

SOME REMARKS ON HILBERT'S INTEGRAL INEQUALITY

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Abstract. In this paper we generalize Hilbert's integral inequality.

1. Introduction

If $f, g \in L^2[0, \infty)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left(\int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right)^{1/2} \quad (1.1)$$

where π is the best value.

The inequality (1.1) is well known as Hilbert's integral inequality.

Yang Bicheng [1] gives a generalization of Hilbert's integral inequality as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty x^{1-\lambda} g^2(x) dx \right\}^{1/2} \quad (1.2)$$

where $B(p, q)$ ($p, q > 0$) is the beta function and $\lambda \in (0, 1]$.

In [1] is proved also the following result:

Let $b > a > 0$ and $0 < \lambda \leq 1$, $f, g \in L^2[a, b]$. Then

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[1 - \left(\frac{a}{b}\right)^{\lambda/4} \right] \times \left(\int_a^b x^{1-\lambda} f^2(x) dx \int_a^b x^{1-\lambda} g^2(x) dx \right)^{1/2}. \quad (1.3)$$

In this paper we generalize inequality (1.2) and improve inequality (1.3).

Mathematics subject classification (2000): 26D15.

Key words and phrases: Hilbert's integral inequality.

2. Main results

A generalization of the inequality (1.1) is given in the following theorem.

THEOREM 1. *Let n be a natural number, $n \geq 2$. If $f_i \in L^n[0, \infty)$, $i = 1, 2, \dots, n$ then*

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1 + x_2 + \cdots + x_n)^\lambda} dx_1 dx_2 \cdots dx_n \\ & \leq \frac{\Gamma^2\left(\frac{2-n+\lambda}{2}\right)}{\Gamma(\lambda)} \left(\prod_{k=1}^n \int_0^\infty x^{n-1-\lambda} f_k^n(x) dx \right)^{1/n} \end{aligned} \quad (2.1)$$

for any $\lambda \in (n-2, n-1]$ where $\Gamma(x)$ ($x \geq 0$) is gamma-function.

Proof. The following equality holds:

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1 + x_2 + \cdots + x_n)^\lambda} dx_1 dx_2 \cdots dx_n \\ & = \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1) \left(\frac{x_1}{x_2}\right)^{\frac{\alpha}{n}} f_2(x_2) \left(\frac{x_2}{x_3}\right)^{\frac{\alpha}{n}} \cdots f_n(x_n) \left(\frac{x_n}{x_1}\right)^{\frac{\alpha}{n}}}{(x_1 + x_2 + \cdots + x_n)^\lambda} dx_1 dx_2 \cdots dx_n, \end{aligned} \quad (2.2)$$

where $\alpha = \frac{-\lambda + n}{2}$.

By Cauchy's inequality and (2.2), we have

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1 + x_2 + \cdots + x_n)^\lambda} dx_1 dx_2 \cdots dx_n \\ & \leq \prod_{k=1}^n \left(\int_0^\infty \cdots \int_0^\infty \left(\frac{x_k}{x_{k+1}}\right)^\alpha \frac{f_k^n(x_k)}{(x_1 + \cdots + x_n)^\lambda} dx_1 \cdots dx_n \right)^{\frac{1}{n}}, \end{aligned} \quad (2.3)$$

where $x_{n+1} = x_1$.

We have

$$x_1^\alpha \int_0^\infty \cdots \int_0^\infty \frac{1}{x_2^\alpha (x_1 + \cdots + x_n)^\lambda} dx_2 \cdots dx_n = x_k^\alpha \int_0^\infty \cdots \int_0^\infty \frac{dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n}{x_{k+1}^\alpha (x_1 + \cdots + x_n)^\lambda},$$

for $k = 1, 2, \dots, n$.

Putting $x_k = x_1 u_k$, $k = 2, \dots, n$, we obtain

$$\begin{aligned} & x_1^\alpha \int_0^\infty \cdots \int_0^\infty \frac{1}{x_2^\alpha (x_1 + \cdots + x_n)^\lambda} dx_2 \cdots dx_n \\ & = x_1^{n-1-\lambda} \int_0^\infty \cdots \int_0^\infty \frac{u_2^{-\alpha}}{(1 + u_2 + \cdots + u_n)^\lambda} du_2 \cdots du_n. \end{aligned} \quad (2.4)$$

But

$$\int_0^\infty \cdots \int_0^\infty \frac{u_2^{-\alpha}}{(1 + u_2 + \cdots + u_n)^\lambda} du_2 \cdots du_n = \frac{\Gamma^2\left(\frac{2-n+\lambda}{2}\right)}{\Gamma(\lambda)}. \tag{2.5}$$

From relations (2.5), (2.4) and (2.3) we obtain inequality (2.1).

REMARK 1. For $n = 2$ inequality (2.1) becomes (1.1).

REMARK 2. For $\lambda = n - 1$ we obtain the following inequality

$$\int_0^\infty \cdots \int_0^\infty \frac{f_1(x_1) \cdots f_n(x_n)}{(x_1 + \cdots + x_n)^{n-1}} dx_1 \cdots dx_n \leq \frac{\pi}{(n-2)!} \left(\prod_{k=1}^n \int_0^\infty f_k^n(x) dx \right)^{\frac{1}{n}}.$$

This inequality can be considered as Hilbert's integral inequality for n functions from $L^n[0, \infty)$.

The following theorem is an improvement of inequality (1.2).

THEOREM 2. i) Let a be the a real number such that $a > 1$. If $f, g \in L^2\left[\frac{1}{a}, a\right]$, then

$$\int_{\frac{1}{a}}^a \int_{\frac{1}{a}}^a \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq k_a(\lambda) \left(\int_{\frac{1}{a}}^a x^{1-\lambda} f^2(x) dx \int_{\frac{1}{a}}^a x^{1-\lambda} g^2(x) dx \right)^{1/2}, \tag{2.6}$$

where

$$k_a(\lambda) = \int_{\frac{1}{a}}^a \frac{x^{\frac{\lambda-1}{2}}}{(1+x)^\lambda} dx \quad (\lambda \in \mathbf{R}).$$

ii) Let $0 < a < b$. If $f, g \in L^2[a, b]$, then

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq k_{\sqrt{\frac{b}{a}}}(\lambda) \left(\int_a^b x^{1-\lambda} f^2(x) dx \int_a^b x^{1-\lambda} g^2(x) dx \right)^{1/2}. \tag{2.7}$$

Proof. i) By Cauchy's inequality we have:

$$\int_{\frac{1}{a}}^a \int_{\frac{1}{a}}^a \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \leq \left(\int_{\frac{1}{a}}^a w(x) f^2(x) dx \int_{\frac{1}{a}}^a w(x) g^2(x) dx \right)^{1/2}, \tag{2.8}$$

where

$$w(x) = x^{1-\lambda} \int_{\frac{1}{ax}}^{\frac{a}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt.$$

The function h , $h : \left[\frac{1}{a}, a \right] \rightarrow \mathbf{R}_+$

$$h(x) = \int_{\frac{1}{ax}}^{\frac{a}{x}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt$$

is strictly increasing on $\left[\frac{1}{a}, 1 \right]$ and is strictly decreasing on $[1, a]$ because

$$h'(x) = a^{\frac{\lambda}{2}} x^{\frac{\lambda-2}{2}} \left(\frac{1}{(ax+1)^\lambda} - \frac{1}{(x+a)^\lambda} \right).$$

Hence

$$h(x) \leq h(1) = k_a(\lambda).$$

Using this inequality from (2.8) we obtain inequality (2.6).

ii) Putting $x = x_1\sqrt{ab}$ and $y = y_1\sqrt{ab}$ we have

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda} dx dy = (ab)^{1-\frac{\lambda}{2}} \int_{\sqrt{\frac{b}{a}}}^{\sqrt{\frac{a}{b}}} \int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{f(x_1)g(y_1)}{(x_1+y_1)^\lambda} dx_1 dy_1.$$

Now inequality (2.7) follows by inequality (2.6).

REMARK. We have

$$k \sqrt{\frac{b}{a}}(\lambda) \leq B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left[1 - \left(\frac{a}{b} \right)^{\lambda/4} \right]. \quad (2.9)$$

For this is enough to prove the inequality:

$$\int_{\frac{1}{a}}^a \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt \leq B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) (1 - a^{-\frac{\lambda}{2}}), \quad a > 1. \quad (2.10)$$

This inequality can be written in the following form:

$$\int_0^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt - \int_0^{\frac{1}{a}} \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt - \int_a^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt \leq B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) - a^{-\frac{\lambda}{2}} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right),$$

or

$$2 \int_a^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt > a^{-\frac{\lambda}{2}} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right). \quad (2.11)$$

On the other hand we have

$$2 \int_a^\infty \frac{t^{\frac{\lambda-2}{2}}}{(1+t)^\lambda} dt = 2a^{\frac{\lambda}{2}} \int_1^\infty \frac{x^{\frac{\lambda-2}{2}} dx}{(1+xa)^\lambda} \geq 2a^{-\frac{\lambda}{2}} \int_1^\infty \frac{x^{\frac{\lambda-2}{2}} dx}{(1+x)^\lambda},$$

and so inequality (2.11) is true.

REFERENCES

- [1] YANG BICHENG, *On Hilbert's integral inequality*, J. Math. Anal. Appl. 220 (1998), 778–785.

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