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ON THE SYMMETRIC HILBERT'S INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, it is shown that some improvements on Hilbert's inequality for double series can be established by means of Schwarz's inequality; Hilbert-Ingham's inequality can be sharpened, Hardy-Littewood's inequality and Fejer-Riesz's inequality can be refined.

1. Preliminaries

Let $\{a_n\}$ and $\{b_n\}$ be complex numbers. For convenience, let us introduce some notations and define some functions as follows:

$$\|x\|_k^2 = \sum_{n=k}^{\infty} |x_n|^2 \qquad S(a,b) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+\frac{1}{2}}$$

$$r(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x_m \overline{x}_n}{\left(m+n+\frac{1}{2}\right)^2} \qquad T(a,b) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m-n}, \quad m \neq n$$

$$u(a,b) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \qquad W(a,b) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1}$$

$$v(a,b) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m-n}, \quad m \neq n$$

In particular, when $b = \overline{a}$, the above notations are reduced to $u(a)$, $v(a)$, $S(a)$, $T(a)$, $W(a)$ etc. In such case, the complex number b_n contained in each equality is replaced by $\overline{a}_n$.

The inequalities of the form

$$|u(a,b)|^2 \leqslant \pi^2 \|a\|_1^2 \|b\|_1^2 \tag{1}$$

$$|v(a,b)|^2 \leqslant \pi^2 \|a\|_1^2 \|b\|_1^2 \tag{2}$$

$$|W(a,b)|^2 \leqslant \pi^2 \|a\|_0^2 \|b\|_0^2 \tag{3}$$

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and

$$|T(a,b)|^2 \leqslant \pi^2 \|a\|_0^2 \|b\|_0^2 \tag{4}$$

are called Hilbert's inequalities (see [1]).

The inequality of the form

$$|S(a,b)|^2 \leqslant \pi^2 \|a\|_0^2 \|b\|_0^2 \tag{5}$$

is called Hilbert-Ingham's inequality (see [2]).

The main aim of this paper is to show that these inequalities can be refined by considering suitable functions.

We shall see that $u(a,b)$ and $v(a,b)$, $S(a,b)$ and $T(a,b)$ or $W(a,b)$ and $T(a,b)$ appear in pairs in the results obtained. And they are obviously the symmetric forms of elegance.

Therefore the new results related to Hilbert's inequality are called the symmetric Hilbert's inequality.

2. Main Results

In order to prove our theorems, we need the following lemma.

LEMMA. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary sequences of complex numbers. If $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$ are absolutely convergent for any $k \geqslant 0$, then*

(i) $\sum_{m=k}^{\infty}\sum_{n=k}^{\infty} a_m b_n$ is absolutely convergent for any $k \geqslant 0$;

(ii) $\sum_{n=k}^{\infty} |a_n|^2$ and $\sum_{n=k}^{\infty} |b_n|^2$ are convergent for any $k \geqslant 0$.

Proof. (i) For any $k \geqslant 0$, we have

$$\sum_{m=k}^{\infty}\sum_{n=k}^{\infty} |a_m b_n| = \sum_{n=k}^{\infty} |a_n| \sum_{n=k}^{\infty} |b_n|.$$

Since $\sum_{n=k}^{\infty} |a_n|$ and $\sum_{n=k}^{\infty} |b_n|$ are convergent for any $k \geqslant 0$, clearly $\sum_{m=k}^{\infty}\sum_{n=k}^{\infty} |a_m b_n|$ is convergent.

(ii) Notice that

$$\sum_{n=k}^{\infty} |a_n|^2 \leqslant \left(\sum_{n=k}^{\infty} |a_n|\right)^2, \qquad \sum_{n=k}^{\infty} |b_n|^2 \leqslant \left(\sum_{n=k}^{\infty} |b_n|\right)^2$$

$\sum_{n=k}^{\infty} |a_n|$ and $\sum_{n=k}^{\infty} |b_n|$ are convergent for any $k \geqslant 0$, hence (ii) is true.

THEOREM 1. *Let* $0 < \sum_{n=1}^{\infty} |a_n| < +\infty$ *and* $0 < \sum_{n=1}^{\infty} |b_n| < +\infty$. *Then*

$$|u(a,b)|^2 + v(a,b)|^2 \leqslant \pi^2 \|a\|_1^2 \|b\|_1^2. \tag{6}$$

Proof. Define the functions f_1 and g_1 respectively by

$$f_1(a,t) = \sum_{m=1}^{\infty} a_m \sin(mt) \quad \text{and} \quad g_1(b,t) = \sum_{n=1}^{\infty} b_n \cos(nt).$$

Then we have

$$f_1(a,t)g_1(b,t) = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n (\sin(m+n)t + \sin(m-n)t).$$

It is easy to deduce that

$$\int_0^{2\pi} t \sin(m+n)t \, dt = -\frac{2\pi}{m+n}$$

and

$$\int_0^{2\pi} t \sin(m-n)t \, dt = -\frac{2\pi}{m-n}, \quad (m \neq n).$$

Since the both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are convergent, by Lemma the double series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m b_n|$ is convergent. In view of the fact that

$$\begin{aligned} |tf_1(a,t)g_1(b,t)| &= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} t a_m b_n \sin(mt)\cos(nt) \right| \\ &\leqslant \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |t||a_m b_n| \leqslant 2\pi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m b_n|, \quad t \in [0, 2\pi]. \end{aligned}$$

Whence $|tf_1(a,t)g_1(b,t)|$ is uniformly convergent in the interval $[0,2\pi]$. Hence the interchange in order of summation and integration can be made. Thus we way write $|u(a,b)+v(a,b)|$ in the form

$$|u(a,b) + v(a,b))| = \frac{1}{\pi} \left| \int_0^{2\pi} tf_1(a,t)g_1(b,t) \, dt \right|. \tag{7}$$

Applying Schwarz's inequality to (7) we have

$$\begin{aligned} |u(a,b) + v(a,b)|^2 &\leqslant \frac{1}{\pi^2} \left(\int_0^{2\pi} (\sqrt{t}|f_1(a,t)|)(\sqrt{t}|g_1(b,t)|) \, dt \right)^2 \\ &\leqslant \frac{1}{\pi^2} \int_0^{2\pi} t|f_1(a,t)|^2 \, dt \int_0^{2\pi} t|g_1(b,t)|^2 \, dt. \end{aligned}$$

Similarly, the interchanges in order of summation and integration are justified in the following equalities.

$$\int_0^{2\pi} t|f_1(a,t)|^2\,dt = \pi^2\|a\|_1^2 \quad \text{and} \quad \int_0^{2\pi} t|g_1(b,t)|^2\,dt = \pi^2\|b\|_1^2.$$

These two equalities can be obtained by some simple computations.

Hence we have

$$|u(a,b)+v(a,b)|^2 \leqslant \pi^2\|a\|_1^2\|b\|_1^2. \tag{8}$$

It is important to notice that

$$u(b,a)=u(a,b) \quad \text{and} \quad v(b,a)=-v(a,b).$$

Interchanging a,b in (8) we have

$$|u(a,b)-v(a,b)|^2 \leqslant \pi^2\|b\|_1^2\|a\|_1^2. \tag{9}$$

Adding (8) and (9), we have

$$|u(a,b)|^2+|v(a,b)|^2 \leqslant \pi^2\|a\|_1^2\|b\|_1^2.$$

Thus we complete the proof of the theorem.

The inequality (6) is obviously an improvement on both the inequality (1) and the inequality (2).

COROLLARY 1. *If* $0<\sum\limits_{n=1}^{\infty}|a_n|<+\infty$, *then*

$$|u(a)|^2+|v(a)|^2 \leqslant \pi^2\|a\|_1^4.$$

THEOREM 2. *Let* $0<\sum\limits_{n=0}^{\infty}|a_n|<+\infty$ *and* $0<\sum\limits_{n=0}^{\infty}|b_n|<+\infty$. *Then*

$$|S(a,b)|^2+|T(a,b)|^2 \leqslant \pi^2\|a\|_0^2\|b\|_0^2-\frac{1}{\pi^2}r(a)r(b) \tag{10}$$

where $r(x)>0$, $x=a,b$.

Proof. Define the functions f_0 and g_0 respectively by

$$f_0(a,t)=\sum_{m=0}^{\infty}a_m\sin\left(m+\frac{1}{4}\right)t \quad \text{and} \quad g_0(b,t)=\sum_{n=0}^{\infty}b_n\cos\left(n+\frac{1}{4}\right)t.$$

In a similar way as the proof of Theorem 1, the double series $t\cdot f_0(a,t)g_0(b,t)$ is uniformly convergent in the interval $[0,2\pi]$. The interchange in order of summation and integration is justified in following relation established.

$$\begin{aligned}|S(a,b)-T(a,b)|^2 &= \frac{1}{\pi^2}\left(\int_0^{2\pi}|tf_0(a,t)g_0(b,t)|\,dt\right)^2\\ &\leqslant \frac{1}{\pi^2}\int_0^{2\pi}t|f_0(a,t)|^2\,dt\int_0^{2\pi}t|g_0(b,t)|^2\,dt.\end{aligned}$$

It follows from Lemma that the double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} ta_m\overline{a}_n \sin\left(m+\frac{1}{4}\right)t \sin\left(n+\frac{1}{4}\right)t$ and $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} tb_m\overline{b}_n \cos\left(m+\frac{1}{4}\right)t \cos\left(n+\frac{1}{4}\right)t$ are uniformly convergent.

It is easy to deduce that

$$\int_0^{2\pi} t|f_0(a,t)|^2\,dt = \pi^2\|a\|_0^2 + r(a) \quad \text{and} \quad \int_0^{2\pi} t|g_0(b,t)|^2\,dt = \pi^2\|b\|_0^2 - r(b).$$

Hence we have

$$|S(a,b)-T(a,b)|^2 \leqslant \frac{1}{\pi^2}(\pi^2\|a\|_0^2 + r(a))(\pi^2\|b\|_0^2 - r(b)). \tag{11}$$

Notice that

$$S(a,b) = S(b,a) \quad \text{and} \quad T(a,b) = -T(b,a).$$

Interchanging a,b in (11) we have

$$|S(a,b)+T(a,b)|^2 \leqslant \frac{1}{\pi^2}(\pi^2\|b\|_0^2 + r(b))(\pi^2\|a\|_0^2 - r(a)). \tag{12}$$

Adding (11) and (12), we obtain after some simplifications

$$|S(a,b)|^2 + |T(a,b)|^2 \leqslant \pi^2\|a\|_0^2\|b\|_0^2 - \frac{1}{\pi^2}r(a)r(b).$$

It is easy to see that $r(x)\in\mathbf{R}$. By our assumptions, $\|a\|_0^2 \neq 0$ and $\|b\|_0^2 \neq 0$,

$$r(x) = \int_0^1 \frac{1}{u}\left(\int_0^u \left|\sum_{n=0}^{\infty} x_n t^{n-\frac{1}{4}}\right|^2 dt\right) du = \sum_{m,n=0}^{\infty} \frac{x_m\overline{x}_n}{\left(m+n+\frac{1}{2}\right)^2}$$

whence $r(x) > 0$, where $x = a,\ b$.

Thus the theorem is proved.

COROLLARY 2. *With the assumptions as Theorem, then*

$$|S(a,b)|^2 \leqslant \pi^2\|a\|_0^2\|b\|_0^2 - \frac{1}{\pi^2}r(a)r(b) \tag{13}$$

and

$$|T(a,b)|^2 \leqslant \pi^2\|a\|_0^2\|b\|_0^2 - \frac{1}{\pi^2}r(a)r(b) \tag{14}$$

where $r(a)r(b) > 0$.

These are immediate consequences of the theorem.

The inequalities (13) and (14) are obviously improvements on the inequalities (4) and (5).

We obtain easily from (13) that

$$|S(a,b)| \leqslant \pi\left(1 - \frac{r(a)r(b)}{\pi^4\|a\|_0^2\|b\|_0^2}\right)^{\frac{1}{2}} (\|a\|_0\|b\|_0).$$

This could be used as an alternative proof of the best nature of the constant π in this inequality by considering

$$\sup\left\{|S(a,b)| : \|a\|_0^2 \leqslant 1, \|b\|_0^2 \leqslant 1\right\} \leqslant \pi \sup\left(1-\frac{r(a)r(b)}{\pi^4\|a\|_0^2\|b\|_0^2}\right)^{\frac{1}{2}}$$
$$= \pi\left(1-\inf\left\{\frac{r(a)r(b)}{\pi^4\|a\|_0^2\|b\|_0^2}\right\}\right)^{\frac{1}{2}} = \pi.$$

COROLLARY 3. *If* $0<\sum_{n=0}^{\infty}|a_n|<+\infty$, *then*

$$|S(a)|^2+|T(a)|^2 \leqslant (\pi\|a\|_0^2)^2-\left(\frac{r(a)}{\pi}\right)^2. \tag{15}$$

Clearly, we have the following results from Corollary 3.

COROLLARY 4. *If* $0<\sum_{n=0}^{\infty}|a_n|<+\infty$, *then*

$$|S(a)|^2 \leqslant (\pi\|a\|_0^2)^2-\left(\frac{r(a)}{\pi}\right)^2 \tag{16}$$

and

$$|T(a)|^2 \leqslant (\pi\|a\|_0^2)^2-\left(\frac{r(a)}{\pi}\right)^2. \tag{17}$$

THEOREM 3. *Let* $0<\sum_{n=0}^{\infty}|a_n|<+\infty$ *and* $0<\sum_{n=0}^{\infty}|b_n|<+\infty$. *Then*

$$|W(a,b)|^2+|T(a,b)|^2 \leqslant (\pi\|a\|_0\|b\|_0)^2. \tag{18}$$

Proof. Define the functions f and g respectively by

$$f(a,t)=\sum_{m=0}^{\infty}a_m\sin\left(m+\frac{1}{2}\right)t \quad \text{and} \quad g(b,t)=\sum_{n=0}^{\infty}b_n\cos\left(n+\frac{1}{2}\right)t.$$

Its proof is similar to the proof of Theorem 1, so it is omitted here. Clearly, the inequality (18) is an improvement on the inequality (3).

COROLLARY 5. *If* $0<\sum_{n=0}^{\infty}|a_n|<+\infty$, *then*

$$|W(a)|^2+|T(a)|^2 \leqslant (\pi\|a\|_0^2)^2. \tag{19}$$

3. Applications

Let $f(t) \in L^2(0,1]$ and $f(t) \neq 0$. If $a_n = \int_0^1 t^n f(t)\,dt$, $n = 0,1,2,\ldots$ then we have the inequality of the form

$$\|a\|_0^2 < \pi \int_0^1 |f(t)|^2 \, dt. \tag{20}$$

This is Hardy-Littlewood inequality (see [1]).

Using Corollary 5 we can obtain a refinement of the inequality (20).

THEOREM 4. *With the assumption as the above described, if* $0 \leqslant \sum_{n=0}^{\infty} |a_n| < +\infty$, *then*

$$\|a\|_0^2 \leqslant \pi(1-a^2) \int_0^1 \sqrt{t}\,|f(t)|^2 \, dt \tag{21}$$

where $\alpha = \dfrac{r(|a|)}{\|a\|_0^2 \pi^2}$, $r(|a|) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \dfrac{|a_m \overline{a}_n|}{\left(m+n+\frac{1}{2}\right)^2}$.

Proof. By our assumption, we have

$$|a_n|^2 \leqslant |a_n| \int_0^1 t^n \, |f(t)| \, dt.$$

Using Schwarz's inequality and Corollary 4 we have

$$\begin{aligned}
\|a\|_0^2 &= \left(\sum_{n=0}^{\infty} \int_0^1 |a_n|\, t^n \, |f(t)| \, dt \right)^2 \\
&= \left(\int_0^1 \left(\sum_{n=0}^{\infty} |a_n|\, t^{n-\frac{1}{4}} \right) \left(t^{\frac{1}{4}} \, |f(t)| \right) dt \right)^2 \\
&\leqslant \int_0^1 \left(\sum_{n=0}^{\infty} |a_n|\, t^{n-\frac{1}{4}} \right)^2 dt \int_0^1 t^{\frac{1}{2}} \, |f(t)|^2 \, dt \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|a_m \overline{a}_n|}{m+n+\frac{1}{2}} \int_0^1 t^{\frac{1}{2}} \, |f(t)|^2 \, dt \\
&\leqslant \left(\pi^2 \|a\|_0^4 - \left(\frac{r(|a|)}{\pi} \right)^2 \right)^{\frac{1}{2}} \int_0^1 t^{\frac{1}{2}} \, |f(t)|^2 \, dt
\end{aligned} \tag{22}$$

where $r(|a|) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \dfrac{|a_m \overline{a}_n|}{\left(m+n+\frac{1}{2}\right)^2}$.

Thus the theorem is proved.

THEOREM 5. *Let $f(z)$ be analytic in the unit dise $|z| \leqslant 1$. If $f \in H_p$, $p > 0$, then*

$$\left(\int_0^1 |f(t)|^p \, dt\right)^2 + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} t\,|f(-e^{it})|^p \, dt\right)^2 \leqslant \left(\frac{1}{2}\int_0^{2\pi} |f(e^{it})|^p \, dt\right)^2. \tag{23}$$

Proof. We first prove the theorem for the ease $p = 2$. Suppose that

$$f(z) = \sum_{m=0}^{\infty} a_m z^m \quad (|z| \leqslant 1).$$

It is easy to verify the relations of the form

$$\|a\|_0^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \, dt,$$

$$w(a) = \int_0^1 |f(t)|^2 \, dt \quad \text{and} \quad -iT(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,|f(-e^{it})|^2 \, dt.$$

These relations are shown as follows:

Consider the function

$$f(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Since $f(z)$ is analytic in $|z| \leqslant 1$, $\sum_{n=0}^{\infty} a_n t^n$ is uniformly convergent in $[-1, 1]$.

Hence we have

$$\int_0^1 |f(t)|^2 \, dt = \int_0^1 \left|\sum_{n=0}^{\infty} a_n t^n\right|^2 dt = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m \overline{a}_n}{m+n+1} = w(a).$$

Consider the function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then we have

$$f(-e^{it}) = \sum_{n=0}^{\infty} a_n \Big(\cos(\pi+t) + i\sin(\pi+t)\Big)^n = \sum_{n=0}^{\infty} a_n(\cos n(\pi+t) + i\sin n(\pi+t)).$$

The series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent in $|z| \leqslant 1$, hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t\,|f(-e^{it})|^2 \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} tf(-e^{it})\overline{f(-e^{it})}\, dt = -i\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m \overline{a}_n}{m-n} = -iT(a).$$

In a similar way, we have $\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \, dt = \|a\|_0^2$.

By Corollary 5, we have

$$|W(a)|^2 + |-iT(a)|^2 = |W(a)|^2 + |T(a)|^2 \leqslant (\pi \|a\|_0^2)^2.$$

Consequently, for the case $p = 2$ the inequality (23) is valid.

If $p \neq 2$, by the decomposition theorem $f(z) = B(z)G(z)$ where $B(z)$ is a Blaschke function and $G(z) \neq 0$, $|B(z)| \leqslant 1$ in $|z| \leqslant 1$, $|B(e^{it})| = 1$.

Owing to $F(Z) = \left(G(Z)\right)^{\frac{p}{2}} \in H_2$. Hence by (23) for $p = 2$, we have

$$\begin{aligned}
&\left(\int_0^1 |f(t)|^p\,dt\right)^2 + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} t\,|f(-e^{it})|^p\,dt\right)^2 \\
&\qquad = \left(\int_0^1 |F(t)|^2\,dt\right)^2 + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} t\,|F(-e^{it})|^2\,dt\right)^2 \\
&\qquad \leqslant \left(\frac{1}{2}\int_0^{2\pi} |F(e^{it})|^2\,dt\right)^2 \\
&\qquad \leqslant \left(\frac{1}{2}\int_0^{2\pi} |G(e^{it})|^p\,dt\right)^2 \\
&\qquad \leqslant \left(\frac{1}{2}\int_0^{2\pi} |f(e^{it})|^p\,dt\right)^2.
\end{aligned}$$

Thus the proof of the theorem is completed.

In the second term of the left-hand side of (23) is replaced by zero, then Fejer-Riesz's inequality (see[3]) of the form

$$\int_0^1 |f(t)|^p\,dt \leqslant \frac{1}{2}\int_0^{2\pi} |f(e^{it})|^p\,dt \tag{24}$$

is obtained. Consequently, the inequality (23) is a refinement of the inequality (24).

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