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A NOTE ON THE LEINDLER'S CLASS S_α, $\alpha > 0$

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Abstract. In this paper we generalize two inequalities of Telyakovskii type (see [2], [5], [6], [7]), by considering the class S_α, $\alpha > 0$ (see [2]) and positive derivate of order α for cosine and sine series. Also, an equivalent form of the Leinder's class S_α, $\alpha > 0$ is given.

1. Introduction

The following class S_r, $r = 0, 1, 2, \dots$, was introduced in [6]. A null seqence $\{a_n\}$ belongs to the class S_r $r = 0, 1, 2, \dots$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=0}^{\infty} n^r A_n^{(r)} < \infty$ and $|\Delta a_n| \leqslant A_n^{(r)}$, for all n.

Specially for $r = 0$, we obtain the Sidon-Telyakovskii class S ([4]), where $A_n^{(0)} = A_n$.

In [5], [6] the author obtained two new L^1-estimates for the r-th derivate of the cosine and sine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx. \tag{1.2}$$

THEOREM A ([6], [7]). *Let the coefficients of the series* (1.1) *belong to the class* S_r $r = 0, 1, 2, \dots$. *Then the* r*-th derivate of the series* (1.1) *is a Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and the following inequality holds:*

$$\int_0^{\pi} |f^{(r)}(x)| dx \leqslant M \sum_{n=0}^{\infty} n^r A_n, \quad \textit{where} \quad 0 < M = M(r) < \infty.$$

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THEOREM B [7]. *Let the coefficients of the series* $g(x)$ *belong to the class* S_r, $r = 0, 1, 2, \ldots$. *Then the following relation holds for* $m = 1, 2, 3, \ldots$

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(r)}(x)|dx \leqslant \sum_{n=1}^{m} |a_n|\, n^{r-1} + O_r \left(\sum_{n=1}^{\infty} n^r A_n \right),$$

where O_r *depends on* r. *Moreover, if* $\sum_{n=1}^{\infty} |a_n|\, n^{r-1} < \infty$, *then the* r*-th derivate of the series* (1.2) *is a Fourier series of some* $g^{(r)} \in L^1(0,\pi)$ *and*

$$\int_{0}^{\pi} |g^{(r)}(x)|dx \leqslant \sum_{n=1}^{\infty} |a_n|\, n^{r-1} + O_r \left(\sum_{n=1}^{\infty} n^r A_n \right).$$

The direct proof of this theorem the author has given in [5]. We note that, for $r = 0$, we obtain the Telyakovskii type inequalities, proved in [4].

Very recently, L. Leindler [2], defined the generalization of the classes Sr, $r = 0, 1, 2, \ldots$, repleacing the positive integer r, by positive real number α, and denoting by S_α, $\alpha > 0$. In the same paper L. Leinder [2], established new proofs of the Theorem A and Theorem B, by proving the following theorem.

THEOREM C [2]. *Let* $\gamma \geqslant \beta > 0$. *If* $\{a_n\}$ *belongs to the class* S_γ *then the sequence* $\{n^\beta a_n\}$ *belongs to the class* $S_{\gamma-\beta}$ *and*

$$\sum_{n=1}^{\infty} n^{\gamma-\beta} A_n^{(\gamma-\beta)} \leqslant (\beta+1) \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)}$$

holds.

This result was proved also by the author in [7] for $\gamma = \beta = r$.

In this paper, we shall extend the Theorem A and Theorem B by considering the positive real derivatives of order α for series (1.1) and (1.2)

The definitions for derivatives of non-integer order, firstly was introduced in [9] by H. Weyl (see also [3] p.263; [10], XII). For $\alpha \geqslant 0$, let us consider the series

$$\sum_{k=1}^{\infty} k^\alpha a_k \cos\left(kx + \frac{\alpha\pi}{2}\right) \tag{1.3}$$

$$\sum_{k=1}^{\infty} k^\alpha a_k \sin\left(kx + \frac{\alpha\pi}{2}\right) \tag{1.4}$$

DEFINITION 1.1. If the series (1.3) and (1.4) are Fourier series of some functions $f^{(\alpha)}$ and $g^{(\alpha)}$ respectivelly, then these functions are called the α-th derivatives of the series f and g.

The following more generally definition is also very used in the literature (see [10]).

Namely, for $\alpha, \lambda \geqslant 0$ let we consider the series

$$\sum_{k=1}^{\infty} k^{\alpha} \left(a_k \cos\left(kx + \frac{\lambda\pi}{2}\right) + b_k \sin\left(kx + \frac{\lambda\pi}{2}\right)\right) . \tag{1.5}$$

If the series (1.5) is Fourier series of some function $\varphi(x)$, then the derivate $f_{\lambda}^{(\alpha)}(x)$ denotes its sum function. If $\lambda = \alpha$, then we get the derivate of order α, and if $\lambda = \alpha - 1$, we get the function, conjugate by the derivate $f^{(\alpha)}(x)$.

Now, let us consider the following class of sequences. We shall say that a sequence $\{a_n\}$ belongs to S_α^2, $\alpha \geqslant 0$, or $\{a_n\} \in S_\alpha^2$ if there exists a null sequence $\{A_n\}$ such that

$$\sum_{k=1}^{\infty} k^{\alpha+1} |\Delta A_k| < \infty \tag{$*$}$$

and $|\Delta a_k| \leqslant A_k$, for all k.

If $\alpha = 0$, we obtain the class S^2, defined by Garrett-Rees-Stanojević (see [1]).

In this note, we shall prove that the classes S_α and S_α^2, are identical.

2. Lemma

For the proof of our new result we need the following well known lemma.

LEMMA 2.1. *If $A_n \downarrow 0$ with $\sum_{n=1}^{\infty} n^{\alpha} A_n < \infty$, $\alpha \geqslant 0$, then $n^{\alpha+1} A_n = o(1)$, $n \to \infty$.*

We note that this lemma was elegant proved by the author in [9] (see Lemma 2, [9]).

3. Main results

By Definition 1.1, and by Theorem C, Theorem A, B (case $r = 0$), in connection with the classes S_α, $\alpha \geqslant 0$, we obtain the following two generalized theorems of the Theorem A and Theorem B.

THEOREM 3.1. *Let the coefficients of the series (1.3) belong to the class S_α, $\alpha \geqslant 0$. Then the series (1.3) is a Fourier series of some $f^{(\alpha)} \in L^1(0,\pi)$ and the following inequality holds:*

$$\int_0^{\pi} |f^{(\alpha)}(x)| dx \leqslant M \sum_{n=0}^{\infty} n^{\alpha} A_n , \qquad \textit{where} \quad 0 < M = M(\alpha) < \infty .$$

THEOREM 3.2. *Let the coefficients of the series* $g(x)$ *belong to the class* S_α, $\alpha \geqslant 0$. *Then the following inequality holds for* $m = 1, 2, 3, \ldots$

$$\int_{\frac{\pi}{m+1}}^{\pi} |g^{(\alpha)}(x)|dx \leqslant \sum_{n=1}^{m} |a_n|\, n^{\alpha-1} + O_\alpha\left(\sum_{n=1}^{\infty} n^\alpha A_n\right),$$

where O_α *depends on* α. *Moreover, if* $\sum_{n=1}^{\infty}|a_n|\, n^{\alpha-1} < \infty$, *then the series* (1.4) *is a Fourier series of some* $g^{(\alpha)} \in L^1(0,\pi)$ *and*

$$\int_{0}^{\pi} |g^{(\alpha)}(x)|dx \leqslant \sum_{n=1}^{\infty} |a_n|\, n^{\alpha-1} + O_\alpha\left(\sum_{n=1}^{\infty} n^\alpha A_n\right),$$

THEOREM 3.3. *For all* $\alpha \geqslant 0$, *the classes* S_α *and* S_α^2 *are identical.*

Proof. Let $\{a_n\} \in S_\alpha^2$. Since $A_n \to 0$, by $(*)$, we obtain

$$n^{\alpha+1}A_n = n^{\alpha+1}\sum_{k=n}^{\infty}\Delta A_k \leqslant n^{\alpha+1}\sum_{k=n}^{\infty}|\Delta A_k| \leqslant \sum_{k=n}^{\infty}k^{\alpha+1}|\Delta A_k| = o(1),$$

$n \to \infty$, i.e. $n^{\alpha+1}A_n = o(1)$, $n \to \infty$.

But

$$\sum_{k=1}^{n} k^\alpha A_k = \sum_{k=1}^{n-1}(\Delta A_k)\sum_{j=1}^{k} j^\alpha + A_n\sum_{j=1}^{n} j^\alpha \leqslant \sum_{k=1}^{n-1}k^{\alpha+1}|\Delta A_k| + n^{\alpha+1}A_n.$$

Letting $n \to \infty$, we obtain $\sum_{n=1}^{\infty} n^\alpha A_n < \infty$.

This implies that $\{a_n\} \in S_\alpha$.

Now, if $\{a_n\} \in S_\alpha$, $\alpha \geqslant 0$, it suffices to show that $(*)$ holds. By partial summation,

$$\sum_{k=1}^{n-1} k^{\alpha+1}|\Delta A_k| = \sum_{k=1}^{n-1} k^{\alpha+1}(\Delta A_k) = \sum_{k=1}^{n}[k^{\alpha+1} - (k-1)^{\alpha+1}]A_k - n^{\alpha+1}A_n$$

$$\leqslant (\alpha+1)\sum_{k=1}^{n} k^\alpha A_k + n^{\alpha+1}A_n.$$

Leting $n \to \infty$ and appliying the Lemma 2.1, we obtain that $(*)$ holds, i.e. $\{a_n\} \in S_\alpha^2$.

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