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CHARACTERIZATIONS OF CHAOTIC ORDER ASSOCIATED WITH THE MOND–SHISHA DIFFERENCE

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Abstract. Recently, Yamazaki showed new order preserving operator inequalities on the usual order and the chaotic order by estimating the lower bound of the difference. Mond and Shisha gave an estimtate of the difference of the arithmetic one to the geometric one, as a converse of the arithmetic-geometric mean inequality. In this paper, by means of the Mond-Shisha difference, we shall put another interpretation on a characterization of the chaotic order associated with the difference by Yamazaki: If $A > 0$, $MI \geqslant B \geqslant mI > 0$ and $h = \frac{M}{m} > 1$, then $\log A \geqslant \log B$ is equivalent to

$$A^p + D(m^p, M^p)I \geqslant B^p \qquad \text{for all } p > 0,$$

where

$$D(m^p, M^p) = \theta M^p + (1-\theta)m^p - M^{p\theta}m^{p(1-\theta)} \quad \text{and} \quad \theta = \log\left(\frac{h^p - 1}{p\log h}\right)\frac{1}{p\log h}.$$

Moreover, inspired by Yamazaki's work, we shall make an attempt to clarify distinction between the usual order and the chaotic order by using the Furuta inequality. Among others, we show the following parametrized order preserving operator inequalities associated with the difference: If $A > 0$ and $MI \geqslant B \geqslant mI > 0$, then for each $\delta \in [0,1]$

$$A^\delta \geqslant B^\delta \quad \text{if and only if} \quad A^{p+\delta} + \frac{1}{m^r}C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta})I \geqslant B^{p+\delta} \quad \text{for } p, r > 0$$

where the case $\delta = 0$ means the chaotic order.

1. Introduction

The Löwner-Heinz theorem says that $A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for all $\alpha \in [0,1]$, but it is well-known that $A \geqslant B$ does not always ensure $A^2 \geqslant B^2$ in general. Yamazaki [22] showed that t^2 is order preserving in the following sense associated with the difference: If A and B are positive operators on H satisfying $MI \geqslant B \geqslant mI > 0$, then

$$A \geqslant B \qquad \text{implies} \qquad A^2 + \frac{(M-m)^2}{4}I \geqslant B^2. \tag{1}$$

Moreover, he showed the following order preserving operator inequality as an extension of (1):

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THEOREM A. *If $A \geqslant B \geqslant 0$ and $MI \geqslant B \geqslant mI > 0$, then*

$$A^p + C(m,M,p)I \geqslant B^p \qquad \text{for all } p > 1,$$

where

$$C(m,M,p) = \frac{mM^p - Mm^p}{M-m}\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\} \geqslant 0 \tag{2}$$

and

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} \geqslant 1. \tag{3}$$

We remark that $K_+(m,M,p)$ is called *Ky Fan-Furuta constant* ([16, 12]).

The order between positive invertible operators A and B defined by $\log A \geqslant \log B$ is said to be the *chaotic order* $A \gg B$ in [5] which is weaker than the usual one $A \geqslant B$. Yamazaki [23] showed new characterizations of the chaotic order via Specht's ratio [20, 2]. Recall that the *logarithmic mean* $L(m,M)$ is defined for $M \geqslant m > 0$ as

$$L(m,M) = \frac{M-m}{\log M - \log m} \ (M > m) \quad \text{and} \quad L(m,m) = m. \tag{4}$$

THEOREM B. *Let A and B be positive invertible operators on H satisfying $MI \geqslant B \geqslant mI > 0$. Put $h = \frac{M}{m} > 1$. Then $\log A \geqslant \log B$ is equivalent to*

$$A^p + L(m^p, M^p)\log M_h(p)I \geqslant B^p \quad \text{for all } p > 0 \tag{5}$$

where $M_h(p)$ is the generalized Specht's ratio ([24]);

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}. \tag{6}$$

On the other hand, Mond and Shisha [19, 18] made an estimate of the difference between the arithmetic mean and the geometric one: For positive numbers $x_1, \cdots, x_n \in [m,M]$ with $M > m > 0$ and $h = \frac{M}{m}$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} + D(m,M) \geqslant \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{7}$$

where

$$D(m,M) = \theta M + (1-\theta)m - M^\theta m^{1-\theta} \quad \text{and} \quad \theta = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h} \tag{8}$$

which we call the *Mond-Shisha difference*. The result by Mond-Shisha urges us toward a new interpretation on Theorem B.

In this paper, by means of the Mond-Shisha difference, we shall put another interpretation on a characterization of the chaotic order associated with the difference in Theorem B. Moreover, inspired by Yamazaki's work [22, 23], we shall propose some interpolation theorems on new order preserving operator inequalities between the usual order and the chaotic order by using the Furuta inequality. Among others, we show the following parametrized order preserving operator inequalities associated with the difference: If $A > 0$ and $MI \geqslant B \geqslant mI > 0$, then for each $\delta \in [0,1]$

$$A^\delta \geqslant B^\delta \quad \text{if and only if} \quad A^{p+\delta} + \frac{1}{m^r}C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta})I \geqslant B^{p+\delta} \ \text{ for } p,r > 0$$

where the case $\delta = 0$ means the chaotic order.

2. Characterization of chaotic order

First of all, we recall a *determinant* for positive operators [3, 2]: The determinant $\Delta_x(A)$ is considered for positive operators A at a unit vector x as a continuous weighted geometric mean with the weight x:

$$\Delta_x(A) = \exp(\log A\, x, x),$$

and the inequality $\Delta_x(A) \leqslant (Ax,x)$ is nothing but the arithmetic-geometric mean inequality. In [3], we obtained the estimate of the difference of (Ax,x) to the determinant $\Delta_x(A)$ by using the Mond-Pečarić method [17]. From this point of veiw, we shall present another interpretation on Theorem B, in which the Mond-Shisha difference plays an essential role.

THEOREM 1. *Let A and B be positive invertible operators on H satisfying $MI \geqslant B \geqslant mI > 0$. Then $\log A \geqslant \log B$ is equivalent to*

$$A^p + D(m^p, M^p)I \geqslant B^p \quad \textit{for all } p > 0, \tag{9}$$

where $D(m,M)$ is defined in (8)

We need the following two lemmas to prove Theorem 1.

LEMMA 2. *The Mond-Shisha difference coincides with the constant (5) of Theorem B via Specht's ratio: If $M > m > 0$, then*

$$D(m^p, M^p) = L(m^p, M^p)\log M_h(p)$$

where $h = \frac{M}{m} > 1$.

Proof. If we put $\theta = \log\left(\frac{h^p-1}{p\log h}\right)\frac{1}{p\log h}$, then we have

$$\begin{aligned} L(m^p, M^p)\log M_h(p) &= \frac{m^p(h^p-1)}{p\log h}\left(\log\left(\frac{h^p-1}{p\log h}\right) + \frac{p\log h}{h^p-1} - 1\right) \\ &= m^p\left(\log\left(\frac{h^p-1}{p\log h}\right)\frac{h^p-1}{p\log h} + 1 - \frac{h^p-1}{p\log h}\right) \\ &= m^p\left(\theta(h^p-1) + 1 - h^{p\theta}\right) \\ &= D(m^p, M^p). \quad \square \end{aligned}$$

We cite the following result in [3, Theorem 10], which is considered as a continuous version of Mond-Shisha result (7):

LEMMA 3. ([3]) *Let A be a positive operator on H satisfying $MI \geqslant A \geqslant mI > 0$. Put $h = \frac{M}{m}$. Then the difference between (Ax,x) and the determinant $\Delta_x(A)$ for A at x is not greater than the Mond-Shisha difference:*

$$(Ax,x) - \Delta_x(A) \leqslant D(m,M),$$

where $D(m,M)$ *is defined in (8) and the equality holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x=\sqrt{1-\log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}}e_m+\sqrt{\log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}}e_M,$$

where e_m *and* e_M *are corresponding unit eigenvectors to* m *and* M *respectively.*

Proof. Put $S=\log A$, $a=L(m,M)$ and $b=\frac{M\log m-m\log M}{\log M-\log m}$, then we have

$$(e^Sx,x)\leqslant a(Sx,x)+b\leqslant e^{(Sx,x)}+a\log a+b-a.$$

The number $a\log a+b-a$ is exactly the Mond-Shisha difference. In fact, $a=\frac{m(h-1)}{\log h}$ and hence we have

$$\begin{aligned}a\log a+b-a&=a\left(\log a+\frac{(M-m)\log m-m(\log M-\log m)}{M-m}-1\right)\\&=\frac{m(h-1)}{\log h}\left(\log(h-1)-\log(\log h)+\frac{\log h}{h-1}-1\right)\\&=m\left((h-1)\log(\frac{h-1}{\log h})\frac{1}{\log h}+1-\frac{h-1}{\log h}\right)\\&=m((h-1)\theta+1-h^{\theta})\\&=D(m,M),\end{aligned}$$

where $\theta=\log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}$.

To verify the equality condition, we can put $x=\sqrt{1-t^2}e_m+te_M$ for a number $0<t<1$. Then it follows that

$$\log m^{1-t^2}M^{t^2}=(Sx,x)=\log a=\log\frac{m(h-1)}{\log h},$$

and so we have

$$t^2=\log(\frac{h-1}{\log h})\frac{1}{\log h}(>0).\quad\square$$

REMARK 1. As for the power mean version of the Mond-Shisha difference, Mond and Shisha [18] showed the following result: Under the assumption of the above lemma, if real numbers $r<s$, $rs\neq 0$, then

$$(A^sx,x)^{1/s}-(A^rx,x)^{1/r}\leqslant\{\theta M^s+(1-\theta)m^s\}^{1/s}-\{\theta M^r+(1-\theta)m^r\}^{1/r}$$

holds for θ which satisfies some conditions. We note that the Mond-Shisha difference is obtained when we put $s=1$ and $r\to 0$.

Proof of Theorem 1. Suppose that $\log A\geqslant\log B$. Since $M^pI\geqslant B^p\geqslant m^pI>0$, it follows that

$$\begin{aligned}(B^px,x)&\leqslant\exp(\log B^p\,x,x)+D(m^p,M^p)&&\text{by Lemma 3}\\&\leqslant\exp(\log A^p\,x,x)+D(m^p,M^p)&&\text{by }\log B\leqslant\log A\\&\leqslant(A^px,x)+D(m^p,M^p)&&\text{by Jensen's inequality}\end{aligned}$$

holds for every unit vector $x \in H$. Hence we have

$$A^p + D(m^p, M^p)I \geqslant B^p \qquad \text{for all } p > 0.$$

Conversely, suppose (9). Since we have

$$\frac{A^p - I}{p} + \frac{1}{p}D(m^p, M^p)I \geqslant \frac{B^p - I}{p},$$

it follows from Lemma 2 and [23, Theorem 2] that

$$\frac{1}{p}D(m^p, M^p) = \frac{1}{p}L(m^p, M^p)\log M_h(p) \to 0 \quad \text{as } p \to 0,$$

so that we have $\log A \geqslant \log B$. $\square$

3. Order preserving inequality associated with the difference

Extensions of Kantorovich type operator inequalities are discussed and a very wide diversity of characterizations both on the usual order and the chaotic order is shown in [1, 6, 8, 12, 13, 15, 24]. On the other hand, Yamazaki [22, 23] obtained new order preserving operator inequalities associated with the difference as both characterizations. From this point of veiw, we shall make an attempt to clarify distinction between the usual order and the chaotic order by using the Furuta inequality.

Related to the extension of the Löwner-Heinz theorem, Furuta established the following ingenious order preserving operator inequality which is now called the Furuta inequality.

THEOREM F (the Furuta inequality [9]).

If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with

$$(1+r)q \geqslant p + r.$$

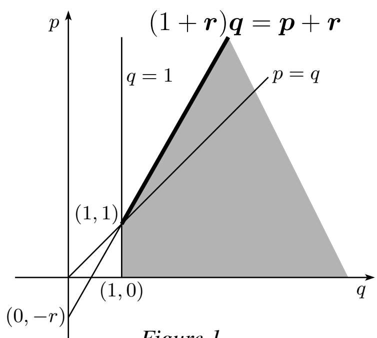

Figure 1

Alternative proofs of Theorem F have been given in [4], [14], and a one-page proof in [10]. The domain drawn for p, q and r in Figure is the best possible one [21] for Theorem F.

Let A and B be positive invertible operators on a Hilbert space H. We consider an order $A^\delta \geqslant B^\delta$ for $\delta \in [0,1]$ which interpolates the usual order $A \geqslant B$ and the chaotic order $A \gg B$ continuously. We consider that the case $\delta = 0$ means the chaotic order since $\lim_{\delta\to 0} \frac{A^\delta - I}{\delta} = \log A$ for a positive invertible operator A.

The following lemma shows that the Furuta inequality connects the usual order and the chaotic one.

LEMMA 4. ([8]) *Let A and B be positive invertible operators on a Hilbert space H. The following statements are mutually equivalent for each $\delta \in [0,1]$:*

(i) $A^\delta \geqslant B^\delta,$ *where the case $\delta = 0$ means $A \gg B$.*

(ii) $\left(B^{\frac{p}{2}} A^{p+\delta} B^{\frac{p}{2}}\right)^{\frac{p+\delta}{2p+\delta}} \geqslant B^{p+\delta}$ *for all $p \geqslant 0$.*

(iii) $\left(B^{\frac{r}{2}} A^{p+\delta} B^{\frac{r}{2}}\right)^{\frac{r+\delta}{p+r+\delta}} \geqslant B^{r+\delta}$ *for all $p \geqslant 0$ and $r \geqslant 0$.*

In fact, Lemma 4 is ensured by the Furuta inequality in the case of $0 < \delta \leqslant 1$ and by [5, 7, 11] in the case of $\delta = 0$. Also, the best possibility of the power $\frac{r}{p+r}$ in (iii) is shown in [25].

Here, we show the following parametrized order preserving operator inequality associated with the difference by using the Furuta inequality and Theorem A:

THEOREM 5-1. *Let A and B be positive invertible operators on H satisfying $MI \geqslant B \geqslant mI > 0$. Then the following assertions are equivalent for each $\delta \in [0,1]$:*

(i) $A^\delta \geqslant B^\delta,$ *where the case $\delta = 0$ means $A \gg B$.*

(ii) $A^p + \dfrac{1}{m^{r-\delta}} C(m^r, M^r, \dfrac{p+r-\delta}{r}) I \geqslant B^p$ *for all $p > \delta$ and $r > \delta$,*

where $C(m,M,p)$ is defined in (2).

Proof. Proof of (i) $\Longrightarrow$ (ii): It follows from Lemma 4 that $A^\delta \geqslant B^\delta$ is equivalent to the following inequality:

$$\left(B^{\frac{r}{2}} A^{p+\delta} B^{\frac{r}{2}}\right)^{\frac{r+\delta}{p+r+\delta}} \geqslant B^{r+\delta} \qquad \text{for all } p > 0 \text{ and } r > 0.$$

Put $A_1 = \left(B^{\frac{r}{2}} A^{p+\delta} B^{\frac{r}{2}}\right)^{\frac{r+\delta}{p+r+\delta}}$ and $B_1 = B^{r+\delta}$, then A_1 and B_1 satisfy $A_1 \geqslant B_1 > 0$ and $M^{r+\delta} I \geqslant B_1 \geqslant m^{r+\delta} I > 0$. Applying Theorem A to A_1 and B_1, we have

$$A_1^{\frac{p+r+\delta}{r+\delta}} + C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta}) I \geqslant B_1^{\frac{p+r+\delta}{r+\delta}}.$$

Therefore we have

$$B^{\frac{r}{2}} A^{p+\delta} B^{\frac{r}{2}} + C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta}) I \geqslant B^{p+r+\delta},$$

so that it follows that

$$A^{p+\delta} + C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta}) B^{-r} \geqslant B^{p+\delta}.$$

Replacing $p+\delta$ by p and $r+\delta$ by r, we have the desired inequality (ii).

Proof of (ii) $\Longrightarrow$ (i): For the case of $\delta > 0$, it follows that $C(m^r, M^r, \frac{p+r-\delta}{r}) \to 0$ as $p \to \delta$, thus we have $A^\delta \geqslant B^\delta$. Also, for the case of $\delta = 0$, it follows from [23] that if we put $r \to 0$, then we have $A^p + L(m^p, M^p)\log M_h(p)I \geqslant B^p$ for all $p > 0$. Therefore it follows from Theorem B that $\log A \geqslant \log B$. □

If we put $\delta = 0$ in Theorem 5-1, then we have the following theorem which is essentially due to Theorem B by Yamazaki:

THEOREM 5-2 (Yamazaki). *Let* $A > 0$ *and* $MI \geqslant B \geqslant mI > 0$. *Then* $\log A \geqslant \log B$ *is equivalent to*

$$A^p + \frac{1}{m^r} C(m^r, M^r, \frac{p+r}{r})I \geqslant B^p \qquad \text{for all } p > 0 \text{ and } r > 0,$$

where $C(m,M,p)$ *is defined in (2).*

If we put $\delta = 1$ in Theorem 5-1, then we have the following characterization of the usual order which is a parallel result to Theorem 5-2.

THEOREM 5-3. *Let* $A > 0$ *and* $MI \geqslant B \geqslant mI > 0$. *Then* $A \geqslant B$ *is equivalent to*

$$A^p + \frac{1}{m^{r-1}} C(m^r, M^r, \frac{p+r-1}{r})I \geqslant B^p \qquad \text{for all } p > 1 \text{ and } r > 1,$$

where $C(m,M,p)$ *is defined in (2).*

Next, Yamazaki [22] showed the following theorem related to Theorem A.

THEOREM Y-1. *If* $A \geqslant B$ *and* $MI \geqslant B \geqslant mI > 0$, *then*

$$A^p + M(M^{p-1} - m^{p-1})I \geqslant A^p + C(m,M,p)I \geqslant B^p \qquad \text{for all } p > 1,$$

where $C(m,M,p)$ *is defined in (2).*

We shall show the following theorem on the chaotic order which is parallel to Theorem Y-1.

THEOREM 6-1. *If* $\log A \geqslant \log B$ *and* $MI \geqslant B \geqslant mI > 0$, *then*

$$A^p + \frac{M}{m}(M^p - m^p)I \geqslant A^p + \frac{1}{m} C(m,M,p+1)I \geqslant B^p \qquad \text{for all } p > 0,$$

where $C(m,M,p)$ *is defined in (2).*

Proof. Since it follows from [24, Theorem 1] and [12] that

$$\left(\frac{M}{m}\right)^p \geqslant K_+(m,M,p+1) \qquad \text{for all } p > 0,$$

we have

$$\frac{1}{m}\frac{mM^{p+1} - Mm^{p+1}}{M-m}\left(\frac{M}{m} - 1\right) \geqslant \frac{1}{m}\frac{mM^{p+1} - Mm^{p+1}}{M-m}\left(K_+(m,M,p+1)^{1/p} - 1\right) \geqslant 0.$$

Therefore we have the first inequality. Also, if we put $\delta = 0$ and $r = 1$ in Theorem 5-1, then we have the second inequality. □

The following theorem is a parametrized order preserving operator inequality which connects Theorem Y-1 and Theorem 6-1.

THEOREM 6-2. *If* $A^\delta \geqslant B^\delta$ *for some* $\delta \in [0,1]$ *and* $MI \geqslant B \geqslant mI > 0$*, then*

$$A^p + \frac{M}{m^{1-\delta}}(M^{p-\delta} - m^{p-\delta})I \geqslant A^p + C(m,M,p+1-\delta)I \geqslant B^p \qquad \text{for all } p > \delta,$$

where $C(m,M,p)$ *is defined in (2).*

Proof. Since it follows from [15, Theorem 3] that

$$\left(\frac{M}{m}\right)^{p-\delta} \geqslant K_+(m,M,p+1-\delta),$$

we can show Theorem 6-2 in the same way as the proof of Theorem 6-1. □

Moreover, Yamazaki [23] showed the following other characterization of the chaotic order.

THEOREM Y-2. *Let* $A > 0$ *and* $MI \geqslant B \geqslant mI > 0$*. Then* $\log A \geqslant \log B$ *is equivalent to*

$$A^p + \frac{(M^p - m^p)^2}{4m^p} I \geqslant B^p \qquad \text{for all } p > 0.$$

By Theorem 5-1, we show the following theorem on the usual order which is parallel to Theorem Y-2.

THEOREM 7-1. *Let* $A > 0$ *and* $MI \geqslant B \geqslant mI > 0$*. If* $A \geqslant B$*, then*

$$A^p + \frac{(M^{p-1} - m^{p-1})^2}{4m^{p-2}} I \geqslant B^p \qquad \text{for all } p > 2.$$

Proof. If we put $\delta = 1$ and $r = p-1 (> 1)$ in Theorem 5-1, then we have $p > 2$ and

$$A^p + \frac{1}{m^{p-2}} C(m^{p-1}, M^{p-1}, 2)I \geqslant B^p \qquad \text{for all } p > 2.$$

By a simple calculation, it follows that

$$A^p + \frac{1}{m^{p-2}} \frac{(M^{p-1} - m^{p-1})^2}{4} I \geqslant B^p \qquad \text{for all } p > 2. \quad \square$$

The following theorem is a parametrized order preserving operator inequality which connects Theorem Y-2 and Theorem 7-1.

THEOREM 7-2. *Let* $A > 0$ *and* $MI \geqslant B \geqslant mI > 0$*. If* $A^\delta \geqslant B^\delta$ *for some* $\delta \in [0,1]$*, then*

$$A^p + \frac{(M^{p-\delta} - m^{p-\delta})^2}{4m^{p-2\delta}} I \geqslant B^p \qquad \text{for all } p > 2\delta.$$

Proof. If we put $r = p - \delta (> \delta)$ in Theorem 5-1, then we have this theorem. □

REMARK 2. Theorem 5-1 interpolates Theorem A and Theorem B by means of the constant $C(m,M,P)$. Let A and B be positive invertible operators and $MI \geqslant B \geqslant mI > 0$. Then the following assertions hold:

(i) $A \geqslant B$ implies $A^p + C(m,M,p)I \geqslant B^p$ for all $p \geqslant 1$.

(ii) $A^\delta \geqslant B^\delta$ implies $A^p + C(m^\delta, M^\delta, \frac{p}{\delta})I \geqslant B^p$ for all $p \geqslant \delta$.

(iii) $\log A \geqslant \log B$ implies $A^p + L(m^p, M^p)\log M_h(p)I \geqslant B^p$ for all $p > 0$.

It follows that the constant of (ii) interpolates the scalar of (i) and (iii) continuously. In fact, if we put $\delta = 1$ in (ii), then we have (i), also if we put $\delta \to 0$ in (ii), then we have

$$\begin{aligned} C(m^\delta, M^\delta, \frac{p}{\delta}) &= \frac{m^\delta M^p - M^\delta m^p}{M^\delta - m^\delta}\{K_+(m^\delta, M^\delta, \frac{p}{\delta})^{\frac{\delta}{p-\delta}} - 1\} \\ &= \frac{\delta}{h^\delta - 1} m^p (h^p - h^\delta) \frac{K_+(m^\delta, M^\delta, \frac{p}{\delta})^{\frac{\delta}{p-\delta}} - 1}{\delta} \\ &\to \frac{1}{\log h}(M^p - m^p)\log M_h(p)^{\frac{1}{p}} \quad (\text{as } \delta \to 0) \\ &= L(m^p, M^p)\log M_h(p), \end{aligned}$$

where $h = \frac{M}{m} > 1$.

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