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CERTAIN IMBEDDINGS OF WEIGHTED SOBOLEV SPACES

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Abstract. We characterize weight functions for which the weighted Sobolev space $W^{1,p}(\Omega, d_M^{\beta})$ [and also $W^{1,p}(\Omega, s_0(d_M))$] is imbedded continuously or compactly into the weighted Lebesgue space $L^q(\Omega, d_M^{\alpha})$ [and also $L^q(\Omega, s_1(d_M))$] where $1 \leqslant q < p < \infty$ and $M \subset \partial\Omega$. Some of the imbeddings are also extended to the higher order weighted Sobolev spaces.

1. Introduction

The purpose of this paper is to study the continuous imbeddings of the type

$$W^{1,p}(\Omega; v) \hookrightarrow L^q(\Omega; w) \tag{1.1}$$

and the corresponding compact imbeddings

$$W^{1,p}(\Omega; v) \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{1.2}$$

for certain specific weights v, w and the domains $\Omega \in \mathbf{R^N}$.

We shall be concerned with the weights which are some form of the distance d_M from $M \subset \partial\Omega$, the boundary of Ω.

Three situations arise :

(a) the distance, to be denoted by d, is taken from the whole boundary $\partial\Omega$;
(b) the distance, to be denoted by d_0, is taken from a point $x_0 \in \partial\Omega$; and
(c) the distance, to be denoted by d_M, is taken from a set $M \subset \partial\Omega$.

As regards situation (a), Kufner [5] has studied the imbedding (1.1) when $p = q$ and with weights which are either power type d^{α} or some function of the distance d. For $1 \leqslant p \leqslant q < \infty$, Gurka and Opic [1] have studied (1.1) and (1.2) with certain more general weights and for $1 \leqslant q < p < \infty$, again, Gurka and Opic [2] studied (1.1) and (1.2) but for power type weights. The case of the weight $s(d)$ has been discussed in [3].

Situation (b) has been dealt by Kufner [5], again, for $p = q$, the weights being power type and continuous imbeddings were considered. The case $1 \leqslant q < p < \infty$

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has been studied in [4] and [3] for respectively power type weights d_0^α and the weights of the type $s(d_0)$.

Regarding situation (c), Kufner [5] conjuctured a sufficient condition for the imbedding (1.1) to hold when $p = q$ and the weights are of power type. This conjucture was proved by Rákosnîk [9].

In the present paper, we further investigate situation (c) in respect of both the imbeddings (1.1) and (1.2). More precisely, we consider weights which are either power type or some function of the distance d_M and obtain conditions which are both necessary as well as sufficient. Consequently, in the particular cases, when $M = \partial\Omega$ or $M = \{x_0\}$, $x_0 \in \partial\Omega$, the situations (a) and (b) are obtained. We also extend some of the imbeddings obtained to the higher order weighted Sobolev spaces.

2. Preliminaries

Let Ω be a domain in $\mathbf{R}^\mathbf{N}$. By a weight function, we mean a function which is finite, positive and measurable almost everywhere on a domain Ω. We shall denote by S, the set of weight functions defined on Ω. Throughout the paper, we shall assume that Ω is a bounded domain in $\mathbf{R}^\mathbf{N}$ with boundary $\partial\Omega$.

For $w \in S$, $1 \leqslant p < \infty$, the weighted Lebesgue space $L^p(\Omega; w)$ is defined by

$$L^p(\Omega; w) = \left\{ u : \int_\Omega |u(x)|^p \, w(x)\, dx < \infty \right\},$$

with the norm

$$\|u\|_{p,w;\Omega} = \left(\int_\Omega |u(x)|^p \, w(x)\, dx \right)^{1/p}.$$

It is noted that with the above norm, $L^p(\Omega; w)$ is a Banach space.

Let $w \in S$ be such that $w \in L^1_{loc}(\Omega)$ and that $w^{-\frac{1}{p}} \in L^{p'}_{loc}(\Omega)$. Here and also throughout the paper $p' = \frac{p}{p-1}$, the conjugate of p. For $1 \leqslant p < \infty$, the weighted Sobolev space $W^{k,p}(\Omega; w)$ is the set of all $u \in L^p(\Omega; w)$ such that all the distributional derivatives up to k^{th} order also belong to $L^p(\Omega; w)$. The norm of the space $W^{k,p}(\Omega; w)$ is given by

$$\|u\|_{k,p,w;\Omega} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|^p_{p,w;\Omega} \right)^{1/p} \tag{2.1}$$

The space $W^{k,p}(\Omega; w)$ is a Banach space with respect to the norm given by (2.1). Further, we shall denote by $W^{k,p}_0(\Omega; w)$, the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega; w)$ with respect to the norm (2.1). The norm of $W^{k,p}_0(\Omega; w)$ is, again, given by (2.1). It is noted, in general, that $W^{k,p}_0(\Omega; w) \subset W^{k,p}(\Omega; w)$ and that $W^{k,p}_0(\mathbf{R}^N; w) = W^{k,p}(\mathbf{R}^N; w)$.

Let $M \subset \partial\Omega$ be a closed set. Put

$$C_M^\infty(\Omega) = \{u \in C^\infty(\Omega)\, ;\, \operatorname{supp} u \cap M = \emptyset\}.$$

We define by $W_M^{k,p}(\Omega;w)$, the closure of $C_M^\infty(\Omega)$ with respect to the norm (2.1). It can be seen that $C_0^\infty(\Omega) = C_{\partial\Omega}^\infty(\Omega)$ and that $W_0^{k,p}(\Omega;w) = W_{\partial\Omega}^{k,p}(\Omega;w)$.

Set $d_M(x) = \mathrm{dist}\,(x,M)$, $x \in \Omega$. In the particular situations, we shall denote $d_{\partial\Omega}$ by d and $d_{\{x_0\}}$ by d_0. It can be easily verified that any powers d^α, d_0^α and d_M^α, $\alpha \in \mathbf{R}$, are weight functions on Ω. Further, let $s = s(t)$ be a continuous positive funtion defined on $(0,\infty)$ such that either $\lim_{t\to 0} s(t) = 0$ or $\lim_{t\to 0} s(t) = \infty$. Then $s(d)$, $s(d_0)$ and $s(d_M)$ are also weight functions on Ω. For details, one may refer to [5].

A positive continuous function $f \equiv f(t)$ defined on $(0,\infty)$ is said to have *Property* (H) if for every pair of positive constants c_1, c_2, there exists a pair of constants C_1, $C_2 > 0$ such that

$$c_1 \leqslant \frac{t_1}{t_2} \leqslant c_2 \Rightarrow C_1 \leqslant \frac{f(t_1)}{f(t_2)} \leqslant C_2.$$

Put $Q = (0,1)^N$. Define, for $m = 0,1,...,N-1$,

$$Q(m) = \left\{x \in \bar{Q} : x_{m+1} = x_{m+2} = ... = x_N = 0\right\}.$$

We say that a closed subset $M \subset \partial\Omega$ is a manifold of dimension m and that the bounded domain Ω is of class $Q^{0,1}(M)$, if there exists an open covering $\{U_i\}_{i=0}^{\bar{k}}$ ($\bar{k}$ may be finite or infinite) of $\bar{\Omega}$ with the following properties:

(i) $M \subset \bigcup_{i=1}^{\bar{k}} U_i$ and there exists $n_0 \in \{3,4,...\}$ such that every system of n_0+1 sets U_i is disjoint;

(ii) there exists $\delta > 0$ such that $d_M(x) \geqslant \delta,\ x \in U_0$;

(iii) there exist numbers $C_2 \geqslant C_1 > 0$ and a system of one-to-one mappings $T_i : Q \to \Omega \cap U_i;\ i = 1,2,...\bar{k},$ such that $T_i\big(Q(m)\big) = \overline{M \cap U_i}$ and

$$C_1|x-y| \leqslant |T_i(x) - T_i(y)| \leqslant C_2|x-y|, \qquad x,y \in Q;\ i = 1,2,...,\bar{k}.$$

We shall heavily depend upon the following concept of partition of unity:

Let us denote by $\{\phi_0,\phi_1,...,\phi_{\bar{k}}\}$, a partition of unity corresponding to the open covering $\{U_0,U_1,...,U_{\bar{k}}\}$ of $\bar{\Omega}$, i.e., let

$$\phi_i \in C^\infty(\mathbf{R}^N), \qquad]text\mathrm{supp}\,\phi_i \subset U_i, \qquad 0 \leqslant \phi_i(x) \leqslant 1,$$

and

$$\sum_{i=0}^{\bar{k}} \phi_i = 1, \qquad x \in \bar{\Omega}.$$

For $-\infty \leqslant a < b \leqslant \infty$, we denote by $AC(a,b)$, the set of all functions absolutely continuous on every compact interval $[c,d] \subset (a,b)$. Further, we denote by $AC_L(a,b)$ and $AC_R(a,b)$, the sets of all functions $u \in AC(a,b)$ for which, respectively, $\lim_{t\to a^+} u(t) = 0$ and $\lim_{t\to b^-} u(t) = 0$. For further details about the different concepts used one may refer to [5,8,9].

We shall be using the following lemmas in the subsequent sections:

LEMMA 1. *Let* $1 \leqslant p \leqslant q < \infty$, $0 < b < \infty$, $\tau \in \mathbf{R}^+ \cup \{0\}$ *and* $w_1, w_2 \in S$. *Then, the inequality*

$$\left(|u(0)|^q \int_0^\tau w_1(t)\,dt + \int_0^b |u(t)|^q\, w_1(t+\tau)\,dt\right)^{1/q} \leqslant C\left(\int_0^b |u'(t)|^p\, w_2(t+\tau)\,dt\right)^{1/p} \tag{2.2}$$

holds with a constant $C > 0$, *independent of* u, *in the following cases :*
(i) for $u \in AC_L(0,b)$ *if and only if*

$$\sup_{\eta\in(0,b)} \left(\int_\eta^{b+\tau} w_1(t)\,dt\right)^{1/q} \left(\int_0^\eta (w_2(t))^{1-p'}\,dt\right)^{1/p'} < \infty,$$

(ii) for $u \in AC_R(0,b)$ *if and only if*

$$\sup_{\eta\in(0,b)} \left(\int_0^\eta w_1(t)\,dt\right)^{1/q} \left(\int_\eta^{b+\tau} (w_2(t))^{1-p'}\,dt\right)^{1/p'} < \infty.$$

LEMMA 2. *Let* $1 \leqslant q < p < \infty$, $0 < b < \infty$, $\tau \in \mathbf{R}^+ \cup \{0\}$, $w_1, w_2 \in S$ *and* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Then, the inequality (2.2) holds with a constant* $C > 0$, *independent of* u, *in the following cases :*
(i) for $u \in AC_L(0,b)$ *if and only if*

$$\left\{\int_0^{b+\tau}\left[\left(\int_\eta^{b+\tau} w_1(t)\,dt\right)^{1/q}\left(\int_0^\eta (w_2(t))^{1-p'}\,dt\right)^{1/q'}\right]^r (w_2(\eta))^{1-p'}\,d\eta\right\}^{1/r} < \infty,$$

(ii) for $u \in AC_R(0,b)$ *if and only if*

$$\left\{\int_0^{b+\tau}\left[\left(\int_0^\eta w_1(t)\,dt\right)^{1/q}\left(\int_\eta^{b+\tau} (w_2(t))^{1-p'}\,dt\right)^{1/q'}\right]^r (w_2(\eta))^{1-p'}\,d\eta\right\}^{1/r} < \infty. \tag{2.3}$$

REMARK. In Lemmas 1 and 2, we assume that $u(0) = \lim\limits_{t\to 0^+} u(t)$ exists. Further, if $\int_0^\tau w_1(t)\,dt = \infty$, then essentially $u(0) = 0$ and we shall use the convention that $0.\infty = 0$.

Lemmas 1 and 2 are obtained easily by making suitable substitutions in the usual Hardy's inequalities e.g. in Theorems 5.9, 5.10, 6.2, 6.3 of [6].

LEMMA 3 [7]. *Let* Ω *be domain in* $\mathbf{R}^\mathbf{N}$ *and* $1 \leqslant p, q < \infty$. *Let* $\{\Omega_n\}$ *be a sequence of domains in* $\mathbf{R}^N$ *such that* $\Omega_n \subset \Omega_{n+1} \underset{\neq}{\subset} \Omega$, $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$, $n \in \mathbf{N}$. *Further, suppose that for* $w_1, w_2 \in S$

$$W^{1,p}(\Omega_n; w_1) \hookrightarrow\hookrightarrow L^q(\Omega_n; w_2), \quad n \in \mathbf{N}. \tag{2.4}$$

Then, the imbedding

$$W^{1,p}(\Omega;w_1) \hookrightarrow\hookrightarrow L^q(\Omega;w_2)$$

holds, if and only if

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,w_1;\Omega}\leqslant 1} \|u\|_{q,w_2;\Omega^n} = 0,$$

where $\Omega^n = \Omega\backslash\Omega_n$.

3. Results concerning power type weights

THEOREM 1. *Let* $1 \leqslant q < p < \infty$, $\alpha,\beta \in \mathbf{R}$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the imbedding*

$$W^{1,p}(\Omega;d_M^\beta) \hookrightarrow L^q(\Omega;d_M^\alpha)$$

holds if and only if

$$\begin{cases} \qquad\qquad \beta > p-N+m, & \alpha > (\beta+N-m-1)\dfrac{q}{p} - \dfrac{q}{p'} - N+m, \\ or & \\ -N+m < \beta \leqslant p-N+m, & \alpha > -N+m, \\ or & \\ \qquad\qquad \beta \leqslant -N+m, & \alpha > (\beta+N-m-1)\dfrac{q}{p} - \dfrac{q}{p'} - N+m. \end{cases} \tag{3.1}$$

Proof. Let us first assume that (3.1) holds. Using the partition of unity, it is sufficient to prove the result for $V_i (\equiv U_i \cap \Omega)$ instead of Ω.

Case 1. $i \geqslant 1$

In view of the definition of $Q^{0,1}(M)$, we have

$$C_1 d_{Q(m)}(y) \leqslant d_M(T_i(y)) \leqslant C_2 d_{Q(m)}(y), \qquad y \in Q$$

and consequently, the mapping $u \mapsto v$, $v(y) = u(T_i(y))$ is a topological isomorphism between $L^q(V_i)$ and $L^q(Q)$ or $W^{1,p}(V_i)$ and $W^{1,p}(Q)$. Therefore, there exist positive constants C_3, C_4, C_5, C_6 such that

$$C_3\|v\|_{q,d_{Q(m)}^\alpha;Q} \leqslant \|u\|_{q,d_M^\alpha;V_i} \leqslant C_4\|v\|_{q,d_{Q(m)}^\alpha;Q} \tag{3.2}$$

and

$$C_5\|v\|_{1,p,d_{Q(m)}^\beta;Q} \leqslant \|u\|_{1,p,d_M^\beta;V_i} \leqslant C_6\|v\|_{1,p,d_{Q(m)}^\beta;Q} \tag{3.3}$$

Defining v to be zero for $y_i \geqslant 1,\ i = m+1,...,N$, using cylindrical coordinates $y \in Q \mapsto (y', \theta, \bar{r})$, where $y' = (y_1, y_2, ..., y_m)$, $\theta = (\theta_1, \theta_2, ...\theta_{N-m-1})$, $\bar{r} = d_{Q(m)}$ and applying Fubini's theorem, we get

$$\|v\|^q_{q,d^\alpha_{Q(m)};Q} = \int\limits_{(0,1)^m} \left[\int\limits_{(0,\pi/2)^{N-m-1}} \left(\int_0^\infty |v(y', \theta, \bar{r})|^q\, \bar{r}^{N-m-1}\, \bar{r}^\alpha\, d\bar{r} \right) d\theta \right] dy' \tag{3.4}$$

Since $v = 0$, for large $\bar{r}$, we can restrict the upper limit of the inner integral in (3.4) to a finite number, say b, and consequently, $v \in AC_R(0,b)$. In view of the condition (3.1), Lemma 2 (with $\tau = 0,\ w_1(t) = t^{\alpha+N-m-1}$ and $w_2(t) = t^{\beta+N-m-1}$) gives that the inequality

$$\left(\int_0^b |v(y', \theta, \bar{r})|^q\, \bar{r}^{N-m-1}\, \bar{r}^\alpha\, d\bar{r} \right)^{1/q} \leqslant C \left(\int_0^b |\frac{\partial}{\partial \bar{r}} v(y', \theta, \bar{r})|^p\, \bar{r}^{N-m-1}\, \bar{r}^\beta\, d\bar{r} \right)^{1/p} \tag{3.5}$$

Now, by (3.4), (3.5), Hölder's inequality and using Fubini's theorem again, we get the estimate

$$\|v\|_{q,d^\alpha_{Q(m)};Q} \leqslant C \|v\|_{1,p,d^\beta_{Q(m)};Q},$$

which further, using (3.2) and (3.3), gives

$$\|v\|_{q,d^\alpha_M;V_i} \leqslant C \|v\|_{1,p,d^\beta_M;V_i}. \tag{3.6}$$

Case 2. $i = 0$

In view of the definition of $Q^{0,1}(M)$, there exist constants $C_7,\ C_8 > 0$ such that

$$0 < C_7 \leqslant d_M(y) \leqslant C_8, \qquad y \in U_0$$

and consequently, we can use the results which hold for classical Sobolev spaces $W^{1,p}(U_0)$ establishing the estimate (3.6).

Conversely, assume that (3.1) is not true for some α, β. Then, (2.3) (with $\tau = 0,\ w_1(t) = t^{\alpha+N-m-1}$ and $w_2(t) = \tau^{\beta+N-m-1}$) does not hold and consequently, by Lemma 2, the inequality

$$\left(\int_0^b |u(t)|^q\, t^{\alpha+N-m-1}\, dt \right)^{1/q} \leqslant C \left(\int_0^b |u'(t)|^p\, t^{\beta+N-m-1}\, dt \right)^{1/p}$$

does not hold. Thus, in view of Remark 9.2 in [2], there exists a sequence $\{u_n\}$ in $C^\infty((0,b))$ with $b \notin \operatorname{supp} u_n$ such that

$$\int_0^b |u_n'(t)|^p\, t^{\beta+N-m-1}\, dt = 1, \qquad n \in \mathbf{N} \tag{3.7}$$

and

$$\int_0^b |u_n(t)|^q\, t^{\alpha+N-m-1}\, dt \to \infty \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Write $z = (z_1, z_2, ..., z_{N-1}) \in \mathbf{R}^{\mathbf{N}-1}$. Consider a function $\phi \in C^\infty(\mathbf{R}^{\mathbf{N}-1})$ satisfying

$$\begin{cases} \{0 \leqslant \phi(z) \leqslant 1 & , z \in \mathbf{R}^{\mathbf{N}-1} \\ \phi(z) = 1 & , z_i < \frac{1}{2} \text{ for all } i = 1, 2, \ldots, m \\ \phi(z) = 0 & , z_i > \frac{3}{4} \text{ for some } i = 1, 2, \ldots, m \end{cases} \tag{3.9}$$

For $y = (y_1, \ldots, y_m, y_{m+1}, \ldots, y_N) \in \Omega$, define a sequence $\{v_n\}$ such that $v_n(y) = 0$ whenever $y_i \geqslant 1$ for any $i = m+1, \ldots, N$. Introduce cylindrical coordinates $y \in Q \longmapsto (y', \theta, \bar{r})$ where $y' = (y_1, \ldots, y_m)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_{N-m-1})$ and $\bar{r} = d_{Q(m)}$. As $v_n \equiv 0$ for large $\bar{r}$, we can restrict the upper limit of $\bar{r}$ to a finite number, say λ. Set

$$v_n(y) \equiv v_n(y', \theta, \bar{r}) = \phi(y', \theta) u_n(\bar{r}),$$

where u_n is as given in (3.7) and (3.8) with λ instead of b and consequently $u_n \in C^\infty((0, \lambda))$ with $\lambda \notin \operatorname{supp} u_n$. Now, on using the estimate $|\phi| \leqslant 1$ and Fubini's theorem, we get

$$\begin{aligned} \| v_n \|^p_{p, d^\beta_{Q(m)}; Q} &= \int_{\{(0,1)^m \times (0, \frac{\pi}{2})^{N-m-1} \times (0,\lambda)\} \cap \operatorname{supp} \phi} |\phi(y', \theta) u_n(\bar{r})|^p \, \bar{r}^{\beta+N-m-1} \, d\bar{r} \, d\theta \, dy' \\ &\leqslant C \int_0^\lambda |u_n(\bar{r})|^p \, \bar{r}^{\beta+N-m-1} \, d\bar{r} \end{aligned} \tag{3.10}$$

Working on the same lines as above, it can be shown that

$$\| \nabla v_n \|^p_{p, d^\beta_{Q(m)}; Q} \leqslant C \left(\int_0^\lambda |u_n(\bar{r})|^p \, \bar{r}^{\beta+N-m-1} \, d\bar{r} + \int_0^\lambda |u_n'(\bar{r})|^p \, \bar{r}^{\beta+N-m-1} \, d\bar{r} \right) \tag{3.11}$$

Now, from (3.7), (3.10), (3.11) and Lemma 1 (with $\tau = 0, b = \lambda, p = q$, $w_1(t) = w_2(t) = t^{\beta+N-m-1}$), we find that the sequence $\{v_n\}$ belongs to $W^{1,p}(Q; d^\beta_{Q(m)})$ and is bounded in $W^{1,p}(Q; d^\beta_{Q(m)})$-norm.

Further, using (3.9), we obtain as earlier that

$$\| v_n \|^q_{q, d^\alpha_{Q(m)}; Q} \geqslant C\left(\frac{1}{2}\right)^{N-1} \int_0^\lambda |u_n(\bar{r})|^q \, \bar{r}^{\alpha+N-m-1} \, d\bar{r},$$

which from (3.8) yields that the sequence $\{v_n\}$ is unbounded in $L^q(Q; d^\beta_{Q(m)})$-norm. Thus the space $W^{1,p}(Q; d^\beta_{Q(m)})$ is not continuously imbedded in the space $L^q(Q; d^\alpha_{Q(m)})$. The proof is now complete in view of the discussion towards the begining of the proof of this theorem regarding topological isomorphism between spaces.

REMARK. In the particular situations, the results can be obtained when, in Theorem 1, $M = \partial\Omega$ or $M = \{x_0\}$. Some similar results are avilable in the litetrature, e.g., see [2, Theorem 9.3] for $\kappa = 1$ and [5] for $p = q$. Also, the case $p = q$ of Theorem 1 has been studied by Rákosník [9] but he gave only a sufficient condition for the corresponding imbedding to hold.

In view of the above remark, we have the following corollaries of Theorem 1 :

COROLLARY 1. *Let* $1 \leqslant q < p < \infty$, $\alpha, \beta \in \mathbf{R}, \Omega \in Q^{0,1}(\partial\Omega)$. *Then, the imbedding*

$$W^{1,p}(\Omega; d^{\beta}) \hookrightarrow L^q(\Omega; d^{\alpha})$$

holds if and only if

$$\beta > p-1, \qquad \alpha > \beta \frac{q}{p} - \frac{q}{p'} - 1,$$

or

$$-1 \leqslant \beta \leqslant p-1, \qquad \alpha > -1,$$

or

$$\beta \leqslant -1, \qquad \alpha > \beta \frac{q}{p} - \frac{q}{p'} - 1.$$

COROLLARY 2. *Let* $1 \leqslant q < p < \infty$, $\alpha, \beta \in \mathbf{R}, \Omega \in Q^{0,1}(\{x_0\})$, *Then, the imbedding*

$$W^{1,p}(\Omega; d_0^{\beta}) \hookrightarrow L^q(\Omega; d_0^{\alpha})$$

holds if and only if

$$\beta > p-N, \qquad \alpha > (\beta+N-1)\frac{q}{p} - \frac{q}{p'} - N,$$

or

$$-N < \beta \leqslant p-N, \qquad \alpha > -N,$$

or

$$\beta \leqslant -N, \qquad \alpha > (\beta+N-1)\frac{q}{p} - \frac{q}{p'} - N.$$

THEOREM 2. *Let* $1 \leqslant q < p < \infty$, $\alpha, \beta \in \mathbf{R}$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the imbedding*

$$W_M^{1,p}(\Omega; d_M^{\beta}) \hookrightarrow L^q(\Omega; d_M^{\alpha})$$

holds if and only if

$$\beta \in \mathbf{R}, \qquad \alpha > (\beta+N-m-1)\frac{q}{p} - \frac{q}{p'} - N + m.$$

Proof. The proof is analogous to that of Theorem 1 with the observation that as $v \in C_M^{\infty}(Q)$, $v \in AC_L(0,b)$ as well.

THEOREM 3. *Let* $1 \leqslant q < p < \infty$, $\alpha, \beta \in \mathbf{R}$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the imbedding*

$$W_0^{1,p}(\Omega; d_M^{\beta}) \hookrightarrow L^q(\Omega; d_M^{\alpha}) \tag{3.12}$$

holds if and only if

$$\beta \in \mathbf{R}, \qquad \alpha > \beta \frac{q}{p} - \frac{q}{p'} - 1. \tag{3.13}$$

Proof. Let us first assume that (3.13) holds. Using the partition of unity, it is sufficient to prove the result for $V_i (\equiv U_i \cap \Omega)$ instead of Ω and that too only for $i \geqslant 1$. We have

$$\|v\|^q_{q,d^\alpha_{Q(m)};Q} = \int\limits_{(0,1)^{N-1}} \int\limits_0^1 |v(y)|^q \, d^\alpha_{Q(m)}(y) \, dy_l \, dy', \tag{3.14}$$

where $l \in \{m+1, ..., N\}$ and $y' = (y_1, y_2, ..., y_{l-1}, y_{l+1}, ..., y_N)$. It can be proved that for $l \in \{m+1, ..., N\}$ and $y \in Q$, $d_{Q(m)}(y)$ is equivalant to $y_l + \epsilon$, where

$$\epsilon = \left(\sum_{\substack{j=m+1 \\ j \neq l}}^{N} y_j^2 \right)^{1/2}.$$

Thus, there exist constants C_1, $C_2 > 0$, such that

$$C_1(y_l + \epsilon) \leqslant d_{Q(m)}(y) \leqslant C_2(y_l + \epsilon). \tag{3.15}$$

From (3.14), (3.15) and Fubini's Theorem, we get

$$\|v\|^q_{q,d^\alpha_{Q(m)};Q} \leqslant C \int\limits_{(0,1)^{N-1}} \left(\int\limits_0^1 |v(y)|^q \, (y_l + \epsilon)^\alpha \, dy_l \right) dy'.$$

The imbedding (3.12) is now obtained as in the proof of Theorem 1 (with some obvious modifications).

Conversely, let us assume that the condition (3.13) is violated for some α, β. Then, taking $\tau = \epsilon$, $b = 1$, $w_1(t) = t^\alpha$, $w_2(t) = t^\beta$ in Lemma 2, we find that the inequality

$$\left(|u(0)|^q \int_0^\epsilon t^\alpha \, dt + \int_0^1 |u(t)|^q \, (t+\epsilon)^\alpha \, dt \right)^{1/q} \leqslant C \left(\int_0^1 |u'(t)|^p \, (t+\epsilon)^\beta \, dt \right)^{1/p}$$

does not hold. Therefore, in view of Remark 9.2 in [2], there exists a sequence $\{u_n\}$ in $C_0^\infty((0,1))$ such that

$$\int_0^1 |u_n'(t)|^p \, (t+\epsilon)^\beta \, dt = 1, \qquad n \in \mathbf{N}$$

and

$$\int_0^1 |u_n(t)|^q \, (t+\epsilon)^\alpha \, dt \to \infty \quad \text{as} \quad n \to \infty.$$

For $n \in \mathbf{N}$ and $y = (y_1, \dots, y_{l-1}, y_l, y_{l+1}, \dots, y_N) \in \Omega$, define a sequence $\{v_n\}$ as follows :

$$v_n(y) = \begin{cases} \phi(y')u_n(y_l) & , y \in Q \\ 0 & , y \in \Omega \backslash Q \end{cases}$$

where $\phi \in C_0^\infty(\mathbf{R}^{N-1})$ satisfying

$$\begin{cases} 0 \leqslant \phi(z) \leqslant 1 & , z \in \mathbf{R}^{N-1} \\ \phi(z) = 1 & , |z| < \frac{1}{2} \\ \phi(z) = 0 & , |z| > \frac{3}{4} \end{cases}$$

and $y' = (y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_N)$.

If we fix $n \in \mathbf{N}$, then supp $v_n \subset Q$ and there exists a domain G_n such that

$$\operatorname{supp} v_n \subset G_n \subset \bar{G}_n \subset Q.$$

Using (3.15), the estimate $|\phi| \leqslant 1$ and Fubini's theorem, we obtain

$$\begin{aligned} \| v_n \|^p_{p,d^\beta_{Q(m)};Q} &= \int_{Q \cap \operatorname{supp} \phi} |\phi(y')u_n(y_l)|^p d^\beta_{Q(m)}(y)\, dy'\, dy_l \\ &\leqslant \int_0^1 |u_n(y_l)|^p (y_l + \epsilon)^\beta\, dy_l. \end{aligned}$$

The result is now, obtained as in Theorem 1.

REMARK. Results corresponding to Corollaries 1 and 2 can also be obtained in respect of each of the Theorems 2 and 3 which we omit here for conciseness.

REMARK. In what follows, we give the conditions under which the spaces $W^{1,p}(\Omega; d_M^\beta)$, $W_M^{1,p}(\Omega; d_M^\beta)$ or $W_0^{1,p}(\Omega; d_M^\beta)$ are compactly imbedded into the space $L^q(\Omega; d_M^\alpha)$. Suprisingly but true that we obtain the same necessary and sufficient conditions for compactness which were there for the continuous imbeddings of the corresponding spaces. It turns out, in other words, that either the spaces $W^{1,p}(\Omega; d_M^\beta)$, $W_M^{1,p}(\Omega; d_M^\beta)$ and $W_0^{1,p}(\Omega; d_M^\beta)$ are compactly imbedded into the space $L^q(\Omega; d_M^\alpha)$ or they are not even imbedded continuously. The condition $1 \leqslant q < p < \infty$ is playing an important role for this fact.

THEOREM 4. *Let* $1 \leqslant q < p < \infty$, α, $\beta \in \mathbf{R}$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the compact imbedding*

$$W^{1,p}(\Omega; d_M^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d_M^\alpha) \tag{3.16}$$

holds if and only if (3.1) holds.

Proof. Let the imbedding (3.16) hold. Then, it is continuous as well and by Theorem 1, (3.1) holds. To prove the converse, let $\{\Omega_n\}$, $\Omega_n \in Q^{0,1}(M)$, be a sequence of domains such that

$$\left\{x \in \Omega;\ d_M(x) > \frac{1}{n}\right\} \subset \Omega_n \subset \left\{x \in \Omega;\ d_M(x) > \frac{1}{n+1}\right\} \tag{3.17}$$

Clearly, $\Omega_n \subset \Omega_{n+1} \underset{\neq}{\subset} \Omega$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Let us first assume that

$$\beta > p - N + m, \qquad \alpha > (\beta + N - m - 1)\frac{q}{p} - \frac{q}{p'} - N + m. \tag{3.18}$$

Then, there exists $\delta > 0$ such that the numbers β and $\bar{\alpha} := (\alpha - \delta)$ satisfy (3.18) with α replaced by $\bar{\alpha}$. Consequently, by Theorem 1, the continuous imbedding

$$W^{1,p}(\Omega; d_M^{\beta}) \hookrightarrow L^q(\Omega; d_M^{\bar{\alpha}})$$

holds, i.e., there exists $C > 0$ such that for $u \in W^{1,p}(\Omega; d_M^{\beta})$, we have

$$\|u\|_{q,d_M^{\bar{\alpha}};\Omega} \leqslant C\|u\|_{1,p,d_M^{\beta};\Omega}. \tag{3.19}$$

Let $u \in W^{1,p}(\Omega; d_M^{\beta})$. Then, by (3.17) and (3.19), we get

$$\begin{aligned}\|u\|^q_{q,d_M^{\alpha};\Omega^n} &\leqslant \frac{1}{n^{\delta}}\,\|u\|^q_{q,d_M^{\bar{\alpha}};\Omega^n}\\ &\leqslant \frac{C^q}{n^{\delta}}\,\|u\|^q_{1,p,d_M^{\beta};\Omega}.\end{aligned}$$

This gives

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,d_M^{\beta};\Omega}\leqslant 1} \|u\|^q_{q,d_M^{\alpha};\Omega^n} = 0.$$

Now, as $1 \leqslant q < p < \infty$, the local imbeddings (2.4) are guaranteed (*cf.* [4]) and consequently, by Lemma 3, the imbedding (3.16) holds.

Similarly, it can be shown that the imbedding (3.16) also holds if the other two estimates in (3.1) hold.

REMARK. Analogously, it can be proved that the imbeddings in Corollaries 1 and 2 are also compact under the corresponding conditions.

Working on the lines of theorem 4, we can obtain the conditions for the compactness of the spaces $W_M^{1,p}(\Omega; d_M^{\beta})$ and $W_0^{1,p}(\Omega; d_M^{\beta})$ into $L^q(\Omega; d_M^{\alpha})$ and further special cases (corresponding to Corollaries 1 and 2) can be derived.

It is also possible, in certain situations, to extend the continuous imbedding $W^{1,p}(\Omega; d_M^{\beta}) \hookrightarrow L^q(\Omega; d_M^{\alpha})$ to the higher order weighted Sobolev spaces. More precisely, we have

THEOREM 7. *Let* $1 \leqslant q < p < \infty$, $\alpha, \beta \in \mathbf{R}$, $\alpha \geqslant \beta$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, for* $\beta > p - N + m$, *the imbedding*

$$W^{k,p}(\Omega; d_M^{\beta}) \hookrightarrow W^{s,q}(\Omega; d_M^{\alpha})$$

holds if and only if

$$\alpha > (\beta + N - m - 1)\,\frac{q}{p} - \frac{q}{p'} - (k - s - 1)q - N + m,$$

where k *and* s *are non-negative integers.*

Proof. Take $\beta = \beta_0$, $p = p_0$ and let $u \in W^{k,p_0}(\Omega; d_M^{\beta_0})$. Then, using Theorem 1 for functions $D^{\alpha}u$, $|\alpha| = k-1$, we find that, for $\beta_0 > p_0 - N + m$, $1 \leqslant p_1 < p_0 < \infty$, the imbedding

$$W^{k,p_0}(\Omega; d_M^{\beta_0}) \hookrightarrow W^{k-1,p_1}(\Omega; d_M^{\beta_1})$$

holds if and only if

$$\beta_1 > (\beta_0 + N - m - 1)\frac{p_1}{p_0} - \frac{p_1}{p_0'} - N + m. \tag{3.20}$$

Let $\beta_1 \geqslant \beta_0$. Then, $\beta_1 \geqslant \beta_0 > p_0 - N + m > p_1 - N + m$, and, again, by Theorem 1, for $1 \leqslant p_2 < p_1 < \infty$, the imbedding

$$W^{k-1,p_1}(\Omega; d_M^{\beta_1}) \hookrightarrow W^{k-2,p_2}(\Omega; d_M^{\beta_2})$$

holds if and only if

$$\beta_2 > (\beta_1 + N - m - 1)\frac{p_2}{p_1} - \frac{p_2}{p_1'} - N + m. \tag{3.21}$$

From (3.20) and (3.21), we obtain that for $\beta_0 > p_0 - N + m$ and $1 \leqslant p_2 < p_0 < \infty$, the imbedding

$$W^{k,p_0}(\Omega; d_M^{\beta_0}) \hookrightarrow W^{k-2,p_2}(\Omega; d_M^{\beta_2})$$

holds if and only if

$$\beta_2 > (\beta_0 + N - m - 1)\frac{p_2}{p_0} - \frac{p_2}{p_0'} - p_2 - N + m.$$

Further, let $\beta_2 \geqslant \beta_1$. Similar arguments yield that for $\beta_0 > p_0 - N + m$ and $1 \leqslant p_3 < p_0 < \infty$, the imbedding

$$W^{k,p_0}(\Omega; d_M^{\beta_0}) \hookrightarrow W^{k-3,p_3}(\Omega; d_M^{\beta_3})$$

holds if and only if

$$\beta_3 > (\beta_0 + N - m - 1)\frac{p_3}{p_0} - \frac{p_3}{p_0'} - 2p_3 - N + m.$$

Continuing in this way, we obtain that for $\beta_0 > p_0 - N + m$ and $1 \leqslant p_r < p_0 < \infty$, the imbedding

$$W^{k,p_0}(\Omega; d_M^{\beta_0}) \hookrightarrow W^{k-r,p_r}(\Omega; d_M^{\beta_r})$$

holds if and only if

$$\beta_r > (\beta_0 + N - m - 1)\frac{p_r}{p_0} - \frac{p_r}{p_0'} - (r-1)p_r - N + m.$$

Taking $\beta_r = \alpha, p_r = q, k - r = s$, we get that for $\beta > p - N + m$ and $1 \leqslant q < p < \infty$, the imbedding

$$W^{k,p}(\Omega; d_M^{\beta}) \hookrightarrow W^{s,q}(\Omega; d_M^{\alpha})$$

holds if and only if

$$\alpha > (\beta + N - m - 1)\frac{q}{p} - \frac{q}{p'} - (k - s - 1)q - N + m.$$

REMARK. Results in corollaries 1 and 2 can be formulated for higher order Sobolev spaces. The compact imbeddings in higher order Sobolev spaces are also obtained in view of the fact that the composition of a compact map and a continuous map is compact.

4. General weights

In this section, we shall assume that the functions s_0 and s_1 satisfy Property (H). Moreover, throughout the section we assume that $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

THEOREM 8. *Let* $1 \leqslant q < p < \infty$, $0 < b < \infty$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$, $M \subset \partial\Omega$ *be a manifold of dimension* m *and*

$$\int_0^c t^{N-m-1} s_1(t)dt < \infty, \qquad c > 0. \tag{4.1}$$

Then, the imbedding

$$W^{1,p}(\Omega; s_1(d_M)) \hookrightarrow L^q(\Omega; s_0(d_M))$$

holds if and only if

$$\left\{\int_0^b \left[\left(\int_0^\eta t^{N-m-1} s_0(t)\,dt\right)^{1/q} \left(\int_\eta^b \left(t^{N-m-1} s_1(t)\right)^{1-p'} dt\right)^{1/q'}\right]^r \left(\eta^{N-m-1} s_1(\eta)\right)^{1-p'} d\eta\right\}^{1/r} < \infty. \tag{4.2}$$

Proof. In view of the definition of $Q^{0,1}(M)$ and using the fact that the functions s_0 and s_1 satisfy Property (H), there exist constants C_1, C_2, C_3, C_4 such that

$$C_1\|v\|_{q,s_0(d_{Q(m)});Q} \leqslant \|u\|_{q,s_0(d_M);V_i} \leqslant C_2\|v\|_{q,s_0(d_{Q(m)});Q} \tag{4.3}$$

and

$$C_3\|v\|_{1,p,s_1(d_{Q(m)});Q} \leqslant \|u\|_{1,p,s_1(d_M);V_i} \leqslant C_4\|v\|_{1,p,s_1(d_{Q(m)});Q} \tag{4.4}$$

The proof, now, is step by step same as that of Theorem 1 if we use (4.3) and (4.4) instead of (3.2) and (3.3), respectively, and using Lemma 2 with $\tau = 0$, $w_1(t) = t^{\alpha+N-m-1}s_0(t)$, $w_2(t) = t^{\beta+N-m-1}s_1(t)$ instead of $\tau = 0$, $w_1(t) = t^{\alpha+N-m-1}$, $w_2(t) = t^{\beta+N-m-1}$ respectively.

THEOREM 9. *Let* $1 \leqslant q < p < \infty$, $0 < b < \infty$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the imbedding*

$$W^{1,p}_M(\Omega; s_1(d_M)) \hookrightarrow L^q(\Omega; s_0(d_M))$$

holds if and only if (4.1) or

$$\left\{\int_0^b \left[\left(\int_\eta^b t^{N-m-1} s_0(t)\,dt\right)^{1/q} \left(\int_0^\eta \left(t^{N-m-1} s_1(t)\right)^{1-p'} dt\right)^{1/q'}\right]^r \left(\eta^{N-m-1} s_1(\eta)\right)^{1-p'} d\eta\right\}^{1/r} < \infty,$$

hold.

Proof. It follows on the lines of that of Theorem 8 with the observation that as $v \in C_M^\infty(Q)$, $v \in AC_L(0,b)$ as well.

THEOREM 10. *Let* $1 \leqslant q < p < \infty$, $0 < b < \infty$, $0 \leqslant m \leqslant N-1$, $\Omega \in Q^{0,1}(M)$ *and* $M \subset \partial\Omega$ *be a manifold of dimension* m. *Then, the imbedding*

$$W_0^{1,p}\left(\Omega; s_1(d_M)\right) \hookrightarrow L^q\left(\Omega; s_0(d_M)\right)$$

holds if and only if

$$\left\{\int_0^{1+\epsilon}\left[\left(\int_\eta^{1+\epsilon} s_0(t)\,dt\right)^{1/q}\left(\int_0^\eta (s_1(t))^{1-p'}\,dt\right)^{1/q'}\right]^r (s_1(\eta))^{1-p'}\,d\eta\right\}^{1/r} < \infty$$

or

$$\left\{\int_0^{1+\epsilon}\left[\left(\int_0^{\eta} s_0(t)\,dt\right)^{1/q}\left(\int_\eta^{1+\epsilon} (s_1(t))^{1-p'}\,dt\right)^{1/q'}\right]^r (s_1(\eta))^{1-p'}\,d\eta\right\}^{1/r} < \infty$$

hold, where the number ϵ *is as given in the proof of Theorem 3.*

Proof. It can be shown, as in Theorem 3, that $d_{Q(m)}(y)$ is equivalant to $y_l+\epsilon$. Since functions s_0 and s_1 satisfy Property (H), there exist constants C_1, C_2, C_3, $C_4 > 0$ such that

$$C_1\, s_0(y_l+\epsilon) \leqslant s_0\left(d_{Q(m)}(y)\right) \leqslant C_2\, s_0(y_l+\epsilon)$$

and

$$C_3\, s_1(y_l+\epsilon) \leqslant s_1\left(d_{Q(m)}(y)\right) \leqslant C_4\, s_1(y_l+\epsilon).$$

Now, we argue as in Theorem 8 to get the result.

Let us, now, mention certain specific situations by way of examples which suit Theorems 8, 9 or 10.

EXAMPLE 1. Take either

$$s_0(t) = \alpha\, t^{q-N+m}, \qquad s_1(t) = \beta^{(1-p)}\, t^{(1-p)(1-q')-N+m+1} \tag{4.5}$$

with $2 < q < p < \infty$, $\alpha, \beta > 0$, or

$$s_0(t) = \alpha\, t^{q-N+m}, \qquad s_1(t) = \beta^{(1-p)}\, t^{q'-pq'+p-N+m} \tag{4.6}$$

with $1 < q < p < \infty$, $\alpha, \beta > 0$.

Each of the above functions satisfies Property (H). Also, the weights s_0 and s_1 in (4.5) satisfy conditions of Theorems 8 and 9. Consequently, we have the following corresponding continuous imbeddings:

$$W^{1,p}\left(\Omega; \beta^{(1-p)}\, t^{(1-p)(1-q')-N+m+1}\right) \hookrightarrow L^q\left(\Omega; \alpha\, t^{q-N+m}\right)$$

$$W_M^{1,p}\left(\Omega;\beta^{(1-p)}\,t^{(1-p)(1-q')-N+m+1}\right)\hookrightarrow L^q\left(\Omega;\alpha\,t^{q-N+m}\right)$$

Further, we note that s_1 in (4.6) does not satisfy (4.1) while both the weights in (4.6) satisfy conditions of Theorem 9. Therefore, the existence of the imbedding

$$W^{1,p}\left(\Omega;\beta^{(1-p)}\,t^{q'-pq'+p-N+m}\right)\hookrightarrow L^q\left(\Omega;\alpha\,t^{q-N+m}\right)$$

is not guaranteed while the imbedding

$$W_M^{1,p}\left(\Omega;\beta^{(1-p)}\,t^{q'-pq'+p-N+m}\right)\hookrightarrow L^q\left(\Omega;\alpha\,t^{q-N+m}\right)$$

holds.

EXAMPLE 2. Take either

$$s_0(t)=\alpha\,t^{q-1},\qquad s_1(t)=\beta^{(1-p)}\,t^{(1-p)(1-q')}$$

with $2<q<p<\infty$, $\alpha,\beta>0$, or

$$s_0(t)=\alpha\,t^{q-1},\qquad s_1(t)=\beta^{(1-p)}\,t^{(1-p)(q'-1)}$$

with $1<q<p<\infty$, $\alpha,\beta>0$.

In this case, conditions of Theorem 10 are fulfilled and we have the corresponding continuous imbeddings.

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