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DERIVATIVES OF GENERALIZED MEANS

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Abstract. A generalized mean is a reflexive positive function of two positive variables. We study the partial derivatives of first and second order of a generalized mean. We also prove that a homogenous generalized mean is a mean on a limited interval. The results are useful in the researches related to the generalizations of the arithmetic-geometric mean.

1. Means and generalized means

A *mean* is defined usually as a function $M:\mathbf{R}_+^2\to\mathbf{R}_+$ which has the property

$$\min(a,b)\leqslant M(a,b)\leqslant \max(a,b),\forall a,b\geqslant 0.$$

Of course, each mean M is *reflexive*, i.e.

$$M(a,a)=a,\ \forall a\geqslant 0.$$

Sometimes this weaker condition is taken as a definition of means (see [2]), but we call it a generalized mean. Thus, by *generalized mean* we understand a reflexive function $M:\mathbf{R}_+^2\to\mathbf{R}_+$.

A (generalized) mean is called *symmetric* if

$$M(b,a)=M(a,b),\forall a,b\geqslant 0.$$

It is called *homogeneous* (of degree one) if

$$M(ta,tb)=tM(a,b),\forall a,b,t\geqslant 0.$$

Some important examples of means are the weighted power means $P_{n,\lambda}$ defined by

$$P_{n,\lambda}(a,b)=\begin{cases}[\lambda\cdot a^n+(1-\lambda)\cdot b^n]^{1/n}, & n\neq 0\\ a^\lambda\cdot b^{1-\lambda}, & n=0\end{cases},$$

with $\lambda\in[0,1]$ fixed. For $\lambda=1/2$ we get the (symmetric) power means $P_n=P_{n,1/2}$. As the most known special cases we note the arithmetic mean $A=P_1$, the geometric mean $G=P_0$, and the harmonic mean $H=P_{-1}$.

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We give here a list of other well known means (see [3] and [6]):

— the extended logarithmic mean

$$L_r(a,b) = \left(\frac{1}{r}\cdot\frac{a^r - b^r}{\log a - \log b}\right)^{\frac{1}{r}} \quad \text{for } r \neq 0;$$

— the identric mean

$$I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}};$$

— the extended mean

$$E_{r,s}(a,b) = \left(\frac{s}{r}\cdot\frac{a^r - b^r}{a^s - b^s}\right)^{\frac{1}{r-s}}$$

for $rs(r-s) \neq 0$, while

$$E_{r,r}(a,b) = I_r(a,b) = [I(a^r,b^r)]^{\frac{1}{r}} \text{ for } r \neq 0$$

$E_{r,0} = L_r$, and $E_{0,0} = G$;

— the Gini mean

$$G_{r,s}(a,b) = \left(\frac{a^r + b^r}{a^s + b^s}\right)^{\frac{1}{r-s}}$$

for $r \neq s$ and

$$G_{r,r}(a,b) = \exp\left(\frac{a^r\cdot\log a + b^r\cdot\log b}{a^r + b^r}\right);$$

— the Moskovitz mean

$$M_r(a,b) = \frac{a\cdot b^r + b\cdot a^r}{a^r + b^r}.$$

We can compose the (generalized) means M, N and P obtaining the (generalized) mean $M(N,P)$ given by

$$M(N,P)(a,b) = M(N(a,b),P(a,b)), \forall a,b \geqslant 0.$$

We shall use in what follows the mean Q defined on this way by

$$Q = P_{4,3/4}(P_4, G).$$

that is

$$Q(a,b) = \left(\frac{3}{4}\cdot\frac{a^4+b^4}{2} + \frac{a^2b^2}{4}\right)^{1/4}.$$

In [5] are considered the following expressions:

$$M_1 = A\left(\frac{G}{P_2}\right)^2, \quad M_2 = A\left(\frac{G}{Q}\right)^2, \quad M_3 = A\frac{P_2}{G}, \quad M_4 = A\left(\frac{P_2}{G}\right)^2,$$

$$M_5 = A\left(\frac{A}{G}\right)^2, \quad M_6 = A\frac{G}{P_2} \text{ and } M_7 = A\left(\frac{Q}{G}\right)^2.$$

Of course, if M, N and P are generalized means and the function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is such that $f(1) = 1$ then $M \cdot f(N/P)$ is also a generalized mean. Thus, all the above expressions are generalized means.

Another less usual example is the following. Looking after generalized means of harmonic type, we found the expressions

$$H_k(a,b) = \frac{2ab}{a+b} + k \cdot \frac{a^2+b^2-2ab}{a+b}.$$

They define symmetric generalized means. For $k \in [0,1]$ they are even means. For instance $H_0 = H, H_{1/2} = A$ and $H_1 = C = G_{2,1}$ (the contraharmonic mean).

Finally we remember the exponential mean (see [6])

$$T(a,b) = \frac{a \cdot e^a - b \cdot e^b}{e^a - e^b} - 1.$$

2. First order partial derivatives

Regarding the first order partial derivatives of means, we have the following results.

THEOREM 1. *If M is a differentiable generalized mean then*

$$M'_a(c,c) + M'_b(c,c) = 1. \tag{1}$$

Proof. Indeed, Taylor's formula of degree one for M gives

$$M(a+t, b+t) = M(a,b) + t[M'_a(a,b) + M'_b(a,b)] + O(t^2),$$

for t in a neighborhood of zero. Taking $a = b = c$ we get

$$c + t = c + t[M'_a(c,c) + M'_b(c,c)] + O(t^2),$$

which gives (1). □

REMARK 1. This doesn't mean that the derivatives are positive.

EXAMPLE 1. For the generalized mean M defined by

$$M(a,b) = \frac{b(a+3b)}{3a+b}$$

we have

$$M'_a(a,b) = -\frac{8b^2}{(3a+b)^2}$$

thus

$$M'_a(c,c) = -1/2 < 0.$$

REMARK 2. For a mean, this cannot happen as we prove in the following

THEOREM 2. *If M is a differentiable mean then*

$$M'_a(c,c) \geqslant 0. \tag{2}$$

Proof. For $t>0$ we have

$$M(c,c) = c \leqslant M(c+t,c) \leqslant c+t$$

and so

$$M'_a(c,c) = \lim_{t\to 0, t>0} \frac{M(c+t,c)-M(c,c)}{t} \geqslant 0. \quad \square$$

REMARK 3. Similarly we prove that

$$M'_b(c,c) \geqslant 0.$$

Using (1) we deduce that

$$0 \leqslant M'_a(c,c) \leqslant 1.$$

This property doesn't hold in an arbitrary point.

EXAMPLE 2. For the contraharmonic mean

$$C(a,b) = \frac{a^2+b^2}{a+b},$$

we have

$$C'_a(a,b) = \frac{a^2+2ab-b^2}{(a+b)^2} < 0$$

if

$$0 \leqslant a < (\sqrt{2}-1)b.$$

REMARK 4. If M is symmetric we know the value of the first order partial derivatives for $a=b$, even if M is a generalized mean.

THEOREM 3. *If M is a symmetric differentiable generalized mean then*

$$M'_a(c,c) = M'_b(c,c) = 1/2. \tag{3}$$

Proof. If M is symmetric we have

$$M'_a(c,c) = \lim_{t\to 0} \frac{M(c+t,c)-M(c,c)}{t} = \lim_{t\to 0} \frac{M(c,c+t)-M(c,c)}{t} = M'_b(c,c),$$

thus (1) gives (3). $\square$

REMARK 5. For a mean the relation (3) was proved in [4]. This result is used in [4] and [6] for proving the convergence of double sequences of Gauss type. The common limit of such double sequences defines generalizations of the arithmetic-geometric mean.

REMARK 6. Another application of the above results is related to the following theorem.

THEOREM 4. *If M is a differentiable homogeneous generalized mean, such that*

$$M_b'(1,1) = p \in (0,1), \tag{4}$$

then there exists a constant $T > 1$ such that

$$a \leqslant M(a,b) \leqslant b, \tag{5}$$

for

$$1 \leqslant \frac{b}{a} \leqslant T.$$

Proof. As M is homogeneous, the relation (5) is equivalent with

$$1 \leqslant M\left(1, \frac{b}{a}\right) \leqslant \frac{b}{a}.$$

If we denote $t = b/a$, we have the conditions

$$f(t) = M(1,t) - 1 \geqslant 0$$

and

$$g(t) = t - M(1,t) \geqslant 0.$$

But

$$f'(1) = M_b'(1,1) = p > 0$$

so that we can find a $T' > 1$ such that $f(t) \geqslant 0$ for $1 \leqslant t \leqslant T'$. Similarly

$$g'(1) = 1 - M_b'(1,1) = 1 - p > 0,$$

thus $g(t) \geqslant 0$ for $1 \leqslant t \leqslant T''$, with $T'' > 1$. We take $T = \min\{T', T''\}$. □

REMARK 7. We have the same property also for $a > b$. Such a result was proved for some integral means in [1]. As was remarked there, it means that a generalized mean, with the above properties is in fact a mean, but only on a limited interval. We have evalueted the value of T for some of the generalized means defined above.

Generalized mean	T
$A(G/Q)^2$	3.951 ...
AP_2/G	5.570 ...
$A(P_2/G)^2$	2.414 ...
A^3/G^2	4.236 ...
$A(Q/G)^2$	1.913 ...

REMARK 8. The condition (4) is of course necessary for the validity of the above theorem, as follows from the previous results.

EXAMPLE 3. The generalized mean M defined by

$$M(a,b) = b\frac{pa+(1-p)b}{qa+(1-q)b};\ p,q \in (0,1),$$

is a mean if and only if $p \geqslant q$. Indeed, as

$$M'_a(c,c) = p - q,$$

if $p < q$, it cannot be a mean (even on limited interval). For $p \geqslant q$ it is easy to verify that it defines a mean.

3. Second order partial derivatives

Other generalizations of the arithmetic-geometric mean can be obtained starting from its integral representation (see [7] and [5]). In the study of these new generalizations are useful the following results.

THEOREM 5. *If M is a twice differentiable generalized mean then*

$$M''_{a^2}(c,c) + 2M''_{ab}(c,c) + M''_{a^2}(c,c) = 0. \tag{6}$$

Proof. We can use the same idea as in the previous proofs. Indeed, Taylor's formula of degree two for M gives

$$\begin{aligned} M(a+t,b+t) &= M(a,b) + t[M'_a(a,b) + M'_b(a,b)] \\ &\quad + t^2[M''_{a^2}(a,b) + 2M''_{ab}(a,b) + M''_{a^2}(a,b)]/2 + O(t^3), \end{aligned}$$

for t in a neighborhood of zero. Taking $a = b = c$ and using the formula (1) we get (6). □

THEOREM 6. *If M is a symmetric generalized mean then*

$$M''_{ab}(c,c) = -M''_{a^2}(c,c). \tag{7}$$

Proof. As above $M''_{a^2}(c,c) = M''_{b^2}(c,c)$ and so (6) gives (7). □

In what follows we shall show that most of the "usual" symmetric means have the property.

$$M''_{a^2}(c,c) = \frac{\alpha}{c},\ \alpha \in \mathbf{R}. \tag{8}$$

By direct (but difficult) computation we get the following values of α in (8):

M	$E_{r,s}$	$G_{r,s}$	M_r
α	$\frac{r+s-3}{12}$	$\frac{r+s-1}{4}$	$-\frac{r}{2}$

Of course, we have here included the values of L_r, I_r and $P_r = G_{r,0} = E_{2r,r}$.

For the generalized means of Kim we have the following values:

R	M_1	M_2	M_3	M_4	M_5	M_6	M_7
α	-1	$-\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

.

In the case of generalized means of harmonic type, H_k we have $\alpha = \alpha_k = k-1/2$.

But not all the symmetric generalized means have this property. For example we have

$$T''_{a^2}(c,c) = \frac{1}{6},$$

thus it is not of the type (8).

On the other hand a non symmetric mean can also have the above property. For example, in the case of $P_{n,\lambda}$ we have (8) with

$$\alpha = \lambda(1-\lambda)(n-1).$$

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