf{Z}\} \qquad \forall j \in \mathbf{Z}$$
$$\tilde{W}_j = \text{span}_V\{\tilde{\psi}_{jk},\ k \in \mathbf{Z}\} \qquad \forall j \in \mathbf{Z}$$

with

$$(\psi_{jk}, \tilde{\psi}_{j'k'}) = \delta_{jj'}\delta_{kk'} \qquad \forall j, j' \in \mathbf{Z},\ \forall k, k' \in \mathbf{Z}.$$

From (2.1), (2.2) we have two decompositions of V:

$$V = \bigoplus_{j\in\mathbf{Z}} W_j = \bigoplus_{j\in\mathbf{Z}} \tilde{W}_j,$$

which imply that, for each $v \in V$, we can write

$$v = \sum_{j,k\in\mathbf{Z}} \hat{v}_{jk}\psi_{jk} = \sum_{j,k\in\mathbf{Z}} \hat{\hat{v}}_{jk}\tilde{\psi}_{jk}$$

where

$$\hat{v}_{jk} = (v, \tilde{\psi}_{jk}) \qquad \hat{\hat{v}}_{jk} = (v, \psi_{jk}).$$

The bases $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ and $\{\tilde{\psi}_{jk}\}_{j,k\in\mathbf{Z}}$ are called *hierarchical bases* with respect to the multilevel decomposition of V. It is obvious from (2.2) that functions $v_{j+1} \in V_{j+1}$ and $\tilde{v}_{j+1} \in \tilde{V}_{j+1}$ could be expressed in the following ways

$$v_{j+1} = v_j + w_j \qquad \tilde{v}_{j+1} = \tilde{v}_j + \tilde{w}_j$$

where $v_j \in V_j$, $w_j \in W_j$, $\tilde{v}_j \in \tilde{V}_j$ and $\tilde{w}_j \in \tilde{W}_j$; w_j and $\tilde{w}_j$ are the *details* needed to obtain approximations of level $j+1$ from the approximations of level j.

The construction of functions φ, $\tilde{\varphi}$, ψ, $\tilde{\psi}$ satisfying simultaneously all the preceding properties is, of course, a non- trivial task; let us address the reader to, for example, [5] for full details of the theory.

A fundamental example of biorthogonal wavelet decomposition is given by the spline-wavelet decomposition of $L^2(\mathbf{R})$. In particular we may consider the spline-wavelet decomposition of order one, where $\varphi(x) = (1-|x|)_+$ and $\psi(x)$ is the so called *mexican hat function*

$$\psi(x) = -\frac{1}{4}\varphi(2x+1) - \frac{1}{2}\varphi(2x) + \frac{3}{2}\varphi(2x-1) - \frac{1}{2}\varphi(2x-2) - \frac{1}{4}\varphi(2x-3).$$

Beginning from the previous setting we are able to construct a multilevel decomposition on an interval I of the real line, that is to consider $V = L^2(I)$ (see [1] and [4]). To get a multilevel decomposition on $V = L^2(\mathbf{R}^2)$ or other relevant space on $\mathbf{R}^2$, a way is to consider a tensorial product of the decompositions of $L^2(\mathbf{R})$. Namely, we can define in general

$$V_j(\mathbf{R}^2_{xy}) = V_j(\mathbf{R}_x) \otimes V_j(\mathbf{R}_y)$$

and consequently

$$W_j(\mathbf{R}^2) = (V_j(\mathbf{R}_x) \otimes W_j(\mathbf{R}_y)) \oplus (W_j(\mathbf{R}_x) \otimes V_j(\mathbf{R}_y)) \oplus (W_j(\mathbf{R}_x) \otimes W_j(\mathbf{R}_y)).$$

In particular, we can consider the spline decomposition of order one in $\mathbf{R}^2$, with

$$V_j(\mathbf{R}^2_{xy}) = \operatorname{span}_{V\otimes V}\{\varphi_{jkk'}(x,y) = \varphi_{jk}(x)\varphi_{jk'}(y),\ k,k' \in \mathbf{Z}\}$$

starting from $\varphi(x) = (1-|x|)_+$, and define $W_j(\mathbf{R}^2)$ consequently in terms of the mexican hat function $\psi(x)$ (see, for example, [6] or [4]).

3. The strengthened Cauchy-Schwarz inequality for biorthogonal wavelets

We begin by recalling the result of De Rossi [8]. Under general hypotheses on φ and ψ (cf. for example [6]), the strengthened Cauchy-Schwarz inequality was there proved for one-dimensional biorthogonal wavelets, namely

$$\exists\gamma \in [0,1) \quad \text{such} \quad \text{that} \quad |(v,w)| \leqslant \gamma||v||\,||w||$$

for all $v \in V_j(\mathbf{R})$ and $w \in W_j(\mathbf{R})$, according to the definition of section 2; the constant γ does not depend on j. Scalar product and norm are in the H^1 sense, namely $(v,w) = \int v'(x)\overline{w'(x)}dx$; the method in [8] gives easily the result in L^2 with standard L^2 norms also. At this moment, a general proof of the same inequality for wavelets in the two-dimensional case seems not available in literature. As a first step in this direction, we begin to present here a result concerning spline-wavelets decomposition of order one in $\mathbf{R}^2$, defined by tensor product as at the end of section 2.

We shall limit ourselves to estimate γ when $w \in E_3$, the 3-dimensional vector subspace of $W_0(\mathbf{R}^2)$ given by

$$w(x,y) = c_1\psi(x)\varphi(y) + c_2\varphi(x)\psi(y) + c_3\psi(x)\psi(y) \tag{3.1}$$

and $v \in F_{16}$, the 16-dimensional vector subspace of $V_0(\mathbf{R}^2)$ spanned by all the $\varphi_{0kk'}(x,y)$ whose support has non-empty intersection with the region $[-1,2]\times[-1,2]$ including the support of the preceding $w(x,y)$, namely

$$v(x,y) = \sum_{k=-1}^{2}\sum_{k'=-1}^{2} c_{kk'}\varphi_{0k}(x)\varphi_{0k'}(y). \tag{3.2}$$

Let us observe that, in computing (v,w) for w as in (3.1) and a generic $v \in V_0$, only terms of the form (3.2) actually matter, since $(v,w) = 0$ for $v = \varphi_{0k}(x)\varphi_{0k'}(y) \notin F_{16}$.

We shall prove:

THEOREM 3.1. *Define*

$$\gamma = \sup \frac{|(v,w)|}{||v||\,||w||} \tag{3.3}$$

v running in F_{16}, w in E_3. Then

$$\begin{aligned} 0.2536286367 \leqslant &\gamma \leqslant 0.4168274363 \quad \text{in } L^2 \text{ norm},\\ 0.1740776559 \leqslant &\gamma \leqslant 0.2729316204 \quad \text{in } H^1 \text{ norm}. \end{aligned} \tag{3.4}$$

By a change of variables the result can be easily extended to corresponding subspaces of same dimension of $V_j(\mathbf{R}^2)$ and $W_j(\mathbf{R}^2)$. Concerning the general case $v \in V_j(\mathbf{R}^2)$, $w \in W_j(\mathbf{R}^2)$ (order one spline-wavelets in $\mathbf{R}^2$), the lower bounds in (3.4) keeps obviously valid, whereas we cannot draw any conclusion about the upper bounds. We think however that the particular case we have treated is representative of the general situation, namely we expect that (3.4) keeps valid for γ defined by (3.3) with $v \in V_j(\mathbf{R}^2)$, $w \in W_j(\mathbf{R}^2)$.

As for the proof of theorem 3.1, in principle it would be possible to obtain γ explicitly, by reducing the problem to the computation of a maximum in a finite dimensional setting; however, due to the high number of variables, this proceeding presents evident difficulties. Instead, we give here a quick proof of the bounds (3.4), by combining arguments of functional analysis and some numerical computations, performed with Maple V Release 6.

Let us begin with the following preliminary result, valid in any Hilbert space V.

LEMMA 3.2. *Let v, w_j, $j = 1, \dots, n$, belong to V, write $w = \sum_{j=1}^n w_j$ and assume there exist non-negative constants $\gamma_j < 1$, δ_{jh} with $\delta_{jh} = \delta_{hj}$, for $j = 1, \dots, n$, $h = 1, \dots, n$ and $\max_j \sum_{h \neq j} \delta_{jh} < 1$, such that*

$$|(v, w_j)| \leqslant \gamma_j ||v|| \, ||w_j||, \qquad j = 1, \dots, n,$$
$$|(w_j, w_h)| \leqslant \delta_{jh} ||w_j|| \, ||w_h||, \qquad j, h = 1, \dots, n.$$

Then

$$|(v, w)| \leqslant \gamma ||v|| \, ||w||$$

with

$$\gamma = \frac{\sqrt{\sum_{j=1}^n \gamma_j^2}}{\sqrt{1 - \max_j \sum_{h, h \neq j} \delta_{jh}}}.$$

Proof. We have

$$|(v, w)| \leqslant \sum_{j=1}^n |(v, w_j)| \leqslant \sum_{j=1}^n \gamma_j ||v|| \, ||w_j||$$
$$\leqslant ||v|| \sqrt{\sum_{j=1}^n \gamma_j^2} \sqrt{\sum_{j=1}^n ||w_j||^2}.$$

To get the conclusion, it will be now sufficient to prove that

$$\sqrt{1 - \max_j \sum_{h, h \neq j} \delta_{jh}} \sqrt{\sum_{j=1}^n ||w_j||^2} \leqslant ||w||. \tag{3.5}$$

In fact

$$||w||^2 = \sum_{j=1}^n ||w_j||^2 + \sum_{\substack{j,h \\ h \neq j}} (w_j, w_h),$$

hence

$$\begin{aligned}\sum_{j=1}^{n} ||w_j||^2 &\leqslant ||w||^2 + \sum_{\substack{j,h\\ h\neq j}} |(w_j, w_h)| \\ &\leqslant ||w||^2 + \frac{1}{2}\sum_{\substack{j,h\\ h\neq j}} 2\delta_{jh}||w_j||\,||w_h|| \\ &\leqslant ||w||^2 + \frac{1}{2}\sum_{\substack{j,h\\ h\neq j}} \delta_{jh}(||w_j||^2 + ||w_h||^2) \\ &= ||w||^2 + \sum_{j}\left(\sum_{h,h\neq j} \delta_{jh}||w_j||^2\right).\end{aligned}$$

Therefore

$$\sum_{j=1}^{n}\left(1 - \sum_{h,h\neq j}\delta_{jh}\right)||w_j||^2 \leqslant ||w||^2$$

and (3.5) follows. The proof of the lemma is therefore concluded. □

The preceding lemma gives immediately the following two corollaries.

COROLLARY 3.3. *Assume in the preceding lemma* $\delta_{jh} = 0$ *for* $j,h = 1,\dots,n$, $h\neq j$, *i.e. the* w_j *are orthogonal each other, then in the conclusion*

$$\gamma = \sqrt{\sum_{j=1}^{n}\gamma_j^2}.$$

REMARK. The previous result is sharp in the following sense. Under the orthogonality assumption of corollary 3.3, if $\gamma_j = |(v,w_j)|/||v||\,||w_j||$, then considering the orthogonal projection of v on the space spanned by w_j:

$$\tilde{v} = \sum_{j=1}^{n}\frac{(v,w_j)w_j}{||w_j||^2}$$

we have

$$\gamma = \frac{|(v,\tilde{v})|}{||v||\,||\tilde{v}||} = \sqrt{\sum_{j=1}^{n}\gamma_j^2}.$$

COROLLARY 3.4. *Assume in the preceding lemma* $\gamma_j \leqslant \tilde{\gamma} < 1/n$, $\delta_{jh} \leqslant \tilde{\gamma} < 1/n$ *for all* $j = 1,\dots,n$, $h = 1,\dots,n$, $h\neq j$. *Then for* γ *in the conclusion, we have*

$$\gamma \leqslant \sqrt{\frac{n\tilde{\gamma}}{\frac{1}{\tilde{\gamma}} - n + 1}} < 1.$$

Proof of theorem 3.1. Let us write $\varphi_k(x) = \varphi(x-k)$, $\psi_k(x) = \psi(x-k)$, $k \in \mathbf{Z}$, in particular $\varphi_0 = \varphi$, $\psi_0 = \psi$. It will be convenient to introduce an orthogonal basis in F_{16}. Namely in L^2 norm we shall write

$$v(x,y) = \sum_{k=-1}^{2} \sum_{j=-1}^{2} c_{kj} \varphi_k^\sharp(x) \varphi_j^\sharp(y) \tag{3.6}$$

where $\varphi_{-1}^\sharp = \varphi_{-1}$, $\varphi_1^\sharp = \varphi_1$ and

$$\varphi_0^\sharp = \varphi_0 - \frac{1}{4}\varphi_{-1} - \frac{1}{4}\varphi_1$$
$$\varphi_2^\sharp = \varphi_2 - \frac{1}{4}\varphi_1 + \frac{1}{14}\varphi_0^\sharp$$

are orthogonal in $L^2(\mathbf{R})$, assuring by tensor product the orthogonality of $\varphi_{jk}^\sharp(x,y) = \varphi_j^\sharp(x)\varphi_k^\sharp(y)$ in $L^2(\mathbf{R^2})$. In H^1 norm we have similar orthogonal decomposition with $\varphi_{-1}^\sharp = \varphi_{-1}$, $\varphi_1^\sharp = \varphi_1$ and

$$\varphi_0^\sharp = \varphi_0 + \frac{1}{2}\varphi_{-1} + \frac{1}{2}\varphi_1$$
$$\varphi_2^\sharp = \varphi_2 + \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_0^\sharp.$$

Let us begin by computing in the one-dimensional case

$$\gamma_{-1} = \frac{|(\varphi_{-1}^\sharp, \psi_0)|}{||\varphi_{-1}^\sharp||\,||\psi_0||} = \begin{cases} 0.1178511303 & \text{in } L^2 \\ 0.08703882800 & \text{in } H^1, \end{cases}$$

$$\gamma_0 = \frac{|(\varphi_0^\sharp, \psi_0)|}{||\varphi_0^\sharp||\,||\psi_0||} = \begin{cases} 0.1259881577 & \text{in } L^2 \\ 0.1230914910 & \text{in } H^1, \end{cases}$$

$$\gamma_1 = \frac{|(\varphi_1^\sharp, \psi_0)|}{||\varphi_1^\sharp||\,||\psi_0||} = \begin{cases} 0.1178511303 & \text{in } L^2 \\ 0.08703882800 & \text{in } H^1, \end{cases}$$

$$\gamma_2 = \frac{|(\varphi_2^\sharp, \psi_0)|}{||\varphi_2^\sharp||\,||\psi_0||} = \begin{cases} 0.1437939210 & \text{in } L^2 \\ 0.0 & \text{in } H^1. \end{cases}$$

Let us observe that for $w_1(x,y) = \psi_0(x)\psi_0(y)$

$$\frac{|(\varphi_{jk}^\sharp, w_1)|}{||\varphi_{jk}^\sharp||\,||w_1||} = \frac{|(\varphi_j^\sharp, \psi_0)||(\varphi_k^\sharp, \psi_0)|}{||\varphi_j^\sharp||\,||\psi_0||\,||\varphi_k^\sharp||\,||\psi_0||} = \gamma_j\gamma_k.$$

We now begin to estimate, using corollary 3.3 and taking $v \in F_{16}$ as in (3.6):

$$\frac{|(v, w_1)|}{||v||\,||w_1||} \leqslant G_1$$

with

$$G_1 = \sqrt{\sum_{j=-1}^{2}\sum_{k=-1}^{2} \gamma_j^2\gamma_k^2} = \sum_{j=-1}^{2}\gamma_j^2.$$

We have

$$G_1 = \begin{cases} 0.06432748535 & \text{in} \quad L^2, \\ 0.03030303028 & \text{in} \quad H^1. \end{cases}$$

Let us consider now $w_2(x,y) = \psi_0(x)\varphi_0(y)$ and observe that in this case

$$\frac{|(\varphi_{jk}^\sharp, w_2)|}{||\varphi_{jk}^\sharp||\,||w_2||} = \frac{|(\varphi_j^\sharp, \psi_0)|}{||\varphi_j^\sharp||\,||\psi_0||}\,\frac{|(\varphi_k^\sharp, \varphi_0)|}{||\varphi_k^\sharp||\,||\varphi_0||} = \gamma_j\tilde{\gamma}_k$$

with

$$\tilde{\gamma}_k = \frac{|(\varphi_k^\sharp, \varphi_0)|}{||\varphi_k^\sharp||\,||\varphi_0||}.$$

Note that by Bessel identity

$$\sum_{k=-1}^{2}\tilde{\gamma}_k^2 = 1.$$

Therefore for $v \in F_{16}$

$$\frac{|(v, w_2)|}{||v||\,||w_2||} \leqslant G_2$$

with

$$G_2 = \sqrt{\sum_{j=-1}^{2}\sum_{k=-1}^{2} \gamma_j^2\tilde{\gamma}_k^2} = \sqrt{\sum_{j=-1}^{2}\gamma_j^2}$$

and then we have

$$G_2 = \begin{cases} 0.2536286367 & \text{in} \quad L^2, \\ 0.1740776559 & \text{in} \quad H^1. \end{cases}$$

In the same way we argue on $w_3(x,y) = \varphi_0(x)\psi_0(y)$ obtaining the bound

$$G_3 = G_2.$$

At this moment we compute

$$\delta_{12} = \frac{|(w_1, w_2)|}{||w_1||\,||w_2||} = \frac{|(\psi_0, \varphi_0)|}{||\varphi_0||\,||\psi_0||} = \begin{cases} 0.1178511303 & \text{in} \quad L^2 \\ 0.08703882796 & \text{in} \quad H^1, \end{cases}$$

$$\delta_{13} = \delta_{12},$$

$$\delta_{23} = \frac{|(w_2, w_3)|}{||w_2||\,||w_3||} = \left(\frac{|(\psi_0, \varphi_0)|}{||\varphi_0||\,||\psi_0||}\right)^2 = (\delta_{12})^2.$$

We finally consider (v, w) with $v \in F_{16}$ and $w = c_1w_1 + c_2w_2 + c_3w_3 \in E_3$ as before. Since G_1, G_2, G_3, δ_{12}, δ_{13}, δ_{23} are estimated by $1/3$, we may apply corollary 3.4 with $n = 3$. More precisely from lemma 3.2 we obtain the values of γ:

$$\gamma = \frac{\sqrt{G_1^2 + G_2^2 + G_3^2}}{\sqrt{1 - 2\delta_{12}}} = \begin{cases} 0.4168274363 & \text{in} \quad L^2 \\ 0.2729316204 & \text{in} \quad H^1. \end{cases}$$

This gives the upper bounds in theorem 3.1. To get the lower bounds, we fix $w = w_2$ and apply the remark after corollary 3.3 to the projection of w_2 on F_{16}; we obtain then the values corresponding to G_2. □

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