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INTEGRAL GENERALIZED MEANS

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Abstract. Generalizing the integral representation of the arithmetic-geometric mean, some authors characterized more means by integrals. In this paper we extend a modified method used by Y.-H. Kim to construct generalized means.

1. Introduction

Let us denote

$$r_{n,\theta}(a,b) = (a^n \cos^2\theta + b^n \sin^2\theta)^{1/n},\ n \neq 0$$

and

$$r_{0,\theta}(a,b) = \lim_{n\to 0} r_{n,\theta}(a,b) = a^{\cos^2\theta} b^{\sin^2\theta}.$$

As was proved in [9], if $p : \mathbf{R}_+ \to \mathbf{R}$ is a strictly monotonic function, then

$$M_{p,n}(a,b) = p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{n,\theta}(a,b))d\theta\right)$$

defines a symmetric mean. A well known example is given by the arithmetic-geometric mean of Gauss (see [2]), where $n = 2$ and $p(x) = x^{-1}$. H. Haruki considered in [4] an arbitrary p (also for $n = 2$). Then the values $n = -1$ and $n = 1$ where studied in [5] and [6]. The general case (of arbitrary n) was studied in [9] and then in [10] and [12].

In [13] is considered the expression

$$H_{p,n}(a,b) = \frac{1}{H(a,b)} \cdot p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r^2_{2n,\theta}(a,b))d\theta\right).$$

For $n = \pm 1$ it was already studied in [7]. This is a generalized mean not a mean. In this paper we shall replace the mean H by an arbitrary generalized mean. We shall study the same problems as those from [7], [9], [10], [12] and [13].

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2. Means and generalized means

A *mean* is defined usually as a function $M : \mathbf{R}_+^2 \to \mathbf{R}_+$ which has the property

$$\min(a,b) \leqslant M(a,b) \leqslant \max(a,b), \forall a,b \geqslant 0.$$

Of course, each mean M is *reflexive*. Sometimes this weaker condition is taken as a definition of means (see [1]). We call it a generalized mean. Thus, by *generalized mean* we understand a function $M : \mathbf{R}_+^2 \to \mathbf{R}_+$ with the property

$$M(a,a) = a, \ \forall a \geqslant 0.$$

A (generalized) mean is called *symmetric* if

$$M(b,a) = M(a,b), \ \forall a,b \geqslant 0.$$

In what follows we shall use weighted power means $P_{n,\lambda}$ defined by

$$P_{n,\lambda}(a,b) = \begin{cases} [\lambda \cdot a^n + (1-\lambda)\cdot b^n]^{1/n}, & n \neq 0 \\ a^\lambda \cdot b^{1-\lambda}, & n = 0 \end{cases},$$

with $\lambda \in [0,1]$ fixed. For $\lambda = 1/2$ we get the (symmetric) power means $P_n = P_{n,1/2}$. The most important special cases are the arithmetic mean $A = P_1$, the geometric mean $G = P_0$, and the harmonic mean $H = P_{-1}$.

We can compose three (generalized) means M, N and P obtaining the (generalized) mean $M(N,P)$ given by

$$M(N,P)(a,b) = M(N(a,b),P(a,b)), \ \forall a,b \geqslant 0.$$

We shall use in what follows the mean Q defined in [7] by

$$Q(a,b) = \left(\frac{3}{4}\cdot\frac{a^4+b^4}{2} + \frac{a^2b^2}{4}\right)^{1/4},$$

that is

$$Q = P_{4,3/4}(P_4, G).$$

3. An integral generalized mean

Given an arbitrary generalized mean N and a bijection p, let us consider the expression

$$N_{p,n}(a,b) = \frac{1}{N(a,b)}\cdot p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} p(r_{2n,\theta}^2(a,b))d\theta\right).$$

If we denote $q(x) = p(x^2)$, we get

$$N_{p,n} = M_{q,2n}^2/N$$

thus it is a generalized mean. The question is when it reduces at a given (generalized) mean R. If we denote

$$f(a,b;p,n) = \frac{1}{2\pi}\int_0^{2\pi} p(r^2_{2n,\theta}(a,b))d\theta,$$

the condition $N_{p,n} = R$ can be written as

$$f(a,b;p,n) = p(N(a,b)\cdot R(a,b)). \tag{1}$$

By direct computation, as in [9] we can prove the following

LEMMA 1. *If p has a continuous second order derivative, then f has the following partial derivatives*

$$f''_{a^2}(c,c;p,n) = [3c^2p''(c^2)+(n+1)p'(c^2)]/2$$

and

$$f''_{ab}(c,c;p,n) = [c^2p''(c^2)-(n-1)p'(c^2)]/2,$$

where c is an arbitrary positive number.

Using it we can prove

THEOREM 1. *If $N_{p,n} = R$ then p satisfies the differential equations*

$$c^2p''(c^2)\left\{3-2\left[N'_a(c,c)+R'_a(c,c)\right]^2\right\}+p'(c^2) \\ \cdot\{n+1-2c\left[N''_{a^2}(c,c)+R''_{a^2}(c,c)\right]-4N'_a(c,c)R'_a(c,c)\} = 0 \tag{2}$$

and

$$c^2p''(c^2)\left\{1-2\left[N'_a(c,c)+R'_a(c,c)\right]\left[N'_b(c,c)+R'_b(c,c)\right]\right\} \\ +p'(c^2)\cdot\{1-n-2c\left[N''_{ab}(c,c)+R''_{ab}(c,c)\right] \\ -2[N'_a(c,c)R'_b(c,c)+N'_b(c,c)R'_a(c,c)]\} = 0. \tag{3}$$

Proof. We calculate the partial derivatives of order two with respect to a of both members of the relation (1). We get

$$f'_a(a,b;p.n) = p'(N(a,b)\cdot R(a,b))\cdot[N'_a(a,b)\cdot R(a,b)+N(a,b)\cdot R'_a(a,b)]$$

and then

$$f''_{a^2}(a,b;p.n) = p''(N(a,b)\cdot R(a,b))\cdot[N'_a(a,b)\cdot R(a,b)+N(a,b)\cdot R'_a(a,b)]^2 \\ +p'(N(a,b)\cdot R(a,b))\cdot[N''_{a^2}(a,b)\cdot R(a,b) \\ +2N'_a(a,b)\cdot R'_a(a,b)+N(a,b)\cdot R''_{a^2}(a,b)].$$

For $a=b=c$, taking into account the lemma, we obtain

$$[3c^2p''(c^2)+(n+1)p'(c^2)]/2 = p''(c^2)\cdot c^2\cdot\left[N'_a(c,c)+\cdot R'_a(c,c)\right]^2 \\ +p'(c^2)\cdot\left[N''_{a^2}(c,c)\cdot c+2\cdot N'_a(c,c)\cdot R'_a(c,c)+c\cdot R''_{a^2}(c,c)\right]$$

which gives (2). In a similar way we can prove (6). □

As it is proved in [11] we have the following

LEMMA 2. *If N is a symmetric generalized mean then*

$$N'_a(c,c) = 1/2 \tag{4}$$

and

$$N''_{ab}(c,c) = -N''_{a^2}(c,c). \tag{5}$$

REMARK 1. The relation (4) was proved for a symmetric mean N in [3].

THEOREM 2. *If $N_{p,n} = R$, where N and R are symmetric generalized means, then p satisfies the differential equation*

$$c^2 p''(c^2) + p'(c^2)\left\{n - 2c\left[N''_{a^2}(c,c) + R''_{a^2}(c,c)\right]\right\} = 0. \tag{6}$$

Proof. Using (4) and (2) we get (6). Taking into account (5), (3) also reduces at (6). □

THEOREM 3. *If $H_{p,n} = R$, where R is a symmetric generalized mean, then p satisfies the differential equation*

$$c^2 p''(c^2) + p'(c^2)\cdot\left[n + 1 - 2cR''_{a^2}(c,c)\right] = 0. \tag{7}$$

Proof. We have

$$H''{}_{a^2}(c,c) = -\frac{1}{2c}$$

thus (6) reduces at (7). □

REMARK 2. In [11] it is shown that most of the symmetric generalized means verify the hypotheses of the following

THEOREM 4. *If $N_{p,n} = R$ where N and R are symmetric generalized means such that*

$$R''_{a^2}(c,c) = \frac{\alpha}{c},\ N''_{a^2}(c,c) = \frac{\beta}{c} \text{ with } \alpha + \beta \neq \frac{n}{2},$$

then

$$p(x) = A \cdot x^{2\alpha+2\beta-n+1} + B,$$

where A and B are arbitrary constants.

Proof. The equation (3) becomes in this case

$$c^2 p''(c^2) + p'(c^2)\cdot(n - 2\alpha - 2\beta) = 0.$$

Denoting $c^2 = x$ and $p' = z$ we get the equation

$$x \cdot z'(x) + (n - 2\alpha - 2\beta)\cdot z(x) = 0$$

with the solution

$$z(x) = D \cdot x^{2\alpha+2\beta-n},$$

which, by integration, gives the expression of p. □

In the special case $N = H$ we have

THEOREM 5. *If* $H_{p,n} = R$ *where* R *is a symmetric generalized mean and*

$$R''_{a^2}(c,c) = \frac{\alpha}{c}, \text{ with } \alpha \neq \frac{n+1}{2},$$

then

$$p(x) = A \cdot x^{2\alpha - n} + B,$$

where A *and* B *are arbitrary constants.*

REMARK 3. For $n = \pm 1$ the last theorem gives the results from [7]. There it is proved also that in these cases the conditions are sufficient, thus we get the integral representation of the following generalized means:

$$M_1 = A\left(\frac{G}{P_2}\right)^2, \quad M_2 = A\left(\frac{G}{Q}\right)^2, \quad M_3 = A\frac{P_2}{G}, \quad M_4 = A\left(\frac{P_2}{G}\right)^2,$$

$$M_5 = A\left(\frac{A}{G}\right)^2, \quad M_6 = A\frac{G}{P_2}, \text{ and } M_7 = A\left(\frac{Q}{G}\right)^2.$$

In [11] it is shown that all these generalized means satisfy the conditions of the last theorem with the following values of α:

R	A	H	M$_1$	M$_2$	M$_3$	M$_4$	M$_5$	M$_6$	M$_7$
α	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

REMARK 4. For arbitrary n, we get the results from [13] where $R = P_q$. Using the values of α for other means N and R, we get new results.

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