Mathematical
Inequalities
& Applications
Volume 5, Number 4 (2002), 755–763

THE MEAN INEQUALITY OF RANDOM VARIABLES

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(*communicated by S. Puntanen*)

Abstract. In this paper, we prove an inequality about random variables. The inequality extends and compliments some existing results in the literature on Kantorovich type inequalities.

1. Introduction

The theory of means and their inequalities are very basic and important in many fields including mathematics, statistics, physics, and economics. Motivated by different concerns, there are numerous ways to introduce mean values. In probability and statistics, the most commonly used mean is expectation. In this paper, we define the other means of random variable, which are arithmetic mean and geometric mean of random variable. We discuss the relationship among the three kinds of means of random variable and prove a new mean inequality of random variables. As its application, we point out several useful inequalities are special cases of the mean inequality of random variables.

2. A new mean inequality of random variable

DEFINITION 1. We say that $\inf_x\{x:\ P(\xi \leqslant x) = 1\}$ and $\sup_x\{x:\ P(\xi \geqslant x) = 1\}$ are respectively the supremum and infimum of random variable ξ. We simply express $\inf_x\{x:\ P(\xi \leqslant x) = 1\}$ and $\sup_x\{x:\ P(\xi \geqslant x) = 1\}$ as $\sup \xi$ and $\inf \xi$.

If ξ is bounded random variable, the $\sup \xi$ and $\inf \xi$ are finite valued and sole.

DEFINITION 2. If ξ is bounded, we define the arithmetic mean $A(\xi)$ of the random variable ξ by

$$A(\xi) = \frac{\sup \xi + \inf \xi}{2}.$$

If $\inf \xi \geqslant 0$ as well, we define the geometric mean $G(\xi)$ of the random variable ξ by

$$G(\xi) = \sqrt{\sup \xi \cdot \inf \xi}.$$

Mathematics subject classification (2000): Primary 60E15, 26D20.

Key words and phrases: inequality, arithmetic mean, geometric mean, expectation.

The author was supported by SF of Jiangsu Institute of Petrochemical Technology.

DEFINITION 3. If both ξ and η are bounded random variable, we define the independent arithmetic mean $\overline{A}(\xi\eta)$ of the product of two random variables ξ and η by

$$\overline{A}(\xi\eta) = \frac{\sup\xi \cdot \sup\eta + \inf\xi \cdot \inf\eta}{2}.$$

It is easy to see that if ξ and η are independent, the independent arithmetic mean of the product of two random variables equals to the arithmetic mean of the product of two random variables. That is $\overline{A}(\xi\eta) = A(\xi\eta)$.

DEFINITION 4. If both ξ and η are bounded random variable; $\inf\xi \geqslant 0$, $\inf\eta \geqslant 0$, we define the independent geometric mean $\overline{G}(\xi\eta)$ of the product of two random variables ξ and η by

$$\overline{G}(\xi\eta) = \sqrt{\sup\xi \cdot \inf\xi \cdot \sup\eta \cdot \inf\eta}.$$

It is easy to see that if ξ and η are independent, the independent geometric mean of the product of two random variables equals to the geometric mean of the product of two random variables. That is $\overline{G}(\xi\eta) = G(\xi\eta)$.

EXAMPLE 1. Suppose $\sup\xi = 5$, $\inf\xi = 3$, $\eta = \frac{1}{\xi}$ then

$$\overline{A}\left(\frac{1}{\xi}\xi\right) = \frac{\sup\frac{1}{\xi} \cdot \sup\xi + \inf\frac{1}{\xi} \cdot \inf\xi}{2} = \frac{\frac{1}{3} \cdot 5 + \frac{1}{5} \cdot 3}{2} = \frac{17}{15},$$

$$A\left(\frac{1}{\xi}\xi\right) = A(1) = \frac{\sup 1 + \inf 1}{2} = 1,$$

$$\overline{G}\left(\frac{1}{\xi}\xi\right) = \sqrt{\sup\frac{1}{\xi} \cdot \inf\frac{1}{\xi} \cdot \sup\xi \cdot \inf\xi} = \sqrt{\frac{1}{3} \cdot \frac{1}{5} \cdot 3 \cdot 5} = 1,$$

$$G\left(\frac{1}{\xi}\xi\right) = G(1) = \sqrt{\sup 1 \cdot \inf 1} = 1.$$

EXAMPLE 2. Suppose the joint probability distribution of (ξ, η) is

$$P\{(\xi,\eta) = (1,2)\} = \frac{1}{7}, \quad P\{(\xi,\eta) = (1,3)\} = \frac{1}{7},$$

$$P\{(\xi,\eta) = (2,1)\} = \frac{1}{7}, \quad P\{(\xi,\eta) = (2,2)\} = \frac{1}{7}, \quad P\{(\xi,\eta) = (2,3)\} = \frac{1}{7},$$

$$P\{(\xi,\eta) = (4,1)\} = \frac{1}{7}, \quad P\{(\xi,\eta) = (4,2)\} = \frac{1}{7},$$

then

$$\overline{A}(\xi\eta) = \frac{\sup\xi \cdot \sup\eta + \inf\xi \cdot \inf\eta}{2} = \frac{4 \cdot 3 + 1 \cdot 1}{2} = \frac{13}{2},$$

$$A(\xi\eta) = \frac{\sup(\xi\eta) + \inf(\xi\eta)}{2} = \frac{8+2}{2} = 5,$$

$$\overline{G}(\xi\eta) = \sqrt{\sup\xi \cdot \inf\xi \cdot \sup\eta \cdot \inf\eta} = \sqrt{4\cdot 1\cdot 3\cdot 1} = \sqrt{12},$$
$$G(\xi\eta) = \sqrt{\sup(\xi\eta)\cdot\inf(\xi\eta)} = \sqrt{8\cdot 2} = \sqrt{16} = 4.$$

LEMMA 2.1. *Let ξ be bounded random variable, $M = \sup\xi$, $m = \inf\xi$, $f(x)$ be strictly increasing function on $[m,M]$. Then the infimum and supremum of random variable $f(\xi)$ are respectively $f(m)$ and $f(M)$, if $f(x)$ be strictly decreasing function on $[m,M]$. Then the infimum and supremum of random variable $f(\xi)$ are respectively $f(M)$ and $f(m)$.*

Proof. We prove $f(M)$ is the supremum of random variable $f(\xi)$ only while $f(x)$ is strictly increasing function on $[m,M]$. Others are similar. Because $M = \inf_x\{x : P(\xi \leqslant x) = 1\}$ and $P(\xi \leqslant x)$ is a right-continuous function for all $x \in \mathbf{R}$, we have $M \in \{x:\ P(\xi \leqslant x) = 1\}$. According to the definition of M, we conclude that

$$P(\xi \leqslant M) = 1, \quad P(\xi \leqslant M') < 1 \quad \text{for all} \quad M' < M.$$

Thus,

$$P(f(\xi) \leqslant f(M)) = 1, \quad P(f(\xi) \leqslant f(M')) < 1 \quad \text{for all} \quad M' < M.$$

That is $\sup f(\xi) = f(M)$.

COROLLARY. *Let ξ be bounded random variable, $M = \sup\xi$, $m = \inf\xi$, $f(x)$ be strictly monotone function on $[m,M]$. Then $A(f(\xi)) = \dfrac{f(M)+f(m)}{2}$.*

Under the condition $f(x) \geqslant 0$, $x \in [m,M]$, we have $G(f(\xi)) = \sqrt{f(M)f(m)}$.

Now we give the main result of this paper.

THEOREM 2.1. *Let ξ and η be bounded random variable. If $\inf\xi > 0$ and $\inf\eta > 0$, then*

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leqslant \frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}. \tag{1}$$

Equality holds if and only if

$$P\left\{\left(\frac{\xi}{\eta} = \frac{a}{B}\right) \cup \left(\frac{\xi}{\eta} = \frac{A}{b}\right)\right\} = 1 \quad \text{and} \quad G(\eta^2)E\xi^2 = G(\xi^2)E\eta^2, \tag{2}$$

where $A = \sup\xi$, $B = \sup\eta$, $a = \inf\xi$, $b = \inf\eta$.

Proof. Obviously, under the condition of the theorem, a, A, b, B are all finite numbers. Because of $P\left\{\left(\frac{\xi}{\eta} - \frac{a}{B}\right)\left(\frac{\eta}{\xi} - \frac{b}{A}\right) \geqslant 0\right\} = 1$, so $P\{(B\xi - a\eta)(A\eta - b\xi) \geqslant 0\} = 1$, therefor $E(B\xi - a\eta)(A\eta - b\xi) \geqslant 0$. Expansion, we obtain

$$(BA + ab)E(\xi\eta) \geqslant BbE\xi^2 + AaE\eta^2 \geqslant 2(BbAaE\xi^2E\eta^2)^{\frac{1}{2}}.$$

That is

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leqslant \frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}.$$

Now we prove the equality holds $\iff$ $P\left\{\left(\frac{\xi}{\eta}=\frac{a}{B}\right)\cup\left(\frac{\xi}{\eta}=\frac{A}{b}\right)\right\}=1$, and $G(\eta^2)E\xi^2=G(\xi^2)E\eta^2$.

($\Leftarrow$) Suppose $P\left\{\left(\frac{\xi}{\eta}=\frac{a}{B}\right)\cup\left(\frac{\xi}{\eta}=\frac{A}{b}\right)\right\}=1$ and $G(\eta^2)E\xi^2=G(\xi^2)E\eta^2$.

Since $P\left\{\left(\frac{\xi}{\eta}=\frac{a}{B}\right)\cup\left(\frac{\xi}{\eta}=\frac{A}{b}\right)\right\}=1$, it follows that,

$$E(B\xi-a\eta)(A\eta-b\xi)=0. \tag{3}$$

From $G(\eta^2)E\xi^2=G(\xi^2)E\eta^2$, we obtain

$$G(\eta^2)E\xi^2+G(\xi^2)E\eta^2=2\left[G(\eta^2)E\xi^2G(\xi^2)E\eta^2\right]^{\frac{1}{2}}.$$

That is

$$BbE\xi^2+AaE\eta^2=2(BbAaE\xi^2E\eta^2)^{\frac{1}{2}}. \tag{4}$$

(3) and (4) follow that

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} = \frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}.$$

($\Rightarrow$) Next we suppose

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} = \frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}.$$

First we notice that $P\{(B\xi-a\eta)(A\eta-b\xi)\geqslant 0\}=1$. According to the proof of the inequality, we have

$$BbE\xi^2+AaE\eta^2=2(BbAaE\xi^2E\eta^2)^{\frac{1}{2}} \quad \text{and} \quad E(B\xi-a\eta)(A\eta-b\xi)=0.$$

That is

$$G(\eta^2)E\xi^2=G(\xi^2)E\eta^2 \quad \text{and} \quad P\{(B\xi-a\eta)(A\eta-b\xi)=0\}=1.$$

So

$$P\left\{\left(\frac{\xi}{\eta}=\frac{a}{B}\right)\cup\left(\frac{\xi}{\eta}=\frac{A}{b}\right)\right\}=1 \quad \text{and} \quad G(\eta^2)E\xi^2=G(\xi^2)E\eta^2.$$

COROLLARY 1. *Let* ξ *be bounded random variable. If* $\inf\xi>0$, *then*

$$\frac{E\xi^2}{E^2(\xi)} \leqslant \frac{A^2(\xi)}{G^2(\xi)}. \tag{5}$$

Equality holds if and only if $P\{(\xi=a)\cup(\xi=A)\}=1$ *and* $E\xi^2=G(\xi^2)$, *where* $A=\sup\xi$, $a=\inf\xi$.

Proof. Obviously, under the condition of the corollary, a, A are all finite numbers. Let $\eta=1$ in (1) and (2), we have the result.

COROLLARY 2. *Let* ξ *be bounded random variable and* $\inf \xi > 0$. *Then we have*

$$\frac{\operatorname{var} \xi}{E^2(\xi)} \leqslant \frac{A^2(\xi) - G^2(\xi)}{G^2(\xi)}.$$

Equality holds if and only if $P\{(\xi = a) \cup (\xi = A)\} = 1$ *and* $E\xi^2 = G(\xi^2)$.

Proof. Both sides of the inequality (5) minus 1 and notice $\operatorname{var} \xi = E\xi^2 - E^2\xi$, so we can have the results.

COROLLARY 3. *Under the condition of the theorem 2.1, if* ξ *and* η *are independent, we have inequality*

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leqslant \frac{A^2(\xi\eta)}{G^2(\xi\eta)}.$$

Equality holds if and only if $P\left\{\left(\frac{\xi}{\eta} = \frac{a}{B}\right) \cup \left(\frac{\xi}{\eta} = \frac{A}{b}\right)\right\} = 1$ *and* $G(\eta^2)E\xi^2 = G(\xi^2)E\eta^2$.

Proof. If ξ and η are independent, we have $\overline{A}(\xi\eta) = A(\xi\eta)$ and $\overline{G}(\xi\eta) = G(\xi\eta)$. The result is obvious.

3. Some applications of the new inequality

Now we use the mean inequality of random variable to prove some inequalities. First we give a lemma.

LEMMA 3.1. *If* $\inf \xi > 0$, *let* $M' < +\infty$ *and* $m' > 0$ *be respectively the upper bound and lower bound of random variable* ξ. *That is* $P(\xi \leqslant M') = 1$ *and* $P(\xi \geqslant m') = 1$. *Then*

$$\frac{A(\xi)}{G(\xi)} \leqslant \frac{\frac{m'+M'}{2}}{\sqrt{m'M'}}.$$

Proof. Because of

$$f(m', M') = \frac{\frac{m'+M'}{2}}{\sqrt{m'M'}} = \frac{1}{2}\left(\sqrt{\frac{m'}{M'}} + \sqrt{\frac{M'}{m'}}\right),$$

let $t = \sqrt{\frac{m'}{M'}}$, then $0 < t \leqslant 1$ and $f(m', M') = g(t) = \frac{1}{2}\left(t + \frac{1}{t}\right)$; $g'(t) = \frac{1}{2}\left(1 - \frac{1}{t^2}\right)$. So $g(t)$ is strictly decreasing function on $(0, 1]$. For the maximum value of t is $\sqrt{\frac{\inf \xi}{\sup \xi}}$, we have $g\left(\sqrt{\frac{\inf \xi}{\sup \xi}}\right) \leqslant g(t)$. That is $\frac{A(\xi)}{G(\xi)} \leqslant \frac{\frac{m'+M'}{2}}{\sqrt{m'M'}}$.

THEOREM 3.1. **(Kantorovich inequalities [1])**. *Let A be a positive Hermitian matrix, λ_1 and λ_n are respectively the maximum eigenvalue and minimum eigenvalue of A. Then, for all vector $x \neq 0$*

$$\frac{x^*Axx^*A^{-1}x}{(x^*x)^2} \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}. \tag{6}$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_k \neq \lambda_{k-1}$, $\lambda_n = \lambda_{n-1} = \cdots = \lambda_s \neq \lambda_{s-1}$, equality holds if and only if

$$x = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k + d_n\phi_n + d_{n-1}\phi_{n-1} + \cdots + d_s\phi_s,$$

where $\varphi_1, \cdots, \varphi_k$ are unit orthonormal eigenvectors corresponding to eigenvalue λ_1 and $\phi_n, \cdots, \phi_s$ are orthonormal eigenvectors corresponding to eigenvalue λ_n. $c_1, \cdots, c_k$ and $d_n, \cdots, d_s$ are any real numbers which satisfy $\sum\limits_{i=1}^{k} c_i^2 = \frac{1}{2}$, $\sum\limits_{i=s}^{n} d_i^2 = \frac{1}{2}$.

Proof. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be eigenvalues of A. $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. There is a Hermitian matrix U that satisfies $A = U^*\Lambda U$. Let $y = Ux$, $p_i = \dfrac{|y_i|^2}{\sum\limits_{i=1}^{n}|y_i|^2}$, $i = 1, \cdots, n$. Then

$$\frac{x^*Axx^*A^{-1}x}{(x^*x)^2} = \frac{x^*U^*\Lambda Uxx^*U^*\Lambda^{-1}Ux}{(x^*U^*Ux)^2} = \frac{y^*\Lambda yy^*\Lambda^{-1}y}{(y^*y)^2} = \left(\sum_{i=1}^{n}\lambda_i p_i\right)\left(\sum_{i=1}^{n}\lambda_i^{-1}p_i\right).$$

The problem transfers to proof

$$\left(\sum_{i=1}^{n}\lambda_i p_i\right)\left(\sum_{i=1}^{n}\lambda_i^{-1}p_i\right) \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}, \tag{7}$$

for all $p_i \geqslant 0$, $\sum\limits_{i=1}^{n} p_i = 1$.

Let $l = \max\{i : p_i > 0\}$, $m = \min\{i : p_i > 0\}$.

We define random variable ζ, $p(\zeta = \lambda_i) = p_i$, $i = 1, \cdots, n$. Suppose $\xi = \zeta^{\frac{1}{2}}$, $\eta = \zeta^{-\frac{1}{2}}$. Notice that λ_1 and λ_n are the upper bound and lower bound of random variable ζ and $\overline{A}(\xi\eta) = \dfrac{\lambda_l^{\frac{1}{2}}\lambda_m^{-\frac{1}{2}} + \lambda_m^{\frac{1}{2}}\lambda_l^{-\frac{1}{2}}}{2}$, $\overline{G}(\xi\eta) = \sqrt{\lambda_l^{\frac{1}{2}}\lambda_m^{\frac{1}{2}}\lambda_m^{-\frac{1}{2}}\lambda_l^{-\frac{1}{2}}} = 1$. According to the lemma, we have

$$\frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)} = \left(\frac{\lambda_l^{\frac{1}{2}}\lambda_m^{-\frac{1}{2}} + \lambda_m^{\frac{1}{2}}\lambda_l^{-\frac{1}{2}}}{2}\right)^2 = \frac{(\lambda_l+\lambda_m)^2}{4\lambda_l\lambda_m} \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}.$$

Then the inequality (6) can be expressed by

$$\frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}.$$

This is the result of the theorem 2.1. Now we prove the equality holds $\Longleftrightarrow \frac{x}{||x||} = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k + d_n\phi_n + d_{n-1}\phi_{n-1} + \cdots + d_s\phi_s$, where $\varphi_1, \cdots, \varphi_k$ are unit orthonormal eigenvectors corresponding to eigenvalue λ_1 and $\phi_n, \cdots, \phi_s$ are unit orthonormal eigenvectors corresponding to eigenvalue λ_n. $c_1, \cdots, c_k$ and $d_n, \cdots, d_s$ are any real numbers which satisfy $\sum_{i=1}^{k} c_i^2 = \frac{1}{2}$, $\sum_{i=s}^{n} d_i^2 = \frac{1}{2}$.

($\Leftarrow$) Suppose $\frac{x}{||x||} = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k + d_n\phi_n + d_{n-1}\phi_{n-1} + \cdots + d_s\phi_s$.

Under the conditions, the distribution of random variable ζ is $p(\zeta = \lambda_1) = \frac{1}{2}$, $p(\zeta = \lambda_n) = \frac{1}{2}$. Let $\xi = \zeta^{\frac{1}{2}}$, $\eta = \zeta^{-\frac{1}{2}}$, we have

$$\frac{x^*Axx^*A^{-1}x}{(x^*x)^2} = \frac{E\xi^2 \cdot E\eta^2}{E^2(\xi\eta)} = \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}.$$

($\Rightarrow$) Next, we suppose $\frac{x^*Axx^*A^{-1}x}{(x^*x)^2} = \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}$.

According to the proof of the theorem 2.1, we have

$$E\left(\lambda_n^{-\frac{1}{2}}\zeta^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\right)\left(\lambda_1^{\frac{1}{2}}\zeta^{-\frac{1}{2}} - \lambda_1^{-\frac{1}{2}}\zeta^{\frac{1}{2}}\right) = 0, \quad \lambda_n^{-\frac{1}{2}}\lambda_1^{-\frac{1}{2}}E\zeta = \lambda_n^{\frac{1}{2}}\lambda_1^{\frac{1}{2}}E\zeta^{-1}.$$

Notice $P\left\{\left(\lambda_n^{-\frac{1}{2}}\zeta^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\right)\left(\lambda_1^{\frac{1}{2}}\zeta^{-\frac{1}{2}} - \lambda_1^{-\frac{1}{2}}\zeta^{\frac{1}{2}}\right) \geqslant 0\right\} = 1$. So $p_i = 0$, ($i = k+1, \cdots, s-1$). That is $\frac{x}{||x||} = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k + d_n\phi_n + d_{n-1}\phi_{n-1} + \cdots + d_s\phi_s$, where $\varphi_1, \cdots, \varphi_k$ are unit orthonormal eigenvectors corresponding to eigenvalue λ_1 and $\phi_n, \cdots, \phi_s$ are unit orthonormal eigenvectors corresponding to eigenvalue λ_n. Suppose $\sum_{i=1}^{k} c_i^2 = w^2$, $\sum_{i=s}^{n} d_i^2 = 1 - w^2$.

According to $\lambda_n^{-\frac{1}{2}}\lambda_1^{-\frac{1}{2}}E\zeta = \lambda_n^{\frac{1}{2}}\lambda_1^{\frac{1}{2}}E\zeta^{-1}$. We obtain $w^2 = \frac{1}{2}$.

In particular, if $x = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}$, the equalities hold, where φ_1 and φ_2 are unit orthonormal eigenvalues corresponding to eigenvalue λ_1 and λ_n. This is corresponding to $c_1 = \frac{1}{\sqrt{2}}$, $c_i = 0$, ($i = 2, \cdots, k$) and $d_n = \frac{1}{\sqrt{2}}$, $d_j = 0$, ($j = n-1, \cdots, s$).

THEOREM 3.2. **(Greub-Rheinboldt inequality).** *Suppose A and B are tow positive Hermitian matrices, $AB = BA$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \cdots \geqslant \mu_n$ denote the eigenvalues of A and B, respectively. Then, for all vector $x \neq 0$*

$$\frac{x^*A^2xx^*B^2x}{(x^*ABx)^2} \leqslant \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}.$$

Proof. Since $AB = BA$, there is a Hermitian matrix U that satisfies $A = U^*\Lambda U$ and $B = U^*MU$, where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ and $M = \mathrm{diag}(\mu_1, \cdots, \mu_n)$. Let $y = Ux$, $p_i = \frac{|y_i|^2}{\sum_{i=1}^{n} |y_i|^2}$, $i = 1, \cdots, n$. We define the tow random variables by the joint probability distribution of (ξ, η) as follows

$$p\{(\xi,\eta)=(\lambda_i,\mu_j)\}=\begin{cases} p_i, & \text{when} \quad i=j \\ 0, & \text{when} \quad i\neq j \end{cases}.$$

Then the probability distribution of ξ is $P(\xi=\lambda_i)=p_i$, $(i=1,\cdots,n)$, and the probability distribution of η is $P(\eta=\mu_i)=p_i$, $(i=1,\cdots,n)$.

Let $l=\max\{i:p_i>0\}$, $m=\min\{i:p_i>0\}$. Notice that λ_1 and λ_n are the upper bound and lower bound of random variable ξ; μ_1 and μ_n are the upper bound and lower bound of random variable η. We have

$$E\xi^2=\sum_{i=1}^{n}\lambda_i^2 p_i, \quad E\eta^2=\sum_{i=1}^{n}\mu_i^2 p_i, \quad E(\xi\eta)=\sum_{i=1}^{n}\lambda_i\mu_i p_i;$$

$$\overline{A}(\xi\eta)=\frac{\lambda_l\mu_l+\lambda_m\mu_m}{2}, \quad \overline{G}(\xi\eta)=\sqrt{\lambda_l\mu_l\lambda_m\mu_m};$$

$$\frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}=\frac{\left(\frac{\lambda_l\mu_l+\lambda_m\mu_m}{2}\right)^2}{\left(\sqrt{\lambda_l\mu_l\lambda_m\mu_m}\right)^2}=\frac{(\lambda_l\mu_l+\lambda_m\mu_m)^2}{4\lambda_l\lambda_l\mu_m\mu_m}\leqslant\frac{(\lambda_1\mu_1+\lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}.$$

We have

$$\frac{x^*A^2xx^*B^2x}{(x^*ABx)^2}=\frac{x^*U^*\Lambda^2Uxx^*U^*M^2Ux}{(x^*U^*\Lambda UU^*MUx)^2}=\frac{(y^*\Lambda^2y)(y^*M^2y)}{(y^*\Lambda My)^2}=\frac{\left(\frac{y^*\Lambda^2y}{y^*y}\right)\left(\frac{y^*M^2y}{y^*y}\right)}{\left(\frac{y^*\Lambda My}{y^*y}\right)^2}$$

$$=\frac{\left(\sum_{i=1}^{n}\lambda_i^2p_i\right)\left(\sum_{i=1}^{n}\mu_i^2p_i\right)}{\left(\sum_{i=1}^{n}\lambda_i\mu_ip_i\right)^2}=\frac{E\xi^2\cdot E\eta^2}{E^2(\xi\eta)}\leqslant\frac{\overline{A}^2(\xi\eta)}{\overline{G}^2(\xi\eta)}$$

$$=\frac{(\lambda_l\mu_l+\lambda_m\mu_m)^2}{4\lambda_l\lambda_m\mu_l\mu_m}\leqslant\frac{(\lambda_1\mu_1+\lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}.$$

That is

$$\frac{x^*A^2xx^*B^2x}{(x^*ABx)^2}\leqslant\frac{(\lambda_1\mu_1+\lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}.$$

THEOREM 3.3. **(Pólya-Szegö inequality [1])**. *Let* $a_k>0$, $b_k>0$, $(k=1,2,\cdots,n)$, $a=\min a_k$, $A=\max a_k$, $b=\min b_k$, $B=\max b_k$. *Then*

$$\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)\leqslant\frac{1}{4}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)^2\left(\sum_{k=1}^{n}a_kb_k\right)^2.$$

Proof. In inequality (1), let ξ have the distribution $P(\xi = a_k) = \frac{1}{n}$, $k = 1,2,\cdots,n$; $f(a_k) = b_k$, $k = 1,2,\cdots,n$; $\eta = f(\xi)$. Then

$$E\xi^2 = \frac{1}{n}\sum_{k=1}^{n} a_k^2, \quad E\eta^2 = \frac{1}{n}\sum_{k=1}^{n} b_k^2, \quad E(\xi\eta) = \frac{1}{n}\sum_{k=1}^{n} a_k b_k;$$
$$\overline{A}(\xi\eta) = \frac{AB+ab}{2}, \quad \overline{G}(\xi\eta) = \sqrt{AaBb}.$$

The inequality (1) becomes

$$\frac{\frac{1}{n}\left(\sum_{k=1}^{n} a_k^2\right)\cdot\left(\frac{1}{n}\sum_{k=1}^{n} b_k^2\right)}{\left(\frac{1}{n}\sum_{k=1}^{n} a_k b_k\right)^2} \leqslant \frac{\left(\frac{AB+ab}{2}\right)^2}{\left(\sqrt{AaBb}\right)^2}.$$

That is

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \leqslant \frac{1}{4}\left(\sqrt{\frac{AB}{ab}}+\sqrt{\frac{ab}{AB}}\right)^2\left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

We have the result.

The author thanks for the referees' suggestions.

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(Received February 20, 2001)

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