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NONLINEAR GENERALIZED QUASI–VARIATIONAL INEQUALITIES: A FIXED POINT APPROACH

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Abstract. The existence of solutions to quasi-variational inequalities is established using recent fixed point theory.

1. Introduction

In this paper we study quasi-variational inequalities. The existence of solutions of such inequalities will follow from the fixed point theory in the literature due to Agarwal and O'Regan [1, 2], O'Regan [9–11] and Park [12]. This paper was motivated by the very interesting paper of Chen and Park [5] which generalized most known results on variational inequalities in the literature [4, 8, 13]. Moreover since the results in this paper extend those in [5], our theory will include the Hartman-Stampacchia inequality [6] and many more (see [5] for a partial list).

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Let C be a nonempty, convex subset of a Hausdorff topological vector space X. Recall a *polytope* P in C is any convex hull of a nonempty finite subset of C. Of particular importance in this paper will be the class $\mathscr{U}_c^k$ (see [12]). X and Y are Hausdorff topological vector spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. A class $\mathscr{U}$ of maps is defined by the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single valued continuous functions;
(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued; and
(iii) for any polytope P, $F \in \mathscr{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathscr{U}$.

DEFINITION 1.1. $F \in \mathscr{U}_c^k(X,Y)$ if for any compact subset K of X, there is a $G \in \mathscr{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

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DEFINITION 1.2. We say $G \in \mathscr{B}(X,Y)$ if $G : X \to 2^Y$ is such that for any polytope P in X and any continuous function $g : G(P) \to P$, the composition $g\,(G|_P) : P \to 2^P$ has a fixed point.

DEFINITION 1.3. $F \in \mathscr{B}^k(X,Y)$ (i.e. F is $\mathscr{B}^k$-admissible) if $F : X \to 2^Y$ is such that for any compact, convex subset K of X, there exists a closed map $G \in \mathscr{B}(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

A nonempty subset X of a Hausdorff topological vector space E is said to be *admissible* if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h : K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is said to be q-*admissible* if any nonempty compact, convex subset Ω of X is admissible.

In [12] Park proved the following result for $\mathscr{B}^k$-admissible maps (see [12 pp. 807] for an elementary proof).

THEOREM 1.1. *Let E be a Hausdorff topological vector space and X an admissible, convex subset of E. Then any compact map $F \in \mathscr{B}^k(X,X)$ has a fixed point.*

In [10] O'Regan established the following Furi-Pera type result for $\mathscr{U}_c^k$ maps.

THEOREM 1.2. *Let Q be a closed convex subset of a metrizable locally convex topological vector space E with $0 \in Q$. Suppose $F \in \mathscr{U}_c^k(Q,E)$ is a closed, compact map with the following condition satisfied:*

$$(1.1)\qquad \begin{cases} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x,\lambda) \text{ with } x \in \lambda\, F(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then} \\ \{\lambda_j\, F(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F has a fixed point in Q.

In [9] O'Regan established the following fixed point result for noncompact $\mathscr{B}^k$ maps (see also [9] for more general results).

THEOREM 1.3. *Let Ω be a q-admissible, closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F \in \mathscr{B}^k(\Omega,\Omega)$ with the following property holding:*

$$(1.2)\qquad A \subseteq \Omega,\ A = \overline{co}\,(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.}$$

Then F has a fixed point in Ω.

Let (E,d) be a pseudometric space. For $S \subseteq E$, let $B(S,\epsilon) = \{x \in E : d(x,S) \leqslant \epsilon\}$, $\epsilon > 0$, where $d(x,S) = \inf_{y \in Y} d(x,y)$. The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A,\epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F : S \to 2^E$; here $S \subseteq E$. The map F is said to be a countably P-concentrative mapping if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \leqslant \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e. X is not precompact in the pseudonormed space (E,p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (E,p).

In [2, 11] we established the following result for countably P-concentrative $\mathscr{U}_c^k$ maps.

THEOREM 1.4. *Let Ω be a nonempty, closed, convex subset of a Fréchet space E (P is a defining system of seminorms). Suppose $F \in \mathscr{U}_c^k(\Omega,\Omega)$ is a countably P-concentrative mapping. Then F has a fixed point in Ω.*

Finally for completeness we also give the definition of countably condensing maps. Let X be a metric space and $P_B(X)$ the bounded subsets of X. The Kuratowskii measure of noncompactness is the map $\alpha : P_B(X) \to [0,\infty)$ defined by

$$\alpha(A) = \inf\{\epsilon > 0 : \; A \subseteq \cup_{i=1}^n X_i \;\text{ and }\; \operatorname{diam}(X_i) \leqslant \epsilon\};$$

here $A \in P_B(X)$. Let S be a nonempty subset of X and let $H : S \to 2^X$. H is called countably condensing if $H(S)$ is bounded, $\alpha(H(\Omega)) \leqslant \alpha(\Omega)$ for all countably bounded sets Ω of S and $\alpha(H(\Omega)) < \alpha(\Omega)$ for all countably bounded sets Ω of S with $\alpha(\Omega) \neq 0$.

We now state a Furi-Pera type result [2, 11] for countably bounded $\mathscr{U}_c^k$ maps in Hilbert spaces.

THEOREM 1.5. *Let Q be a closed convex subset of a Hilbert space H with $0 \in Q$. Suppose $F \in \mathscr{U}_c^k(Q,H)$ is a closed countably condensing map. In addition assume the following condition holds:*

$$\left\{\begin{array}{l} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x,\lambda) \text{ with } x \in \lambda\, F(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then} \\ \{\lambda_j\, F(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{array}\right. \tag{1.3}$$

Then F has a fixed point in Q.

2. Quasi-variational inequalities

We begin this section by expressing Theorem 1.1 as a variational inequality type theorem. Then two general results will be deduced from our main theorem.

THEOREM 2.1. *Let E and Y be Hausdorff topological vector spaces, Q a convex subset of E, $G : Q \to K(Q)$ (nonempty compact subsets of Q) and $T : Q \to 2^C$ where C is a convex subset of Y. In addition assume the following conditions hold:*

$$f : Q \times C \times Q \to \mathbf{R} \text{ is a upper semicontinuous function} \tag{2.1}$$

$$G \text{ and } T \text{ are compact maps} \tag{2.2}$$

(2.3) $$Q \times C \text{ is an admissible subset of } E \times Y$$

and

(2.4) $$F \in \mathscr{B}^k(Q \times C, Q \times C);$$

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

If in addition

(2.5) $$f(x,y,x) \leqslant 0 \text{ for all } (x,y) \in Q \times C,$$

then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant 0 \text{ for all } z \in G(x_0).$$

Proof. Now [3 pp. 44] guarantees that $\Phi(x,y)$ is nonempty (and compact) for each $(x,y) \in Q \times C$. As a result $F : Q \times C \to 2^{Q \times C}$ and also F is compact since $F(Q \times C) \subseteq G(Q) \times T(Q)$. Now (2.3), (2.4) and Theorem 1.1 guarantee that there exists $(x_0, y_0) \in Q \times C$ with $(x_0, y_0) \in \Phi(x_0, y_0) \times T(x_0)$. That is there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $x_0 \in T(x_0)$ and $f(x_0, y_0, x_0) = M(x_0, y_0)$ (i.e. $f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0)$ for all $z \in G(x_0)$), so we are finished the first part. For the second part assume (2.5) holds, and so the result is immediate from the first part. □

Our next result is for a certain subclass of the $\mathscr{B}^k$ maps, namely the acyclic maps. If X and Y are subsets of Hausdorff topological vector spaces then we say $F \in \mathscr{V}(X,Y)$ if F is upper semicontinuous with nonempty compact acyclic values; of course [12] we know $\mathscr{V}(X,Y) \subseteq \mathscr{B}^k(X,Y)$.

COROLLARY 2.2. *Let* E *and* Y *be Hausdorff topological vector spaces,* Q *a convex subset of* E, $G : Q \to K(Q)$ *and* $T : Q \to K(C)$ *where* C *is a convex subset of* Y. *Suppose* (2.1), (2.2) *and* (2.3) *hold and in addition assume the following conditions are satisfied:*

(2.6) $$G : Q \to 2^Q \text{ is upper semicontinuous}$$

(2.7) $$\begin{cases} M : Q \times C \to Q \text{ is lower semicontinuous} \\ (\text{here } M(x,y) = \max_{w \in G(x)} f(x,y,w)) \end{cases}$$

(2.8) $$T : Q \to 2^C \text{ is upper semicontinuous with acyclic values}$$

and

$$\Phi : Q \times C \to 2^Q \quad \textit{has acyclic values;} \tag{2.9}$$

here

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}.$$

Then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0) \ \textit{for all} \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0, y_0, z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. The result follows from Theorem 2.1 once we show (2.4) holds. In fact we will show $F \in \mathscr{V}(Q \times C, Q \times C)$; here $F(x,y) = \Phi(x,y) \times T(x)$. Clearly $F : Q \times C \to 2^{Q \times C}$ has nonempty, compact, acyclic values from Künneth's theorem, and so it remains to show F is upper semicontinuous. If we show Φ is upper semicontinuous then [3 pp. 472] guarantees that F is upper semicontinuous. To show Φ is upper semicontinuous it suffices ([3 pp. 465], note Φ is compact) to show Φ is closed. Let $\{(x_\alpha, y_\alpha, w_\alpha)\}$ be a net in *graph* (Φ) with $(x_\alpha, y_\alpha, w_\alpha) \to (x, y, w)$. Notice (2.1), (2.2), (2.6) and [3 pp. 473] implies M is upper semicontinuous. This together with (2.7) implies M is continuous. As a result

$$f(x,y,w) \geqslant \limsup f(x_\alpha, y_\alpha, w_\alpha) = \limsup M(x_\alpha, y_\alpha) = M(x,y).$$

Also $w_\alpha \in G(x_\alpha)$ together with $x_\alpha \to x$, $w_\alpha \to w$ and G upper semicontinuous (so G is closed [3 pp. 465]) implies $w \in G(x)$ and $f(x,y,w) \geqslant M(x,y)$. Consequently $f(x,y,w) = M(x,y)$, so $(x,y,w) \in$ *graph* (Φ). □

REMARK 2.1. If (2.7) in Corollary 2.2 is replaced by

$$G \ \text{ is lower semicontinuous} \tag{2.10}$$

and

$$f : Q \times C \times Q \to \mathbf{R} \ \text{ is a lower semicontinuous function,} \tag{2.11}$$

(so G is continuous and f is continuous) then the result in Corollary 2.2 is again true since [3 pp. 472] guarantees that (2.7) is true.

It is easy to generalize Corollary 2.2 for a subclass $\mathscr{A}$ of $\mathscr{B}^k$. If X and Y are subsets of Hausdorff topological vector spaces then we say $F \in \mathscr{A}(X,Y)$ if $F \in \mathscr{B}^k(X,Y)$ and is upper semicontinuous with nonempty compact values and satisfies property (C) (to be specified in the applications considered). Also we assume for subsets X_1 and X_2 of Hausdorff topological vector spaces

$$\begin{cases} \text{if } \ F_1 \in \mathscr{A}(X_1 \times X_2, X_1) \ \text{ and } \ F_2 \in \mathscr{A}(X_1, X_2) \\ \text{then } \ F_3 \in \mathscr{B}^k(X_1 \times X_2, X_1 \times X_2); \end{cases} \tag{2.12}$$

here $F_3(x,y) = F_1(x,y) \times F_2(x)$. A typical example of a class $\mathscr{A}$ is the acyclic maps (i.e. property (C) means the map has acyclic values).

COROLLARY 2.3. *Let E and Y be Hausdorff topological vector spaces, Q a convex subset of E, $G : Q \to K(Q)$ and $T : Q \to K(C)$ where C is a convex subset of Y. Suppose* (2.1), (2.2), (2.3), (2.6), (2.7), (2.12) *hold and in addition assume the following conditions are satisfied:*

(2.13) $T \in \mathscr{B}^k(Q,C)$ *is upper semicontinuous and satisfies property* (C)

and

(2.14) $\Phi \in \mathscr{B}^k(Q \times C, Q),$ *and satisfies property* (C);

here

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}.$$

Then there exists $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0,y_0,z) \leqslant f(x_0,y_0,x_0) \ \textit{for all} \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0,y_0,z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. The result follows from Theorem 2.1 once we show (2.4) holds. As in Corollary 2.2 we have that Φ is upper semicontinuous with nonempty, compact values, so this together with (2.14) implies $\Phi \in \mathscr{A}(Q \times C, Q)$. Also (2.2) and (2.13) guarantees that $T \in \mathscr{A}(Q,C)$. As a result $F \in \mathscr{B}^k(Q \times C, Q \times C)$ from (2.12); here $F(x,y) = \Phi(x,y) \times T(x)$. □

REMARK 2.2. In Corollary 2.3 notice (2.7) can be replaced by (2.10) and (2.11).

REMARK 2.3. Notice Theorem 2 in [5] follows from Corollary 2.3 (with Remark 2.2), so consequently all the results in [5] (e.g. Hartman-Stampacchia [6], Lions-Stampacchia [7], Saigal [14] etc.) follow from Corollary 2.3.

Next we present some variational inequalities where the map F (in Theorem 2.1) is not a self map.

THEOREM 2.4. *Let E and Y be metrizable locally convex topological vector spaces, Q a closed convex subset of E, $G : Q \to K(E)$ and $T : Q \to 2^C$ where C is a closed convex subset of Y. In addition assume the following conditions hold:*

(2.15) $f : Q \times C \times E \to \mathbf{R}$ *is a upper semicontinuous function*

(2.16) G *and* T *are compact maps*

(2.17) $0 \in Q$ *and* $0 \in C$

(2.18) $F \in \mathscr{U}_c^k(Q \times C, E \times Y)$ *is closed* *and*

$$(2.19)\qquad \begin{cases} \textit{if } \{(z_j,\lambda_j)\}_{j=1}^{\infty},\ z_j=(x_j,y_j), \ \textit{ is a sequence in} \\ \partial(Q\times C)\times[0,1] \ \textit{ converging to } \ (z,\lambda),\ z=(x,y), \\ \textit{with } \ (x,y)\in\lambda\, F(x,y) \ \textit{and} \ 0\leqslant\lambda<1, \ \textit{ then} \\ \{\lambda_j\, F(x_j,y_j)\}\subseteq Q\times C \ \textit{ for } \ j \ \textit{ sufficiently large;} \end{cases}$$

here $F(x,y)=\Phi(x,y)\times T(x)$ *for* $(x,y)\in Q\times C$ *with*

$$\Phi(x,y)=\{w\in G(x):\ f(x,y,w)=M(x,y)\}$$

and $M(x,y)=\max_{w\in G(x)} f(x,y,w)$. *Then there exists* $(x_0,y_0)\in Q\times C$, $x_0\in G(x_0)$ *and* $y_0\in T(x_0)$ *with*

$$f(x_0,y_0,z)\leqslant f(x_0,y_0,x_0)\ \textit{ for all } \ z\in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0,y_0)\in Q\times C$, $x_0\in G(x_0)$ *and* $y_0\in T(x_0)$ *with* $f(x_0,y_0,z)\leqslant 0$ *for all* $z\in G(x_0)$.

Proof. As in Theorem 2.1, Φ is well defined and $F: Q\times C\to 2^{E\times C}$ is compact. Now apply Theorem 1.2. □

Next we discuss a subclass $\mathscr{M}$ of $\mathscr{U}_c^k$. If X and Y are subsets of metrizable locally convex topological vector spaces then we say $F\in\mathscr{M}(X,Y)$ if $F\in\mathscr{U}_c^k(X,Y)$ and is upper semicontinuous with nonempty compact values and satisfies property (C). Also we assume for subsets X_1, Y_1, X_2, Y_2 of metrizable locally convex topological vector spaces

$$(2.20)\qquad \begin{cases} \text{if } \ F_1\in\mathscr{M}(X_1\times X_2,Y_1) \ \text{ and } \ F_2\in\mathscr{M}(X_1,Y_2) \\ \text{then } \ F_3\in\mathscr{U}_c^k(X_1\times X_2,Y_1\times Y_2); \end{cases}$$

here $F_3(x,y)=F_1(x,y)\times F_2(x)$.

COROLLARY 2.5. *Let* E *and* Y *be metrizable locally convex topological vector spaces,* Q *a closed convex subset of* E, $G: Q\to K(E)$ *and* $T: Q\to K(C)$ *where* C *is a closed convex subset of* Y. *Suppose* (2.15), (2.16), (2.17), (2.20) *hold and in addition assume the following conditions are satisfied:*

$$(2.21)\qquad G: Q\to 2^E \quad \textit{is upper semicontinuous}$$

$$(2.22)\qquad \begin{cases} M: Q\times C\to E \ \textit{ is lower semicontinuous} \\ (\textit{here } \ M(x,y)=\max_{w\in G(x)} f(x,y,w)) \end{cases}$$

$$(2.23)\qquad T\in\mathscr{U}_c^k(Q,Y) \ \textit{ is upper semicontinuous and satisfies property } \ (C)$$

and

$$(2.24)\qquad \Phi\in\mathscr{U}_c^k(Q\times C,E), \qquad \textit{and satisfies property } \ (C);$$

here

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}.$$

Finally assume (2.19) *holds with* $F(x,y) = \Phi(x,y) \times T(x)$. *Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0) \ \textit{for all} \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0, y_0, z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. The result follows from Theorem 2.4 once we show (2.18) holds. Essentially the same reasoning as in Corollary 2.2 guarantees that Φ is upper semicontinuous with nonempty, compact values, so this together with (2.24) implies $\Phi \in \mathscr{M}(Q \times C, E)$. Also $T \in \mathscr{M}(Q, Y)$. As a result $F \in \mathscr{U}_c^k(Q \times C, E \times Y)$ from (2.20). In addition [3 pp. 472] implies F is upper semicontinuous since both F and Φ are. Thus [3 pp. 465] implies F is closed, so (2.18) holds. □

It is also possible to obtain the analogue of Theorem 2.1 when the maps G and T are not compact.

THEOREM 2.6. *Let* E *and* Y *be Hausdorff topological vector spaces,* Q *a closed convex subset of* E, $G : Q \to K(Q)$ *and* $T : Q \to 2^C$ *where* C *is a closed convex subset of* Y. *Suppose* (2.1) *holds and in addition assume the following conditions are satisfied:*

(2.25) $\qquad Q \times C$ *is a* q*-admissible subset of* $E \times Y$ *with* $z_0 \in Q \times C$

(2.26) $\qquad F \in \mathscr{B}^k(Q \times C, Q \times C)$

and

(2.27) $\qquad A \subseteq Q \times C,\ A = \overline{co}\,(\{z_0\} \cup F(A))$ *implies* A *is compact;*

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0) \ \textit{for all} \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0, y_0, z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. As in Theorem 2.1, Φ is well defined. We can now apply Theorem 1.3. □

REMARK 2.4. It is also possible to obtain an analogue of Corollary 2.3 in this situation. We leave the details to the reader (see Corollary 2.8).

REMARK 2.5. It is possible to replace (2.27) with other noncompactness criteria and use the other results in [9] instead of Theorem 1.3.

If E and Y are Fréchet spaces and F is P-concentrative (P a defining system of seminorms for $E \times Y$) then (2.27) holds. Our next theorem discusses the case when F is countably P-concentrative.

THEOREM 2.7. *Let E and Y be Fréchet spaces (and let P be a defining system of seminorms for $E \times Y$), Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to 2^C$ where C is a closed convex subset of Y. Suppose (2.1) holds and in addition assume the following condition is satisfied:*

(2.28) $\qquad F \in \mathcal{U}_c^k(Q \times C, Q \times C)$ *is countably P-concentrative;*

here $F(x,y) = \Phi(x,y) \times T(x)$ for $(x,y) \in Q \times C$ with

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. Then there exists $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0,y_0,z) \leqslant f(x_0,y_0,x_0) \ \text{ for all } \ z \in G(x_0).$$

If in addition (2.5) holds, then there exists $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with $f(x_0,y_0,z) \leqslant 0$ for all $z \in G(x_0)$.

Proof. As in Theorem 2.1, Φ is well defined. We can now apply Theorem 1.4. □

Next we establish the analogue of Corollary 2.3 in this situation.

COROLLARY 2.8. *Let E and Y be Fréchet spaces (and let P be a defining system of seminorms for $E \times Y$), Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to K(C)$ where C is a closed convex subset of Y. Suppose (2.1), (2.6), (2.7), (2.20) hold and in addition assume the following conditions are satisfied:*

(2.29) $\qquad \Phi \in \mathcal{U}_c^k(Q \times C, Q) \qquad$ *and satisfies property (C)*

(2.30) $\qquad T \in \mathcal{U}_c^k(Q, C)$ *is upper semicontinuous and satisfies property (C)*

and

(2.31) $\qquad F : Q \times C \to 2^{Q \times C}$ *is countably P-concentrative;*

here $F(x,y) = \Phi(x,y) \times T(x)$ for $(x,y) \in Q \times C$ with

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. Then there exists $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0,y_0,z) \leqslant f(x_0,y_0,x_0) \ \text{ for all } \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0, y_0, z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. The result follows from Theorem 2.7 once we show $F \in \mathscr{U}_c^k(Q \times C, Q \times C)$. Essentially the same reasoning as in Corollary 2.2 shows Φ is closed. We *claim* Φ is upper semicontinuous (note we cannot apply [3 pp. 463] as in Corollary 2.2 since Φ is not necessarily compact). To see this notice

$$\Phi(x,y) = G(x) \cap \Lambda(x,y)$$

where

$$\Lambda(x,y) = \{w \in Q : \ f(x,y,w) = M(x,y)\}.$$

The same reasoning as in Corollary 2.2 (except easier) implies Λ is closed. Now since G is upper semicontinuous with nonempty compact values then [3 pp. 470] implies Φ is upper semicontinuous. Thus $\Phi \in \mathscr{M}(Q \times C, C)$. Also $T \in \mathscr{M}(Q, C)$ so (2.20) implies $F \in \mathscr{U}_c^k(Q \times C, Q \times C)$. □

Our final result concerns non self maps in the noncompact situation.

THEOREM 2.9. *Let* E *and* Y *be Hilbert spaces,* Q *a closed convex subset of* E, $G : Q \to K(Q)$ *and* $T : Q \to 2^C$ *where* C *is a closed convex subset of* Y. *Suppose* (2.15), (2.17) *hold and in addition assume the following condition is satisfied:*

(2.32) $F \in \mathscr{U}_c^k(Q \times C, E \times C)$ *is a closed countably condensing map;*

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : \ f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Finally assume* (2.19) *holds. Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leqslant f(x_0, y_0, x_0) \ \text{ for all } \ z \in G(x_0).$$

If in addition (2.5) *holds, then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with* $f(x_0, y_0, z) \leqslant 0$ *for all* $z \in G(x_0)$.

Proof. Apply Theorem 1.5. □

REMARK 2.6. It is also possible to obtain an analogue of Corollary 2.5 in this situation. We leave the details to the reader.

REMARK 2.7. It is also possible to restate Theorem 2.1 (and other theorems in this section) if the $\mathscr{B}^k$ maps are replaced by the KKM maps; in this case we use the fixed point theory from [1, 2, 11].

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