Mathematical
Inequalities
& Applications
Volume 6, Number 1 (2003), 125–132

A NEW CLASS OF GENERALIZED NONLINEAR MIXED QUASI–VARIATIONAL INEQUALITIES IN BANACH SPACES

NAN-JING HUANG, YA-PING FANG AND YEOL JE CHO

(*communicated by R. P. Agarwal*)

Abstract. In this paper, we introduce and study a new class of generalized nonlinear mixed quasi-variational inequalities. Using the KKM technique, we prove the existence and uniqueness of solutions for this class of generalized nonlinear mixed quasi-variational inequalities in reflexive Banach spaces. Our results include the main results of Verma [1], [2] as special cases.

1. Introduction and preliminaries

Variational inequality theory has become a rich source of inspiration in pure and applied mathematics. Variational inequalities not only have stimulated the new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. In recent years, variational inequalities have been generalized and applied in various directions. For details, we refer to [1]–[13] and the references therein.

Recently, Verma [2] introduced and studied a class of monotone onlinear variational inequalities, and proved the existence and uniqueness of solutions for this kind of variational inequalities.

In this paper, a new class of generalized nonlinear mixed quasi-variational inequalities are introduced and studied. Also, the solvability of this class of generalized nonlinear mixed quasi-variational inequalities in reflexive Banach spaces is given. Our results improve and extend the corresponding results of [1] and [2].

Throughout this paper, let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty convex closed subset of X. Denote $\langle \omega, x\rangle = \omega(x)$ for all $\omega \in X^*$ and $x \in X$. Let $S, T : K \to X^*$, $M : X^* \times X^* \to X^*$, $\eta : K \times K \to K$ be four mappings and $f : K \to R \cup \{+\infty\}$ be a proper convex functional. Suppose that η is affine with respect to the first argument satisfying $\eta(u, v) = -\eta(v, u)$ for all $u, v \in K$. It is clear that $\eta(u, u) = 0$ for all $u \in K$. We consider the following problem:

Mathematics subject classification (2000): 47H06, 49J30, 49J40.

Key words and phrases: Generalized nonlinear mixed quasi-variational inequality, Generalized monotone type mapping, Generalized Lipschitz type mapping, the KKM mapping.

The third author was supported from the Korea Science and Engineering Foundation (KOSEF 2000-1-10100-003-3).

Paper MIA-06-12

For any $\omega \in X^*$, find $u \in K$ such that

$$\langle M(Su,Tu) - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant 0 \tag{1.1}$$

for all $v \in K$.

If $M(x,y) = x - y$, then the problem (1.1) is equivalent to the following problem: For any $\omega \in X^*$, find $u \in K$ such that

$$\langle Su - Tu - \omega, \eta(v,u)\rangle + f(v) - f(u) \geqslant 0 \tag{1.2}$$

for all $v \in K$.

If $\eta(u,v) = gu - gv$, then the problem (1.1) is equivalent to the following problem: For any $\omega \in X^*$, find $u \in K$ such that

$$\langle M(Su,Tu) - \omega, gv - gu\rangle + f(v) - f(u) \geqslant 0 \tag{1.3}$$

for all $v \in K$, where $g : K \to K$ is an affine mapping.

REMARK 1.1. For a suitable choice of S, T, M, η and f, the problem (1.1) includes many kinds of known variational inequalities as special cases (see [1], [2] and the references therein).

In the sequel, we give some definitions.

DEFINITION 1.1. A mapping $S : K \to X^*$ is said to be *η-φ-p monotone* with respect to the first argument of $M : X^* \times X^* \to X^*$ if there exist mappings $\varphi : [0,+\infty) \to [0,+\infty)$, $\eta : K \times K \to K$ and a constant $p > 1$ such that

$$\langle M(Su,\cdot) - M(Sv,\cdot), \eta(u,v)\rangle \geqslant \varphi(\|\eta(u,v)\|)\|\eta(u,v)\|^p \tag{1.4}$$

for all $u, v \in K$.

DEFINITION 1.2. A mapping $S : K \to X^*$ is said to be *η-φ-p monotone* if there exist mappings $\varphi : [0,+\infty) \to [0,+\infty)$, $\eta : K \times K \to K$ and a constant $p > 1$ such that

$$\langle Su - Sv, \eta(u,v)\rangle \geqslant \varphi(\|\eta(u,v)\|)\|\eta(u,v)\|^p \tag{1.5}$$

for all $u, v \in K$.

DEFINITION 1.3. A mapping $S : K \to X^*$ is said to be *g-φ-p monotone* with respect to the first argument of M if there exist $\varphi : [0,+\infty) \to [0,+\infty)$, $g : K \to K$ and a constant $p > 1$ such that

$$\langle M(Su,\cdot) - M(Sv,\cdot), gu - gv\rangle \geqslant \varphi(\|gu - gv\|)\|gu - gv\|^p \tag{1.6}$$

for all $u, v \in K$.

REMARK 1.2. For a suitable choice of M, S, φ, η and p, Definition 1.1 includes many known definitions of monotone type mappings as special cases (see [1], [2]).

DEFINITION 1.4. A mapping $T : K \to X^*$ is said to be *η-ψ-p relaxed monotone* with respect to the second argument of $M : X^* \times X^* \to X^*$ if there exist mappings $\psi : [0,+\infty) \to [0,+\infty)$, $\eta : K \times K \to K$ and a constant $p > 1$ such that

$$\langle M(\cdot,Tu) - M(\cdot,Tv), \eta(u,v)\rangle \geqslant -\psi(\|\eta(u,v)\|)\|\eta(u,v)\|^p \tag{1.7}$$

for all $u,v \in K$.

DEFINITION 1.5. A mapping $T : K \to X^*$ is said to be η-ψ-p *Lipschitzian* if there exist mappings $\psi : [0,+\infty) \to [0,+\infty), \eta : K \times K \to K$ and a constant $p > 1$ such that

$$\langle Tu - Tv, \eta(u,v)\rangle \leqslant \psi(\|\eta(u,v)\|)\|\eta(u,v)\|^p \tag{1.8}$$

for all $u,v \in K$.

DEFINITION 1.6. A mapping $T : K \to X^*$ is said to be g-ψ-p *relaxed monotone* with respect to the second argument of $M : X^* \times X^* \to X^*$ if there exist mappings $\psi : [0,+\infty) \to [0,+\infty), g : K \to K$ and a constant $p > 1$ such that

$$\langle M(\cdot,Su) - M(\cdot,Sv), gu - gv\rangle \geqslant -\psi(\|gu - gv\|)\|gu - gv\|^p \tag{1.9}$$

for all $u,v \in K$.

REMARK 1.3. For a suitable choice of T, M, η, ψ and p, Definition 1.4 includes many known definitions of Lipschitzian type and relaxed monotone type mappings as special cases (see [1], [2]).

DEFINITION 1.7. A mapping $S : K \to X^*$ is said to be *hemicontinuous* if, for all $x,y,z \in K$, a mapping $t \mapsto \langle S(x+ty), z\rangle$ is continuous on $[0,1]$.

2. Main results

Now, we give our main results in this paper.

THEOREM 2.1. *Let X be a real reflexive Banach space, X^* be the its dual space and K be a nonempty convex closed subset of X. Let $S,T : K \to X^*$ be two hemicontinuous mappings satisfying (1.4) and (1.7), respectively, where mappings $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ satisfy $\varphi(t) > \psi(t)$ for all $t \geqslant 0$, and there exists a constant $\delta > 0$ such that $\varphi - \psi$ is bounded on $[0,\delta]$. In addition, suppose that $\eta(u,v) = -\eta(v,u)$ for all $u,v \in K$, $\eta : K \times K \to K$ is affine with respect to the first argument, $M : X^* \times X^* \to X^*$ is continuous with respect to the weak* topology of X and $f : K \to R \cup \{+\infty\}$ is a proper convex function. Then, for any $\omega \in X^*$, $u \in K$ is a solution of the problem (1.1) if and only if $u \in K$ is a solution of the following problem:*

Find $u \in K$ such that

$$\begin{aligned}&\langle M(Sv,Tv) - \omega, \eta(v,u)\rangle + f(v) - f(u)\\ &\quad \geqslant (\varphi(\|\eta(v,u)\|) - \psi(\|\eta(v,u)\|))\|\eta(v,u)\|^p\end{aligned} \tag{2.1}$$

for all $v \in K$.

Proof. Suppose that the problem (1.1) holds. Since the mappings S and T satisfy (1.4) and (1.7), respectively, then, for all u, v in K, we have

$$\begin{aligned}
&\langle M(Sv, Tv) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\
&= \langle -\omega, \eta(v, u)\rangle + \langle M(Su, Tv) - M(Sv, Tv), \eta(u, v)\rangle \\
&\quad + \langle M(Su, Tu) - M(Su, Tv), \eta(u, v)\rangle + f(v) - f(u) \\
&\quad - \langle M(Su, Tu), \eta(u, v)\rangle \\
&\geqslant \langle M(Su, Tu) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\
&\quad + (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|))\|\eta(v, u)\|^p.
\end{aligned}$$

Since $u \in K$ is a solution of the problem (1.1), this implies that

$$\langle M(Su, Tu) - \omega, \eta(v, u)\rangle + f(v) - f(u) \geqslant 0$$

for all $v \in K$. Thus, we have

$$\begin{aligned}
&\langle M(Sv, Tv) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\
&\geqslant (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|))\|\eta(v, u)\|^p
\end{aligned}$$

for all $v \in K$, i.e., (2.1) is true.

Conversely, suppose that (2.1) holds. Without loss of generality, choose a point $v \in K$ such that $f(v) < +\infty$ and so $f(u) < +\infty$. Letting $v_t = (1-t)u + tv$, then $v_t \in K$ and $v_t - u = t(v - u)$, where $t \in [0, 1]$. Further, since $\eta : K \times K \to K$ is affine with respect to the first argument and $\eta(u, u) = 0, \eta(u, v) = -\eta(v, u)$ for all $u, v \in K$, we obtain $\eta(v_t, u) = t\eta(v, u)$. It follows from (2.1) that

$$\begin{aligned}
&\langle M(Sv_t, Tv_t) - \omega, \eta(v_t, u)\rangle + f(v_t) - f(u) \\
&= t\langle M(Sv_t, Tv_t) - \omega, \eta(v, u)\rangle + f(v_t) - f(u) \\
&\geqslant (\varphi(\|\eta(v_t, u)\|) - \psi(\|\eta(v_t, u)\|))\|\eta(v_t, u)\|^p \\
&= t^p(\varphi(t\|\eta(v, u)\|) - \psi(t\|\eta(v, u)\|))\|\eta(v, u)\|^p.
\end{aligned} \tag{2.2}$$

Since f is convex and $\eta(v_t, u) = t\eta(v, u)$, from (2.2), it follows that

$$\begin{aligned}
&\langle M(Sv_t, Tv_t) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\
&\geqslant t^{p-1}(\varphi(t\|\eta(v, u)\|) - \psi(t\|\eta(v, u)\|))\|\eta(v, u)\|^p.
\end{aligned}$$

By the continuity of M, the hemicontinuity of S, T and the boundedness of $\varphi - \psi$ on $[0, \delta]$, we have

$$\langle M(Su, Tu) - \omega, \eta(v, u)\rangle + f(v) - f(u) \geqslant 0$$

for all $v \in K$. This completes the proof.

If $M(x, y) = x - y$, then Theorem 2.1 reduces to the following:

THEOREM 2.2. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty convex closed subset of X. Let $S,T : K \to X^*$ be two hemicontinuous mappings satisfying (1.5) and (1.8), respectively. Suppose that the mappings φ, ψ, η, f are the same as in Theorem 2.1. Then $u \in K$ is a solution of the problem (1.2) if and only if $u \in K$ is a solution of the following problem:*

For any $\omega \in X^*$, find $u \in K$ such that

$$\begin{aligned} &\langle Sv - Tv - \omega, \eta(v,u)\rangle + f(v) - f(u) \\ &\geqslant (\varphi(\|\eta(v,u)\|) - \psi(\|\eta(v,u)\|))\|\eta(v,u)\|^p \end{aligned} \tag{2.3}$$

for all $v \in K$.

If $\eta(u,v) = gu - gv$, where $g : K \to K$ is an affine mapping, then Theorem 2.1 reduces to the following:

THEOREM 2.3. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty convex closed subset of X. Let $S,T : K \to X^*$ be two hemicontinuous mappings satisfying (1.6) and (1.9), respectively. Suppose that the mappings φ, ψ, f, M are the same as in Theorem 2.1. If $g : K \to K$ is an affine mapping, then $u \in K$ is a solution of the problem (1.3) if and only if $u \in K$ is a solution of the following problem:*

Find $u \in K$ suth that

$$\begin{aligned} &\langle M(Sv,Tv) - \omega, gv - gu\rangle + f(v) - f(u) \\ &\geqslant (\varphi(\|gv-gu\|) - \psi(\|gv-gu\|))\|gv-gu\|^p \end{aligned} \tag{2.4}$$

for all $v \in K$.

REMARK 2.1. Theorems 2.1$\sim$2.3 improve and extend Theorem 2.1 of [1] and Thereom 2.1 of [2].

In the sequel, we need the following definition and lemma for our further results.

DEFINITION 2.1. A mapping $F : X \to 2^X$ is said to be a *KKM mapping* if, for any $\{x_1, x_2, \ldots, x_n\} \subset X$,

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(X_i).$$

LEMMA 2.1. [14] Let K be a nonempty subset of a topological vector space E and $F : K \to 2^E$ be a *KKM* mapping. If $F(x)$ is closed in E for every x in K and there exists at least a point $x_0 \in K$ such that $F(x_0)$ is compact, then

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

THEOREM 2.4. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty bounded closed convex subset of X. Let S, T, M, φ, ψ and η be the same as in Theorem 2.1. Suppose that $\varphi - \psi$, η are continuous and $f : K \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous. Then the problem (1.1) has a solution. Moreover, if $\eta(u, v) = 0$ implies $u = v$, then the problem (1.1) has unique solution.*

Proof. We first prove the existence of solution of the problem (1.1). Define the set-valued mappings $F, G : K \to 2^K$ by

$$F(v) = \{u \in K : \langle M(Su, Tu) - \omega, \eta(v, u)\rangle + f(v) - f(u) \geqslant 0\}$$

and

$$G(v) = \{u \in K : \langle M(Sv, Tv) - \omega, \eta(v, u)\rangle + f(v) - f(u) \geqslant (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|))\|\eta(v, u)\|^p\}$$

for all $v \in K$, respectively. We show that F is a KKM mapping. Assume that F is not a KKM mapping. Then there exist $\{v_1, v_2, \cdots, v_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^n t_i = 1, \quad v = \sum_{i=1}^n t_i v_i \notin \bigcup_{i=1}^n F(v_i).$$

By the definition of F, we have

$$\langle M(Sv, Tv) - \omega, \eta(v_i, v)\rangle + f(v_i) - f(v) < 0$$

for $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned} 0 &= \langle M(Sv, Tv) - \omega, \eta(v, v)\rangle = \langle M(Sv, Tv) - \omega, \eta(\sum_{i=1}^n t_i v_i, v)\rangle \\ &= \sum_{i=1}^n t_i \langle M(Sv, Tv) - \omega, \eta(v_i, v)\rangle < \sum_{i=1}^n t_i (f(v) - f(v_i)) \\ &= f(v) - \sum_{i=1}^n t_i f(v_i) \leqslant f(v) - f(v) = 0, \end{aligned}$$

which is a contradiction. This implies that F is a KKM mapping. Now we prove that $F(v) \subset G(v)$ for all $v \in K$. Let $u \in F(v)$. Since the mappings S and T satisfy (1.4) and (1.7), respectively, we have

$$\begin{aligned} &\langle M(Sv, Tv) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\ &= \langle M(Su, Tv) - M(Sv, Tv), \eta(u, v)\rangle \\ &\quad + \langle M(Su, Tu) - M(Su, Tv), \eta(u, v)\rangle \\ &\quad + \langle -\omega, \eta(v, u)\rangle - \langle M(Su, Tu), \eta(u, v)\rangle + f(v) - f(u) \\ &\geqslant (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|))\|\eta(v, u)\|^p \\ &\quad + \langle M(Su, Tu) - \omega, \eta(v, u)\rangle + f(v) - f(u) \\ &\geqslant (\varphi(\|\eta(v, u)\|) - \psi(\|\eta(v, u)\|))\|\eta(v, u)\|^p, \end{aligned}$$

for all $v \in K$. This implies that $u \in G(v)$, and so G is also a KKM mapping. It follows from Theorem 2.1 that

$$\bigcap_{v\in K} F(v) = \bigcap_{v\in K} G(v).$$

From the assumption, it follows that $G(v)$ is closed for all v in K. Since K is bounded closed convex, we know that K is weakly compact and so $G(v)$ is weakly compact in K for all $v \in K$. It follows from Lemma 2.6 that

$$\bigcap_{v\in K} F(v) = \bigcap_{v\in K} G(v) \neq \emptyset.$$

Hence, there exists a point $u_0 \in K$ such that

$$\langle M(Su_0, Tu_0) - \omega, \eta(v, u_0)\rangle + f(v) - f(u_0) \geqslant 0$$

for all $v \in K$.

To show the uniqueness of the solution, letting $u_1, u_2 \in K$ be two solutions of the problem (1.1), then we have

$$\langle M(Su_1, Tu_1) - \omega, \eta(v, u_1)\rangle + f(v) - f(u_1) \geqslant 0 \tag{2.5}$$

and

$$\langle M(Su_2, Tu_2) - \omega, \eta(v, u_2)\rangle + f(v) - f(u_2) \geqslant 0 \tag{2.6}$$

for all $v \in K$. Setting $v = u_2$ in (2.5) and $v = u_1$ in (2.6) and adding, we have

$$\langle M(Su_1, Tu_1) - M(Su_2, Tu_2), \eta(u_2, u_1)\rangle \geqslant 0. \tag{2.7}$$

By (1.4) and (1.7), we obtain

$$\begin{aligned}
&\langle M(Su_1, Tu_1) - M(Su_2, Tu_2), \eta(u_2, u_1)\rangle \\
&= \langle M(Su_1, Tu_1) - M(Su_2, Tu_1), \eta(u_2, u_1)\rangle \\
&\quad + \langle M(Su_2, Tu_1) - M(Su_2, Tu_2), \eta(u_2, u_1)\rangle \\
&= -\langle M(Su_1, Tu_1) - M(Su_2, Tu_1), \eta(u_1, u_2)\rangle \\
&\quad - \langle M(Su_2, Tu_1) - M(Su_2, Tu_2), \eta(u_1, u_2)\rangle \\
&\leqslant -(\varphi(\|\eta(u_1, u_2)\|) - \psi(\|\eta(u_1, u_2)\|))\|\eta(u_1, u_2)\|^p.
\end{aligned} \tag{2.8}$$

Since $\varphi(t) > \psi(t)$ for all $t > 0$, it follows from (2.7) and (2.8) that

$$\|\eta(u_1, u_2)\|^p = 0.$$

Furthermore, since $\eta(u, v) = 0$ implies that $u = v$, we have $u_1 = u_2$. This completes the proof.

From Theorem 2.4, we have the following theorems:

THEOREM 2.5. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty bounded closed convex subset of X. Let $S, T, \varphi, \psi, \eta$ and f be the same as in Theorem 2.4. Then the set of the solutions of the problem* (1.2) *is not empty. In addition, if $\eta(u, v) = 0$ implies that $u = v$, then the problem* (1.2) *has unique solution.*

THEOREM 2.6. *Let X be a real reflexive Banach space, X^* be its dual space and K be a nonempty bounded closed convex subset of X. Let the mappings S, T, M, φ, ψ and f be the same as in Theorem 2.3. If $g : K \to K$ is a continuous mapping, then the set of the solutions of the problem (1.3) is not empty. In addition, if $g(u) = g(v)$ implies that $u = v$, then the problem (1.3) has a unique solution.*

REMARK 2.2. Theorems 2.4 $\sim$ 2.6 improve and extend Theorem 2.4 of [2].

REFERENCES

[1] R. U. VERMA, *Nonlinear variational inequalities on convex subsets of Banach spaces*, Appl. Math. Lett. **10(4)** (1997), 25–27.

[2] R. U. VERMA, *On monotone nonlinear variational inequality problems*, Comment. Math. Univ. Carolinae **39(1)** (1998), 91–98.

[3] R. U. VERMA, *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, Z. Angwe. Math. Mech. **77(5)** (1997), 387–391.

[4] J. C. YAO, *Applications of variational inequalities to nonlinear analysis*, Appl. Math. Lett. **4(4)** (1991), 89–92.

[5] F. E. BROWDER, *On the unification of the calculus of variations and the theory of monotone nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 419–425.

[6] F. E. BROWDER AND W. V. PETRYSHYN, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[7] E. ZEIDLER, *Nonlinear functional analysis and its applications IV*, Springer-Verlag, New York, 1988.

[8] N. J. HUANG, *Generalized nonlinear variational inclusions with noncompact valued mapping*, Appl. Math. Lett. **9(3)** (1996), 25–29.

[9] N. J. HUANG, *On the generalized implicit quasi-variational inequalities*, J. Math. Anal. Appl. **216** (1997), 197–210.

[10] N. J. HUANG, *A new completely general class of variational inclusions with noncompact valued mapping*, Comput. Math. Appl. **35(10)** (1998), 9–14.

[11] M. A. NOOR, *Mixed variational inequalities*, Appl. Math. Lett. **3** (1990), 73–75.

[12] M. A. NOOR, *General auxiliary principle for variational inequalities*, PanAmerica Math. **4(1)** (1994), 27–44.

[13] A. H. SIDDIQI, Q. H. ANSARI AND K. R. KAZMI, *On nonlinear variational inequalities*, Indian J. Pure Appl. Math. **25(9)** (1994), 969–973.

[14] K. FAN, *Some properties of convex sets related to fixed point theorem*, Math. Annal. **266** (1984), 519–537.

(Received January 6, 2001)

Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P. R. China

Ya-ping Fang
Department of Mathematics
Sichuan University
Chengdu
Sichuan 610064
P. R. China

Yeol Je Cho
Department of Mathematics
Gyeongsang National University
Chinju 660–701
Korea