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GENERAL POWER INEQUALITIES BETWEEN THE SIDES AND THE CIRCUMSCRIBED AND INSCRIBED RADII RELATED TO THE FUNDAMENTAL TRIANGLE INEQUALITY

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Abstract. In this paper we establish the following general triangle inequality between the lengths of its sides α, β, γ, and the circumscribed and inscribed radii R and r, respectively:

$$\alpha^n + \beta^n + \gamma^n \leqslant 2^{n+1}R^n + 2^n\left(3^{1+\frac{n}{2}} - 2^{n+1}\right)r^n \quad \text{for any} \quad n \geqslant 0.$$

This result extends to the general case the results previously known for $n = 1, 2$ established by W. Blundon [2,3]. Our inequality also extends the fundamental triangle inequality.

1. Introduction

Given three positive quantities α, β and γ there exists a triangle ABC having R and r as its radii of the circumscribed and inscribed circles, respectively, and α, β, γ as its sides if and only if the following famous double inequality is satisfied:

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leqslant p^2 \tag{1.1}$$

and

$$2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \geqslant p^2 \tag{1.2}$$

where $2p = \alpha + \beta + \gamma$ is the semiperimeter. The above inequalities are known today as the *fundamental triangle inequality* [1]. This was first established by E. Rouché and later rediscovered, sometimes in different forms, by many authors, see [1, pp. 1-6] for history and more details. In mid 60's W. J. Blundon [2] has reconsidered this result and shown that the above inequalities are the best possible. Given the form of the triangle inequality Blundon was also interested to find the best linear and respective quadratic inequalities in p, R and r. In [2-3] he established that the following forms are the best possible such inequalities in the class of linear and, respectively, quadratic expressions in r and R.

$$3\sqrt{3}r \leqslant p \leqslant 2R + (3\sqrt{3} - 4)r \tag{1.3}$$

$$16Rr - 5r^2 \leqslant p^2 \leqslant 4R^2 + 4Rr + 3r^2 \tag{1.4}$$

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Blundon inequalities (1.3-4) naturally suggest the question of what is the best inequality involving general power with positive exponent between the fundamental triangle elements, i.e. of the form:

$$\alpha^n + \beta^n + \gamma^n \leqslant u(n)R^n + v(n)r^n \tag{1.5}$$

with $u(n)$ and $v(n)$ to be determined.

In this paper we solve problem (1.5). The method of proof is, however, more general and it may be applicable to other situations. It is related to a recent paper where we showed that a large class of geometric inequalities could be resolved by reducing the demonstration to the case of an isosceles triangle [4]. This idea was extended in the form of a principle of proof called "the universal principle of the isosceles triangle" (or PIT) [5]. The idea of [5] is to use the structural symmetry embodied in the definition of the major elements of a given triangle (radii, semiperimeter, area, bisectors, medians, altitudes, etc.). It was noted that the vast majority of the inequalities constructed with these elements are such that their functional form has a bounded variation in the set of all triangles with the property that their extremum values will be attained in the set of the isosceles triangles. For any such case it is therefore clear that the problem is reduced from the study of the variation of the given function depending on the triangle elements (three variables) to a function of only one variable (e.g. one angle only) and as such is usually an elementary problem. In [5] as a practical application of PIT we established in a simple way the celebrated "fundamental inequality of a triangle" (1.1-2).

The problem (1.5) in the present paper is however, more difficult than the question considered in [5]. This is because of the presence, in addition of the triangle elements, of the parameter $n \geqslant 0$ that range in a continuous, unbounded interval.

2. A general class of inequalities in a triangle related to the fundamental triangle inequality

Here we propose to establish the most general inequality of the form

$$\alpha^n + \beta^n + \gamma^n \leqslant u(n)R^n + v(n)r^n \tag{2.1}$$

where α, β and γ are the lengths of the triangle's sides and r, R are the radius of the inscribed and circumscribed circles, respectively. We consider n to be a positive real number. As mentioned in the introduction inequality (2.1) represents the generalisation of a series of known results in literature as follows. The case $n = 1$ has been studied by W. J. Blundon in [3]. He proved that $u(1) = 4$, $v(1) = 6\sqrt{3} - 8$ are the best constants for this case giving that

$$p \leqslant (3\sqrt{3} - 4)r + 2R \tag{2.2}$$

where p is triangle's semiperimeter. (2.2) is also mentioned in [1, page 4]. The case $n = 2$ has also been studied by Blundon [2]. In this case it was shown that

$$u(2) = 8, \quad v(2) = 4 \tag{2.3}$$

Blundon studied the cases $n = 1$ and $n = 2$ motivated to find out the best linear and, respectively, quadratic inequalities between the sides and the two radii R and r. Many other mathematicians have considered the above questions: M. S. Klamkin, O. Bottema, D. S. Mitrinović, P. Erdos, to cite just a few [1, pp 23-25]. Here we shall extend the above results to the general case.

THEOREM. *Let $n \geqslant 0$ be a real number. Then the best inequality (2.1) valid for any triangle ABC with sides α, β and γ is given by*

$$u(n) = 2^{1+n} \quad \text{and} \quad v(n) = 2^n\left(3^{1+\frac{n}{2}} - 2^{1+n}\right) \tag{2.4}$$

Proof. For a given euclidean triangle ABC denote by a, b, c the measures of its angles in radians chosen so that

$$0 \leqslant c \leqslant b \leqslant a \leqslant \pi \quad \text{and} \quad a+b+c = \pi. \tag{2.5}$$

Before entering the details of the calculations we shall present the strategy of the proof. First we aim to reduce the inequality (2.1) with the constants (2.4) from the case of a general triangle to the one of an isosceles triangle. In the second part we shall establish that (2.1) with (2.4) holds true for the particular case of an isosceles triangle.

Note that inequality (2.1) becomes an equality for $n = 0$. In what follows we shall assume that $n > 0$. Consider $T = \{(a,b,c) | a,b,c \geqslant 0, a+b+c = \pi\} \subset \mathbf{R}^3$ and let $g = g(a,b,c)$ defined by

$$\begin{aligned} g(a,b,c) &= \sin^n(a) + \sin^n(b) + \sin^n(c) - 2^{-n}u(n) \\ &- 2^n v(n)(\sin(a/2)\sin(b/2)\sin(c/2))^n \end{aligned} \tag{2.6}$$

g represents the difference between the left and right–hand side members in (2.1) after using the law of sines and the following known relation for r and R

$$r = 4R\sin(a/2)\sin(b/2)\sin(c/2) \tag{2.7}$$

Therefore the task is to establish that

$$g(a,b,c) \leqslant 0 \tag{2.8}$$

for all $n > 0$ and for all $a \geqslant b \geqslant c \geqslant 0$ such that $a+b+c = \pi$.

2.1. Reduction of the general case to the case of the isosceles triangle

In this part we shall show that any local extremum point of g is attained in the set of the isosceles triangles. Clearly at any extremum point $p_0 = (a_0, b_0, c_0)$ we have the relations

$$\frac{\partial g}{\partial a}_{p_0} = \frac{\partial g}{\partial b}_{p_0} = \frac{\partial g}{\partial c}_{p_0} = 0 \tag{2.9}$$

By performing the calculations for $\dfrac{\partial g}{\partial a}_{p_0}$ in (2.6) we find that the following conditions must hold simultaneously

$$\frac{\sin^n(a)\cos(a)}{\sin(a)} = \frac{2^{n-1}v(n)}{\sin(a/2)}(\sin(a/2)\sin(b/2)\sin(c/2))^n\cos(a/2) \tag{2.10}$$

for $a \neq 0$ and $a \neq \pi$. There are two other similar relations for b and c found by circular symmetry from $\dfrac{\partial g}{\partial b}_{p_0} = \dfrac{\partial g}{\partial c}_{p_0} = 0$, respectively. Note that if $a = 0$ then from the assumed ordering it would give $a = b = c = 0$ and (2.8) follows trivially. Also if $a = \pi$ then $b = c = 0$ and again (2.8) is trivially satisfied. In the following we shall assume that $\pi > a \geqslant b \geqslant c > 0$. Note that from the system of relations (2.10) we find that at p_0 we have the necessary conditions

$$\frac{\sin^n(a)\cos(a)}{1+\cos(a)} = \frac{\sin^n(b)\cos(b)}{1+\cos(b)} = \frac{\sin^n(c)\cos(c)}{1+\cos(c)} \tag{2.11}$$

Now consider the function

$$f(x) = \frac{\sin^n(x)\cos(x)}{1+\cos(x)} \tag{2.12}$$

defined for $0 < x < \pi$ and for $n > 0$. It is straightforward to check the following properties for f:

$$\text{if} \quad n \geqslant 2 : f(x) = 0 \iff x = 0, \quad \frac{\pi}{2}, \quad \pi \tag{2.13.1}$$

$$\text{if} \quad 0 < n < 2 : f(x) = 0 \iff x = 0, \quad \frac{\pi}{2} \tag{2.13.2}$$

$$f > 0 \quad \text{on} \quad 0 < x < \frac{\pi}{2} \quad \text{and} \quad f < 0 \quad \text{on} \quad \frac{\pi}{2} < x < \pi \tag{2.14}$$

For any $n \geqslant 2$ f has a single maximum in the interval $\left(0, \dfrac{\pi}{2}\right)$ at x_1^n such that $\cos(x_1^n) = \dfrac{-\sqrt{4n+1}-1}{2n}$. If $0 < n < 2$ then f still has a single maximum in $\left(0, \dfrac{\pi}{2}\right)$ at x_1^n but now is monotonic decreasing in the range $\left(\dfrac{\pi}{2}, \pi\right)$ with $f \to -\infty$ as $x \to \pi^-$.

In figure 1a there is a plot of the behaviour of f for $n = 4$ whereas figure 1b gives the plot of f for $n = 1$. The properties of f show that any line parallel to the x–axis intersects the graph of f in at most two points. As we have three solutions from (2.11) it necessarily follows that at least two values, e.g. a, b are equal. This therefore establishes that any extremum value taken by g in (2.6) is attained only in the set of the isosceles triangles.

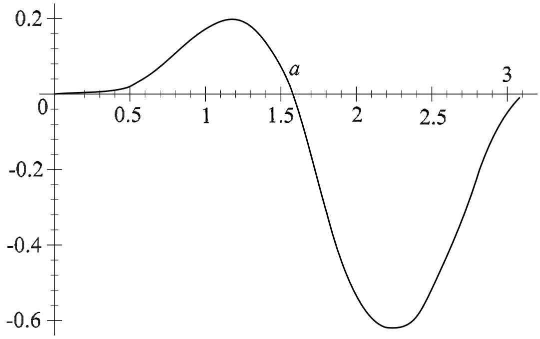

Figure 1a: a plot of f for $n=4$. Note that $f(0)=f\left(\frac{\pi}{2}\right)=f(\pi)=0$. f has a single maximum in $\left(0,\frac{\pi}{2}\right)$ and a single minimum in $\left(\frac{\pi}{2},\pi\right)$.

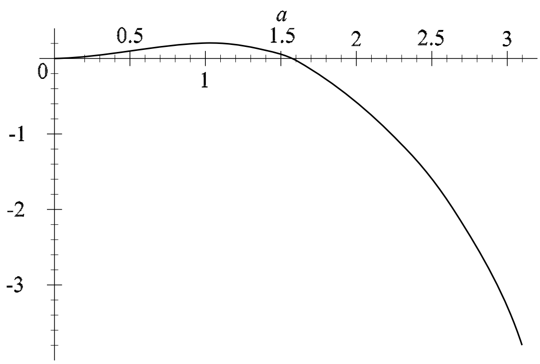

Figure 1b: a plot of f for $n=1$. Note that $f(0)=f\left(\frac{\pi}{2}\right)=0$ and that $\lim\limits_{a\to\pi^-} f=-\infty$. f has a single maximum in $\left(0,\frac{\pi}{2}\right)$.

2.2. Inequality (2.8) for the case of an isosceles triangle

From the first step it follows that in order to establish relation (2.8) it is sufficient to consider only the restriction of the inequality for the case of an isosceles triangle. Therefore we take ABC to be such that the angles satisfy $a=b=t$, $c=\pi-2t$, $t\geqslant\frac{\pi}{3}$. As g is continuous and bounded it will attain its maximum at some point t_0^n. The analysis done at the first step has shown that it is necessary to have that $0<t_o^n<\frac{\pi}{2}$ (otherwise it would allow for two angles greater than $\frac{\pi}{2}$ which is impossible).

We now make the crucial observation that from (2.11) it necessary follows that at t_0^n we have

$$f(t_0^n)=f(\pi-2t_0^n) \tag{2.15}$$

Furthermore the symmetry in the relations (2.10) and (2.15) imply that this is possible only if t_0^n is also the maximum point for the triangle $A_1B_1C_1$ with the angles $(\pi - 2, \pi - 2t, 4t - \pi)$. Thus by evaluating (2.10) for ABC and for $A_1B_1C_1$ we get the condition

$$\sin(a/2)\sin(b/2)\sin(c/2)_{|ABC,t=t_0^n} = \sin(a/2)\sin(b/2)\sin(c/2)_{|A_1B_1C_1,t=\pi-2t_0^n} \quad (2.16)$$

Solving (2.16) then gives that

$$t_0^n = \frac{\pi}{3} \quad \text{or that} \quad t_0^n = \frac{\pi}{2} \qquad (2.17)$$

Therefore we have established that if t_0^n is a point of maximum for g then its value is necessarily taken from the values given by (2.17). It now remains to verify that indeed $t_0^n = \frac{\pi}{3}, \frac{\pi}{2}$ are points of maximum. First it is straightforward that for $u(n)$, $v(n)$ given at (2.4) we have

$$g\left(\frac{\pi}{3}\right) = g\left(\frac{\pi}{2}\right) = 0 \qquad (2.18)$$

for all $n \geqslant 0$. Therefore to finish the proof it suffices to show that at least one of the points at (2.17) is the local maximum of g and therefore is also its global maximum. To do so take $t_1 = \frac{\pi}{3}$. Consider the function

$$h(t) = g(t,t,\pi-2t) = 2\sin(t)^n + \sin(2t)^n - 2^{-n}u(n) - 2v(n)(\sin(t/2)^2\cos(t))^n \quad (2.19)$$

A simple calculation gives that

$$\frac{\partial^2 h}{\partial t^2}_{|t=\frac{\pi}{3}} = n(n^2(1-n)3^{n/2} - 2^{3-n}3^{n/2} + 2^{1-n}3^{2+n/2} - 12) \qquad (2.20)$$

This expression has a maximum for $n = n_1 = -\dfrac{2 + 5\ln(3/4)}{\ln(3/4)} \approx 1.952$. Its value is -1.499. Therefore $\dfrac{\partial^2 h}{\partial t^2}_{|t=\frac{\pi}{3}} \leqslant -1.499 < 0$ which establishes that $t_1 = \dfrac{\pi}{3}$ is a global maximum for g and concludes the proof of the inequality (2.8).

Finally it remains to show that $u(n)$, $v(n)$ are the best constants for the inequality (2.8). This can be established as follows. Take first a (degenerate) triangle ABC with the angles such that $a = b = \frac{\pi}{2}$, $c = 0$. Inserting in (2.6) this gives that

$$u(n) \geqslant 2^{1+n} \qquad (2.21)$$

Therefore the best constant (i.e. the smallest) $u(n)$ is obtained when $u(n) = 2^{1+n}$. Now take ABC to be equilateral $\left(a = b = c = \frac{\pi}{3}\right)$. From (2.6) and the above we obtain

$$v(n) \geqslant 2^n\left(3^{1+\frac{n}{2}} - 2^{1+n}\right) \qquad (2.22)$$

Therefore the best constants are those stated in the Theorem and the proof ends.

COROLLARY. *For* $0 < n \leqslant 1$ *the function* $\sin^n(x)$ *is concave. Hence an application of the Jensen's inequality gives*

$$\sin^n(a) + \sin^n(b) + \sin^n(c) \leqslant 3^{1+n/2} 2^{-n} \tag{2.23}$$

However, in this range we obtain a sharper inequality from our theorem

$$\sin^n(a) + \sin^n(b) + \sin^n(c) \leqslant 2 + (3^{1+n/2} - 2^{1+n}) \left(\frac{r}{R}\right)^n \tag{2.24}$$

because of the Euler inequality $R \geqslant 2r$. *Furthermore this improvement holds true for all* $0 \leqslant n \leqslant n_+ = -1 + \dfrac{1}{\log_3 4 - 1} \approx 2.82$.

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