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EXPONENTIAL STABILITY AND UNSTABILITY OF SEMIGROUPS OF LINEAR OPERATORS IN BANACH SPACES

M. MEGAN, A. L. SASU, B. SASU, A. POGAN

Abstract. Necessary and sufficient conditions for uniform exponential stability and uniform exponential unstability of semigroups of linear operators are given, in terms of Banach function spaces and Banach sequence spaces, respectively. We generalize some well-known theorems proved by Datko, Pazy, Rolewicz and Neerven concerning the exponential stability of C_0-semigroups. We shall obtain the versions of the theorems mentioned above, for the case of uniform exponential unstability of C_0-semigroups.

1. Introduction

Let X be a Banach space. The norm on X and on the space $B(X)$ of all bounded linear operators from X into itself will be denoted by $||\cdot||$.

We recall that a family $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ of bounded linear operators is a *semigroup* on X, if $S(0) = I$ (the identity operator on X) and $S(t+s) = S(t)S(s)$, for all $t,s \geqslant 0$.

DEFINITION 1.1. The semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is called

(i) C_0-*semigroup*, if $\lim\limits_{t\to 0_+} S(t)x = x$, for all $x \in X$;

(ii) *strongly measurable* if for every $x \in X$ the function $S(\cdot)x$ is measurable;

(iii) *injective* if $S(t)$ is an injective operator, for all $t > 0$;

(iv) *exponentially bounded*, if there are $M, \omega > 0$ such that

$$||S(t)|| \leqslant Me^{\omega t}, \quad \forall t \geqslant 0;$$

(v) *uniformly exponentially stable*, if there are $N, \nu > 0$ such that

$$||S(t)|| \leqslant Ne^{-\nu t}, \quad \forall t \geqslant 0;$$

(vi) *uniformly exponentially unstable*, if there are $N, \nu > 0$ such that

$$||S(t)x|| \geqslant Ne^{\nu t}||x||, \quad \forall x \in X,\ \forall t \geqslant 0.$$

REMARK 1.1. Every C_0-semigroup is strongly measurable and exponentially bounded (see [3]).

In the theory of C_0-semigroups in Banach spaces some of the most notable results on exponential stability are due to Datko, Pazy, Rolewicz and Neerven and they are given by the following theorems:

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THEOREM 1.1. *A C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is uniformly exponentially stable if and only if there exists $p \in [1,\infty)$ such that*

$$\int_0^\infty ||S(t)x||^p dt < \infty, \quad \forall x \in X.$$

This result was originally proved by Datko in [2] for $p = 2$ in Hilbert spaces and Pazy (see [17]) obtained the case $p \in [1,\infty)$ in Banach spaces.

A discrete-time theorem of characterization for uniform exponential stability of C_0-semigroups has been formulated by Zabczyk in [19] and it is given by

THEOREM 1.2. *A C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ on the Banach space X is uniformly exponentially stable if and only if there is a strictly increasing continuous convex function $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(0) = 0$ such that for every $x \in X$ there is $\alpha(x) > 0$ with*

$$\sum_{n=0}^{\infty} N(\alpha(x)\,||S(n)x||) < \infty.$$

One of the most important results has been proved in [18] and it is given by

THEOREM 1.3. *The C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is uniformly exponentially stable if and only if there exists a non-decreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(t) > 0$, for all $t > 0$ such that for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\int_0^\infty N(\alpha(x)||S(t)x||)dt < \infty.$$

The above result has been obtained by Rolewicz for the more general case of evolution families. In the original proof, there was an additional condition imposed on the continuity of N, because the author used category theorems. The result has been also treated by Neerven in [13], for the case of semigroups, where he did not assume that the function N is continuous.

An unified treatment of these results has been presented by Neerven in [13], as follows

THEOREM 1.4. *The C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is uniformly exponentially stable if and only if there exists a Banach function space B over $\mathbf{R}_+$ with $\lim\limits_{t\to\infty} |\chi_{[0,t)}|_B = \infty$ such that $||S(\cdot)x|| \in B$, for all $x \in X$.*

The purpose of the present paper is to give an unified treatment for uniform exponential stability and uniform exponential unstability, using the theory of Banach function spaces. For the case of uniform exponential stability our main tool will be the use of functionals on function spaces. We shall prove, that even when we give up at the strong continuity of $\mathbf{S}$ for the uniform exponential stability of $\mathbf{S}$ it is sufficient that one of the conditions from Theorems 1.3. and 1.4. respectively, holds on a subset of

X which is of the second category. Thus a recent results due to Neerven (see [15]) is generalized.

In the second part of the paper we shall present characterizations for uniform exponential unstability of semigroups, in the spirit of Neerven's approach. We shall use the theory of Banach function spaces and Banach sequence spaces, respectively, and the concept of uniform exponential unstability will be characterized in terms of the ownership of some orbits to certain Banach function spaces. We shall obtain the versions of the theorems of Datko, Zabczyk, Rolewicz and Neerven for the case of unstability.

2. Banach function spaces

In this section we shall present some definitions, notations and results about Banach function spaces. Therefore, let (Ω,Σ,μ) be a positive σ-finite measure space. We denote by $M(\mu)$ the linear space of μ-measurable functions $f:\Omega\to\mathbf{C}$, identifying the functions which are equal μ almost everywhere.

DEFINITION 2.1. A *Banach function norm* is a function $N:M(\mu)\to[0,\infty]$ with the following properties:

(i) $N(f)=0$ if and only if $f=0\ \mu$ almost everywhere;
(ii) if $|f|\leqslant|g|\ \mu$ almost everywhere then $N(f)\leqslant N(g)$;
(iii) $N(af)=|a|N(f)$, for all $a\in\mathbf{C}$ and all $f\in M(\mu)$ with $N(f)<\infty$;
(iv) $N(f+g)\leqslant N(f)+N(g)$, for all $f,g\in M(\mu)$.

Let $B=B_N$ be the set defined by $B:=\{f\in M(\mu):|f|_B:=N(f)<\infty\}$. It is easy to see that $(B,|\cdot|_B)$ is a normed linear space. If B is complete then B is called *Banach function space over* Ω.

REMARK 2.1. B is an ideal in $M(\mu)$, i.e. if $|f|\leqslant|g|\ \mu$ almost everywhere and $g\in B$ then also $f\in B$ and $|f|_B\leqslant|g|_B$.

REMARK 2.2. If $f_n\to f$ in norm in B, then there exists a subsequence (f_{k_n}) converging to f pointwise (see [12]).

Let $(\Omega,\Sigma,\mu)=(\mathbf{R}_+,\mathscr{M},m)$, where $\mathscr{M}$ is the σ-algebra of all Lebesgue measurable sets $A\subset\mathbf{R}_+$ and m the Lebesgue measure. For a Banach function space over $\mathbf{R}_+$, we define

$$F_B:\mathbf{R}_+^*\to\overline{\mathbf{R}}_+,\quad F_B(t):=\begin{cases}|\chi_{[0,t)}|_B, & \text{if } \chi_{[0,t)}\in B\\ \infty, & \text{if } \chi_{[0,t)}\notin B\end{cases},$$

where $\chi_{[0,t)}$ denotes the characteristic function of $[0,t)$. The function F_B is called *the fundamental function* of the Banach function space B.

In what follows, we shall denote by $\mathscr{E}(\mathbf{R}_+)$ the set of all Banach function spaces B with the properties $|\chi_{[0,t)}|_B<\infty$, for all $t\geqslant 0$ and $\lim\limits_{t\to\infty}F_B(t)=\infty$.

We shall denote by $\mathscr{L}(\mathbf{R}_+)$ the set of all Banach function spaces B, with the property that for every $\varepsilon > 0$ there exists $t_0 \in \mathbf{R}_+$ such that

$$|\chi_{[t-t_0,t)}|_B \geqslant \varepsilon, \quad \forall t \geqslant t_0.$$

REMARK 2.3. If $B \in \mathscr{L}(\mathbf{R}_+)$ then $\lim\limits_{t\to\infty} F_B(t) = \infty$.

Similary, let $(\Omega, \Sigma, \mu) = (\mathbf{N}, \mathscr{P}(\mathbf{N}), \mu_c)$, where μ_c is the countable measure and let B be a Banach function space over $\mathbf{N}$ (in this case B is called *Banach sequence space*). We define

$$F_B : \mathbf{N}^* \to \overline{\mathbf{R}}_+, \quad F_B(n) := \begin{cases} |\chi_{\{0,\dots,n-1\}}|_B, & \text{if } \chi_{\{0,\dots,n-1\}} \in B \\ \infty, & \text{if } \chi_{\{0,\dots,n-1\}} \notin B \end{cases},$$

called *the fundamental function* of the Banach sequence space B.

We shall denote by $\mathscr{E}(\mathbf{N})$ the set of all Banach sequence spaces with the properties $|\chi_{\{0,\dots,n\}}| < \infty$, for all $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} F_B(n) = \infty$.

We shall denote by $\mathscr{L}(\mathbf{N})$ the set of all Banach sequence spaces B with the property that for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that

$$|\chi_{\{j-n_0,\dots,j\}}|_B \geqslant \varepsilon, \quad \forall j \in \mathbf{N}, j \geqslant n_0.$$

REMARK 2.4. If $B \in \mathscr{L}(\mathbf{N})$ then $\lim\limits_{n\to\infty} F_B(n) = \infty$.

REMARK 2.5. If B is a Banach function space over $\mathbf{R}_+$, which belongs to $\mathscr{L}(\mathbf{R}_+)$ (or to $\mathscr{E}(\mathbf{R}_+)$), then

$$S_B := \{(\alpha_n)_n : \sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)} \in B\},$$

with respect to the norm

$$|(\alpha_n)_n|_{S_B} := |\sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)}|_B,$$

is a Banach sequence space which belongs to $\mathscr{L}(\mathbf{N})$ (to $\mathscr{E}(\mathbf{N})$, respectively).

EXAMPLE 2.1. (*Orlicz spaces*) Let $g : \mathbf{R}_+ \to \overline{\mathbf{R}}_+$ be a non-decreasing, left continuous function, which is not identically 0 or ∞ on $(0,\infty)$. We define the function:

$$Y_g(t) = \int_0^t g(s)\, ds,$$

which is called *the Young function* associated to g.

Let $(\Omega, \Sigma, \mu) \in \{(\mathbf{R}_+, \mathscr{M}, m), (\mathbf{N}, \mathscr{P}(\mathbf{N}), \mu_c)\}$. For every $h : \Omega \to \mathbf{C}$ we consider

$$M_g(h) = \int_\Omega Y_g(|h(\omega)|)\, d\mu.$$

The set of all functions $h : \Omega \to \mathbf{C}$ with the property that there exists $k > 0$ such that $M_g(kh) < \infty$, is easily checked to be a linear space. With respect to the norm

$$|h|_g := inf\,\{k > 0 : M_g(\frac{1}{k}h) \leqslant 1\},$$

it is a Banach function space over Ω called *Orlicz function space*, for $\Omega = \mathbf{R}_+$ and *Orlicz sequence space*, for $\Omega = \mathbf{N}$. For $\Omega = \mathbf{R}_+$ we shall denote it by E_g and for $\Omega = \mathbf{N}$ by O_g, respectively.

REMARK 2.6. Let $p \in [1,\infty]$. The Orlicz function spaces and the Orlicz sequence spaces associated to

$$g_p(t) = p\,t^{p-1}, \text{ if } p \in [1,\infty) \text{ and } g_\infty(t) = \begin{cases} 0, & t \in [0,1] \\ \infty, & t > 1 \end{cases}, \text{ if } p = \infty$$

are $L^p(\mathbf{R}_+,\mathbf{C})$ and $l^p(\mathbf{N},\mathbf{C})$, respectively.

REMARK 2.7. If $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a non-decreasing left continuous function with $g(t) > 0$, for all $t > 0$ and $g(0) = 0$, then the Orlicz function space E_g belongs to $\mathscr{L}(\mathbf{R}_+) \cap \mathscr{E}(\mathbf{R}_+)$ and the Orlicz sequence space O_g belongs to $\mathscr{L}(\mathbf{N}) \cap \mathscr{E}(\mathbf{N})$.

3. Uniform exponential stability of semigroups and functionals on function spaces

In this section, we denote by $\mathscr{S}(\mathbf{R}_+)$ the Fréchet space of all real sequences with the topology of punctual convergence. By $\mathscr{S}^+(\mathbf{R}_+)$ we denote the set of all $s \in \mathscr{S}(\mathbf{R}_+)$ with $s(n) \geqslant 0$, for all $n \in \mathbf{N}$.

Let $\mathscr{F}$ be the set of all functions $F : \mathscr{S}^+(\mathbf{R}_+) \to [0,\infty]$ with the folowing properties

(i) if $s_1, s_2 \in \mathscr{S}^+(\mathbf{R}_+)$ with $s_1 \leqslant s_2$ then $F(s_1) \leqslant F(s_2)$;

(ii) $F(s) = \infty$, for all constant sequence $s \in \mathscr{S}^+(\mathbf{R}_+)$ with $s > 0$;

(iii) $F(\liminf_{n\to\infty} s_n) \leqslant \liminf_{n\to\infty} F(s_n)$, for all $s_n \in \mathscr{S}^+(\mathbf{R}_+)$.

For every semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ and every $x \in X$ we associate the sequence $s_x \in \mathscr{S}^+(\mathbf{R}_+)$ by $s_x(n) = ||S(n)x||$ for all $n \in \mathbf{N}$.

In [11] we have proved the following characterization for uniform exponential stability:

THEOREM 3.1. *An exponentially bounded semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is uniformly exponentially stable if and only if there exist a function $F \in \mathscr{F}$ and a set $A \subset X$ of the second category such that $F(s_x) < \infty$ for all $x \in A$.*

We shall use the above result in order to obtain some characterizations for uniform exponential stability in terms of Banach sequence spaces. After that, we shall present the result for the case of functionals on function spaces and thus, we shall obtain another condition in terms of Banach function spaces.

THEOREM 3.2. *An exponentially bounded semigroup* $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ *is uniformly exponentially stable if and only if there are a Banach sequence space* $B \in \mathscr{E}(\mathbf{N})$ *and a set* $A \subset X$ *of the second category such that* $s_x \in B$*, for all* $x \in A$.

Proof. Necessity. It is sufficient to choose $B = l^1(\mathbf{N}, \mathbf{C})$ and $A = X$.

Sufficiency. For every $m \in \mathbf{N}$ let $F_m : \mathscr{S}^+(\mathbf{R}_+) \to \mathbf{R}_+$, $F_m(s) = |s\chi_{\{0,\dots,m\}}|_B$. Then F_m is continuous, for all $m \in \mathbf{N}$. If $F = \sup\{F_m : m \in \mathbf{N}\}$ it is easy to see that $F \in \mathscr{F}$ and $F(s_x) < \infty$, for all $x \in A$. By Theorem 3.1. it results that $\mathbf{S}$ is uniformly exponentially stable. □

Let $\mathscr{M}(\mathbf{R}_+)$ be the space of all locally bounded and measurable functions $f : \mathbf{R}_+ \to \mathbf{R}$ with the topology of uniform convergence on compact sets. By $\mathscr{M}^+(\mathbf{R}_+)$ we denote the set of all $f \in \mathscr{M}^+(\mathbf{R}_+)$ with $f \geqslant 0$.

Let $\mathscr{G}$ be the set of all functions $G : \mathscr{M}^+(\mathbf{R}_+) \to [0, \infty]$ with the properties

(i) if $f, g \in \mathscr{M}^+(\mathbf{R}_+)$ with $f \leqslant g$ then $G(f) \leqslant G(g)$;

(ii) $G(f) = \infty$, for all constant functions $f \in \mathscr{M}^+(\mathbf{R}_+)$ with $f > 0$;

(iii) $G(\liminf_{n\to\infty} f_n) \leqslant \liminf_{n\to\infty} G(f_n)$, for all $f_n \in \mathscr{M}^+(\mathbf{R}_+)$.

For every strongly measurable semigroup $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ and every $x \in X$ we associate the function $f_x \in \mathscr{M}^+(\mathbf{R}_+)$ by $f_x(t) := ||S(t)x||$.

THEOREM 3.3. *Let* $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ *be a strongly measurable and exponentially bounded semigroup. Then* $\mathbf{S}$ *is uniformly exponentially stable if and only if there exist a subset* $E \subset X$ *of the second category and* $G \in \mathscr{G}$ *such that* $G(f_x) < \infty$*, for all* $x \in E$.

Proof. The necessity is immediate for $E = X$ and

$$G(f) = \int_0^\infty f(t)dt.$$

Sufficiency. For $x \in X$ we consider the function $t_x : \mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$t_x(u) = \frac{1}{Me^\omega}||S(n+1)x||, \quad \forall u \in [n, n+1), \quad \forall n \in \mathbf{N},$$

where M and ω are given by Definition 1.1. Then $t_x(u) \leqslant f_x(u)$, for all $u \in [n, n+1)$ and for all $n \in \mathbf{N}$.

For every sequence $s \in \mathscr{S}^+(\mathbf{R}_+)$ we associate the function $f_s \in \mathscr{M}^+(\mathbf{R}_+)$ defined by $f_s(t) := s(n+1)/(Me^\omega)$, for $t \in [n, n+1)$ and $n \in \mathbf{N}$. If $G \in \mathscr{G}$ then the function $F_G : \mathscr{S}^+(\mathbf{R}_+) \to [0, \infty], F_G(s) := G(f_s)$ belongs to $\mathscr{F}$ and

$$F_G(s_x) = G(t_x) \leqslant G(f_x) < \infty, \quad \forall x \in E.$$

By Theorem 3.1. it follows that $\mathbf{S}$ is uniformly exponentially stable. □

We shall denote by $\mathscr{N}$ the set of all non-decreasing functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ with $f(0) = 0$ and $f(t) > 0$, for all $t > 0$.

COROLLARY 3.1. *A strongly measurable and exponentially bounded semigroup $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ is uniformly exponentially stable if and only if there exist a subset $E \subset X$ of the second category and a function $N \in \mathcal{N}$ such that*

$$\int_0^\infty N(||S(t)x||)dt < \infty, \quad \forall x \in E.$$

Proof. The necessity follows for $E = X$ and $N(t) = t$, for all $t \geqslant 0$.

Sufficiency. Let $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ be the function defined by

$$\Psi(t) = N\left(\frac{t}{Me^{\omega}}\right),$$

where M and ω are given by Definition 1.1. Without loss of generality, we assume that Ψ is left continuous. Then $\Psi \in \mathcal{N}$ and

$$\Psi(||S(n+1)x||) \leqslant \int_n^{n+1} N(||S(t)x||)dt, \quad \forall n \in \mathbf{N}.$$

Hence

$$\sum_{n=0}^{\infty} \Psi(||S(n+1)x||) \leqslant \int_0^\infty N(||S(t)x||)dt < \infty, \quad \forall x \in E.$$

From Theorem 3.2. applied for $B = O_\Psi$ (the Orlicz sequence space generated by Ψ), it results that $\mathbf{S}$ is uniformly exponentially stable. $\square$

THEOREM 3.4. *For every strongly measurable and exponentially bounded semigroup $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ the following statements are equivalent:*

(i) *$\mathbf{S}$ is uniformly exponentially stable;*

(ii) *there are a subset $E \subset X$ of the second category and a Banach function space $B \in \mathscr{E}(\mathbf{R}_+)$ such that $f_x \in B$, for all $x \in E$;*

(iii) *there is a Banach function space $B \in \mathscr{E}(\mathbf{R}_+)$ such that $E_B := \{x \in X : f_x \in B\}$ is of the second category.*

Proof. The implications $(i) \implies (ii)$ and $(ii) \implies (iii)$ being trivial, we shall prove $(iii) \implies (i)$.

Let Q_B be the space of all sequences $s \in \mathscr{S}(\mathbf{R}_+)$ with the property that $\sum_{n=0}^{\infty} s(n+1)\chi_{[n,n+1)} \in B$, which is a Banach space with respect to the norm

$$|s|_{Q_B} = |\sum_{n=0}^{\infty} s(n+1)\chi_{[n,n+1)}|_B + |s(0)|.$$

Since $B \in \mathscr{E}(\mathbf{R}_+)$ we have that $Q_B \in \mathscr{E}(\mathbf{N})$. If M, ω are given by Definition 1.1. we observe that $|s_x|_{Q_B} \leqslant Me^{\omega}|f_x|_B + ||x||$, for all $x \in E_B$. It follows that $E_B \subset A_{S_B}$, so the last set is of the second category.

By Theorem 3.2. we conclude that the semigroup $\mathbf{S}$ is uniformly exponentially stable. $\square$

REMARK 3.1. Theorem 3.4. is an extension of the Neerven's result given by Theorem 1.4.

4. Uniform exponential unstability of semigroups in terms of Banach function spaces

In what follows, we shall present the versions of the theorems from the first section, for the case of uniform exponential unstability of semigroups. We shall also characterize the concept of uniform exponential unstability in terms of Banach sequence spaces and Banach function spaces, respectively, our purpose being to obtain some theorems of Neerven type for uniform exponential unstability. We shall give the versions of the theorems of Zabczyk and Rolewicz, for uniform exponential unstability of semigroups of linear operators.

Let X be a Banach space and let $C = \{x \in X : ||x|| = 1\}$. We shall denote by $\mathscr{N}$ the set of all non-decreasing functions $f : \mathbf{R}_+ \to \mathbf{R}_+$ with $f(0) = 0$ and $f(t) > 0$, for all $t > 0$.

We start with a technical lemma:

LEMMA 4.1. *Let* $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ *be an exponentially bounded semigroup on the Banach space* X. *Then* $\mathbf{S}$ *is uniformly exponentially unstable if and only if there are* $t_0 > 0$ *and* $c > 1$ *such that*

$$||S(t_0)x|| \geqslant c||x||, \quad \forall x \in X.$$

Proof. Necessity is trivial. To prove sufficiency, let $\nu > 0$ such that $c = e^{\nu t_0}$. Let $t > 0$. Then there are $n \in \mathbf{N}$ and $r \in [0, t_0)$ such that $t = nt_0 + r$. It follows

$$e^{\nu t}||x|| \leqslant ||S((n+1)t_0)x|| \leqslant Me^{\omega t_0}||S(t)x||, \quad \forall x \in X,$$

which ends the proof. □

First, we give a characterization for uniform exponential unstability of semigroups, using Banach sequence spaces.

THEOREM 4.1. *Let* $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ *be an exponentially bounded semigroup on the Banach space* X. *Then* $\mathbf{S}$ *is uniformly exponentially unstable if and only if it is injective and there exist a Banach sequence space* $B \in \mathscr{L}(\mathbf{N})$, *a function* $N \in \mathscr{N}$ *and a constant* $K > 0$ *such that for every* $x \in C$ *the mapping*

$$\varphi_x : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_x(n) = N\left(\frac{1}{||S(n)x||}\right)$$

belongs to B *and* $|\varphi_x|_B \leqslant K$, *for all* $x \in C$.

Proof. *Necessity* is immediate for $N(t) = t$, for all $t \geqslant 0$ and $B = l^1(\mathbf{N}, \mathbf{C})$.
Sufficiency. Since $B \in \mathscr{L}(\mathbf{N})$ there is $n_0 \in \mathbf{N}^*$ such that

$$|\chi_{\{j-n_0,\dots,j\}}|_B > \frac{K}{N(1)}, \quad \forall j \geqslant n_0. \tag{4.1}$$

Let $x \in C$. Denote by $\tilde{x} = Lx$, where $L = Me^{\omega n_0}$ and M, ω are given by Definition 1.1. Let $p \in \mathbf{N}$. For every $i \in \{0, \ldots, n_0\}$ we have

$$||S(p+i)x|| \leqslant Me^{\omega i}||S(p)x|| \leqslant ||S(p)\tilde{x}||.$$

It follows that

$$N\left(\frac{1}{||S(p)\tilde{x}||}\right) \leqslant N\left(\frac{1}{||S(p+i)x||}\right), \quad i \in \{0, \ldots, n_0\}.$$

Thus, we obtain that

$$\chi_{\{p,\ldots,p+n_0\}} N\left(\frac{1}{||S(p)\tilde{x}||}\right) \leqslant \varphi_x.$$

Using (4.1) we deduce that

$$\frac{K}{N(1)} N\left(\frac{1}{||S(p)\tilde{x}||}\right) < K.$$

The last inequality implies

$$\frac{1}{||S(p)x||} \leqslant L, \quad \forall x \in C, \forall p \in \mathbf{N}. \tag{4.2}$$

Let $x \in C$ and $\tilde{x} = Lx$. There is $m_0 \in \mathbf{N}^*$ such that

$$|\chi_{\{j-m_0,\ldots,j\}}|_B > \frac{K}{N\left(\frac{1}{2L}\right)}, \quad \forall j \geqslant m_0. \tag{4.3}$$

For every $i \in \{0, \ldots, m_0\}$, using (4.2), we have

$$\frac{||S(i)x||}{||S(m_0)x||} \leqslant L,$$

so

$$N\left(\frac{1}{||S(m_0)\tilde{x}||}\right) \leqslant N\left(\frac{1}{||S(i)x||}\right), \quad \forall i \in \{0, \ldots, m_0\}.$$

It results that

$$\chi_{\{0,\ldots,m_0\}} N\left(\frac{1}{||S(m_0)\tilde{x}||}\right) \leqslant \varphi_x,$$

and hence from (4.3) and $N \in \mathscr{N}$ we deduce that

$$\frac{1}{||S(m_0)\tilde{x}||} < \frac{1}{2L}.$$

It follows that

$$||S(m_0)x|| \geqslant 2, \quad \forall x \in C.$$

From Lemma 4.1. we conclude that $\mathbf{S}$ is uniformly exponentially unstable. $\square$

As a consequence, we obtain a theorem of Zabczyk type, for uniform exponential unstability, given by

COROLLARY 4.1. *Let $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ be an exponentially bounded semigroup on the Banach space X. Then $\mathbf{S}$ is uniformly exponentially unstable if and only if it is injective and there exist a function $N \in \mathscr{N}$ and a constant $K > 0$ such that*

$$\sum_{n=0}^{\infty} N\left(\frac{1}{||S(n)x||}\right) \leqslant K, \quad \forall x \in C.$$

Proof. *Necessity* is immediate for $N(t) = t$, for all $t \geqslant 0$.
Sufficiency follows from Theorem 4.1., for $B = l^1(\mathbf{N}, \mathbf{C})$. □

THEOREM 4.2. *Let $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ be a C_0 -semigroup on the Banach space X. Then $\mathbf{S}$ is uniformly exponentially unstable if and only if it is injective and there exist a Banach function space $B \in \mathscr{L}(\mathbf{R}_+)$, a function $N \in \mathscr{N}$ and a constant $K > 0$ such that for every $x \in C$ the mapping*

$$\psi_x : \mathbf{R}_+ \to \mathbf{R}_+, \quad \psi_x(t) = N\left(\frac{1}{||S(t)x||}\right)$$

belongs to B and $|\psi_x|_B \leqslant K$, for all $x \in C$.

Proof. *Necessity* is trivial.
Sufficiency. Let $x \in C$. If M, ω are given by Definition 1.1. we define

$$\tilde{N} : \mathbf{R}_+ \to \mathbf{R}_+, \quad \tilde{N}(t) = N\left(\frac{t}{Me^{\omega}}\right)$$

and

$$\varphi_x : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_x(n) = \tilde{N}\left(\frac{1}{||S(n)x||}\right).$$

From

$$||S(t)x|| \leqslant Me^{\omega}||S(n)x||, \quad \forall t \in [n, n+1), \forall n \in \mathbf{N},$$

we have that

$$\tilde{N}(\frac{1}{||S(n)x||}) \leqslant N\left(\frac{1}{||S(t)x||}\right), \quad \forall t \in [n, n+1), \forall n \in \mathbf{N}.$$

It follows that

$$\sum_{n=0}^{\infty} \varphi_x(n)\chi_{[n,n+1)} \leqslant \psi_x. \tag{4.4}$$

If S_B is the Banach sequence space associated to B, according to Remark 2.5., then $S_B \in \mathscr{L}(\mathbf{N})$. Moreover, from (4.4) we deduce that $\varphi_x \in S_B$ and

$$|\varphi_x|_{S_B} \leqslant |\psi_x|_B \leqslant K, \quad \forall x \in C.$$

From Theorem 4.1. applied for S_B and for $\tilde{N}$, we finally have that $\mathbf{S}$ is uniformly exponentially unstable. □

We end this section with a condition of Rolewicz type for uniform exponential unstability given by

COROLLARY 4.2. *Let* $\mathbf{S} = \{S(t)\}_{t \geqslant 0}$ *be a* C_0 *- semigroup on the Banach space* X. *Then* $\mathbf{S}$ *is uniformly exponentially unstable if and only if it is injective and there exist a function* $N \in \mathscr{N}$ *and a constant* $K > 0$ *such that*

$$\int_0^\infty N\left(\frac{1}{||S(t)x||}\right) dt \leqslant K, \quad \forall x \in C.$$

Proof. Necessity is immediate for $N(t) = t$, for all $t \geqslant 0$.
Sufficiency follows from Theorem 4.2., for $B = L^1(\mathbf{R}_+, \mathbf{C})$. $\square$

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University of the West
Department of Mathematics
Bd. V. Pârvan 4, 1900
Timişoara, ROMANIA
e-mail: megan@hilbert.math.uvt.ro,
sasu@hilbert.math.uvt.ro

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com