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ERGODIC THEOREMS FOR DYNAMIC RANDOM WALKS

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Abstract. Given any measure-preserving dynamical system $(Y, \mathscr{A}, \mu, T)$ and $g \in L^p(\mu)$, we study convergence of the sequence $\left\{ \frac{1}{n} \sum_{k=1}^{n} g \circ T^{S_k},\ n \geqslant 1 \right\}$ where S_k is a dynamic $\mathbb{Z}^r$-valued random walk generated by another dynamical system, namely an irrational rotation on the d-dimensional torus. In this paper, Van der Corput's inequality and number theory are used for studying ergodic theorems and universally representative random sequences.

1. Introduction. Principal results

Let us consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a sequence $\{S_k,\ k \geqslant 1\}$ of random vectors defined on this space with values in $\mathbb{Z}^r$, $r \geqslant 1$. Let $(Y, \mathscr{A}, \mu, T)$ be a measurable dynamical system, where $(Y, \mathscr{A}, \mu)$ is a probability space and T is an action of $\mathbb{Z}^r$ defined on Y such that $T\mu = \mu$.

Let us introduce the notion of universally representative sequences.

DEFINITION 1.1. A sequence of random vectors $S = \{S_k,\ k \geqslant 1\}$ with values in $\mathbb{Z}^r$, $r \geqslant 1$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is universally representative for L^p, $p > 1$ if there exists $\Omega_o \subset \Omega$ of probability one such that for every $\omega \in \Omega_o$:

For every dynamical system $(Y, \mathscr{A}, \mu, T)$ and $g \in L^p(\mu)$ we have

$$\mu\left\{ y : \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} g(T^{S_k(\omega)} y) \text{ exists} \right\} = 1.$$

For example, when $r = 1$, the sequence $\{p_k + \theta_k,\ k \geqslant 1\}$ where p_k is the k-th prime number and $\{\theta_k,\ k \geqslant 1\}$ a sequence of independent, identically distributed random variables with a moment of strictly positive order, is universally representative for L^p, $p > 1$ (see [22] for more explanations).

M. Lacey, K. Petersen, D. Rudolph and M. Wierdl (see Theorem 5 in [18]) also proved:

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THEOREM 1.2. *Let $X = \{X_k,\ k \geqslant 1\}$ be a sequence of independent, identically distributed random variables such that $\mathbb{E}\,X_1 \neq 0$ and $\mathbb{E}\,(X_1)^2 < \infty$. Then the sequence*

$$S = \left\{ \sum_{j=1}^{k} X_j\ ,\ k \geqslant 1 \right\}$$

is universally representative for $L^p,\ p > 1$.

In the case $r \geqslant 2$, they obtained the following result (see Theorem 7 in [18]) : When $g \in L^2(\mu)$, then

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{N}}} \frac{1}{n} \sum_{k=1}^{n} g \circ T^{S_k} \text{ exists } \mu\text{-a.e.}$$

where $\mathscr{N} = \{[2^{t \log t}],\ t \in \mathbb{N}^*\}$.

There also exists a recent paper of D. Schneider (see [25]) that explains the behaviour of the previous averages for almost-everywhere convergence in terms of conditions on the spectral measure of the operator T.

Let us recall another definition.

DEFINITION 1.3. A sequence of random vectors $S = \{S_k,\ k \geqslant 1\}$ with values in $\mathbb{Z}^r$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is universally 2-representative in mean, if there exists $\Omega_o \subset \Omega$ of probability one, such that for every $\omega \in \Omega_o$:

For every dynamical system $(Y, \mathscr{A}, \mu, T)$ and $g \in L^2(\mu)$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g \circ T^{S_k(\omega)} \text{ exists in } L^2(\mu).$$

In this situation we have the following result (see [8]):

THEOREM 1.4. *Let $X = \{X_k,\ k \geqslant 1\}$ be a sequence of independent, identically distributed random vectors defined on $(\Omega, \mathscr{F}, \mathbb{P})$, with values in $\mathbb{Z}^r$. Let us assume that there exists $\delta > 0$ such that $\mathbb{E}\,||X_1||^{\delta}_{\mathbb{R}^r} < \infty$. Then the sequence of random vectors $S = \{\sum_{j=1}^{k} X_j,\ k \geqslant 1\}$ is universally 2-representative in mean.*

Let us now introduce the dynamic random walk on $\mathbb{Z}^r$. Let $X_i = (X_i^{(1)}, \ldots, X_i^{(r)})$, $i \geqslant 1$, be a sequence of independent random vectors defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $\{-1, +1\}^r$ such that for every $i \geqslant 1$, the random variables $X_i^{(j)}, 1 \leqslant j \leqslant r$ are independent. Let $f_1, \ldots, f_r$ be functions defined on $\mathbb{T}^d$ with values in $[0,1]$ and τ_α the rotation on the d-dimensional torus $\mathbb{T}^d$, associated with the d-dimensional irrational vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ (i.e. $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$), defined by $x = (x_1, \ldots, x_d) \mapsto (x_1 + \alpha_1 \bmod 1, \ldots, x_d + \alpha_d \bmod 1)$. For every $i \geqslant 1$ and every $1 \leqslant j \leqslant r$, the law of the random variable $X_i^{(j)}$ is given by

$$\begin{aligned} \mathbb{P}(X_i^{(j)} = +1) &= f_j(\tau_\alpha^i x) \text{ where } x \in \mathbb{T}^d \text{ is fixed} \\ &= 1 - \mathbb{P}(X_i^{(j)} = -1). \end{aligned}$$

We write

$$S_0 = 0, \ \ S_n = \sum_{i=1}^{n} X_i \text{ for } n \geqslant 1$$

for the $\mathbb{Z}^r$-random walk generated by the family $(X_i)_{i\in\mathbb{N}}$. It is worth remarking that $(S_n)_{n\in\mathbb{N}}$ is a non-homogeneous Markov chain. Besides mathematical interest, the walks we consider here are of some relevance in the statistical mechanics of quasiperiodic systems in the presence of external spatial disorder (see [13] and [15]) and are well studied in [9], [10] in the case $r = 1$. For $r > 1$, the dynamic $\mathbb{Z}^r$-random walks (with a definition more general than the above one) are studied by the first author in [11], in particular a local limit theorem is established. More applications of the dynamic random walks can be found in the recent paper [12]. Our main results are summarised in the following theorems.

THEOREM 1.5. *Let $f_1, \dots, f_r$ be Riemann integrable functions defined on $\mathbb{T}^d$ with values in $[0,1]$ such that for every $j \in \{1, \dots, r\}$, $\int_{\mathbb{T}^d} f_j(t)(1 - f_j(t))dt > 0$. Then, for every $x \in \mathbb{T}^d$ and for every irrational vector α, the dynamic random walk $(S_n)_{n\in\mathbb{N}}$ is universally 2-representative in mean.*

REMARKS. 1. This result holds when T is a $\mathbb{Z}^r$-action of contractions.

2. The fact that the dynamic random walk $(S_n)_{n\in\mathbb{N}}$ is universally 2-representative in mean means that the set Ω_o in Definition 1.3 is the same for every dynamical system $(Y, \mathscr{A}, \mu, T)$. In fact, we have even more here: the set Ω_o is the same for any set of functions $f_j, 1 \leqslant j \leqslant r$ satisfying the conditions of the above theorem.

In the case $r = 1$, with stronger hypotheses on the function f and using the approximation of the irrational vector α by rationals, we show that the dynamic random walk is not universally representative for $L^p, p \geqslant 1$.

THEOREM 1.6. *Let $f : \mathbb{T}^1 \to [0,1]$ be a function of bounded variation such that $\int_{\mathbb{T}^1} f(t)dt = \frac{1}{2}$ and $\int_{\mathbb{T}^1} f(t)(1 - f(t))dt > 0$. Then, for every $x \in \mathbb{T}^1$ and for every irrational α with continuous fraction expansion $[a_0; \dots, a_m, \dots]$ such that the inequality*

$$a_m < m^{1+\epsilon}$$

is satisfied for any m large enough, with $\epsilon > 0$, the dynamic random walk $(S_n)_{n\in\mathbb{N}}$ is **not** *universally representative for $L^p, p \geqslant 1$. For almost every $\omega \in \Omega$, given any ergodic aperiodic measure-preserving transformation T on $(Y, \mathscr{A}, \mu)$, there exists a function $g \in L^1(Y, \mathscr{A}, \mu)$ such that the averages*

$$A_n^{\omega} g(y) = \frac{1}{n} \sum_{k=1}^{n} g(T^{S_k(\omega)} y)$$

diverge a.e. In fact, the sequence has strong sweeping out: given $\epsilon > 0$, we can choose g to be the characteristic function of a set of measure less than ϵ, yet to have

$$\limsup_{n\to\infty} A_n^{\omega} g(y) = 1 \ \ a.e \ \ and \ \ \liminf_{n\to\infty} A_n^{\omega} g(y) = 0 \ a.e.$$

The above one-dimensional result can be generalised in $d > 1$ under some additional hypotheses on the mutual irrationality of the components of $\alpha = (\alpha_1, \ldots, \alpha_d)$. To formulate precisely these results, some definitions are needed. In order to facilitate the exposition, these definitions are postponed until section 2.2. Here, we give the main result, valid in $d \geqslant 1$.

THEOREM 1.7. *Let $f : \mathbb{T}^1 \to [0,1]$ be a function of bounded variation in the sense of Hardy and Krause such that $\int_{\mathbb{T}^d} f(t)dt = \frac{1}{2}$ and $\int_{\mathbb{T}^d} f(t)(1-f(t))dt > 0$. Then, for every $x \in \mathbb{T}^d$ and for every irrational vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ of type η such that $1 \leqslant \eta < 1 + \frac{1}{d}$, the dynamic random walk $(S_n)_{n\in\mathbb{N}}$ is* **not** *universally representative for $L^p, p \geqslant 1$.*

REMARKS. The set of irrational numbers satisfying the hypotheses of Theorem 1.6 corresponds to irrational numbers badly approximated by rationals and is of full measure. It can also be proved that almost every irrational vector is of type 1 (see [10]).

First in section 2 we study the speed of convergence of ergodic averages associated with an irrational rotation on the torus in terms of arithmetic properties of rotation angle. In section 3, a Functional Law of Iterated Logarithm for the dynamic random walk is established and Theorems 1.6 and 1.7 are proved. In section 4, we prove Theorem 1.5 using the following theorem which will be proved in section 5.

THEOREM 1.8. *Let Y_n be a sequence of completely independent random variables with a uniformly bounded positive moment. Then setting,*

$$Z_k(\theta,\omega) = \exp\left[2i\pi\langle\, \theta, Y_k(\omega)\rangle\,\right] - \mathbb{E}\,\exp\left[2i\pi\langle\, \theta, Y_k\rangle\,\right]$$

we have

$$\mathbb{E} \sup_{\theta\in[0,1[^r} \sup_n \left| \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{n} Z_k(\theta,\omega) \right| < \infty\,.$$

In order to prove Theorem 1.5, we will need Van der Corput's inequality and the spectral lemma that we recall here.

THEOREM 1.9 (Van der Corput's Inequality). *Let $(u_k)_{0\leqslant k<n}$ be a finite sequence of n points in a Hilbert space $\mathscr{H}$. If H is an integer between 0 and n-1, then we have*

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} u_k\right\|_{\mathscr{H}}^2 \leqslant \frac{n+H}{n^2(H+1)} \sum_{k=0}^{n-1} ||u_k||_{\mathscr{H}}^2 + 2\frac{n+H}{n^2(H+1)^2} \sum_{h=1}^{H} (H+1-h)\cdot A(n,h),$$

where $A(n,h) = \mathrm{Re}\left(\sum_{k=0}^{n-h-1} \langle\, u_{k+h}, u_k\rangle_{\mathscr{H}}\right)$.

This inequality is easily proved by writing down

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} u_k\right\|_{\mathscr{H}}^2 = \left\|\frac{1}{n}\sum_{k=-H}^{n-1} \left(\frac{1}{H+1}\sum_{h=0}^{H} u_{k+h}\right)\right\|_{\mathscr{H}}^2,$$

with the convention $u_k = 0$ if $k < 0$ or $k \geqslant n$ and by using Cauchy-Schwarz's inequality. This Hilbert space version of Van der Corput's lemma is due to Bergelson (see [1]).

SPECTRAL LEMMA. *Let T be a contraction of a Hilbert space $\mathscr{H}$ and $p(x)$ be a polynomial defined on $D = [0,1[^r$. Then for every f in $\mathscr{H}$, there exists a Borelian positive measure which is bounded on D, denoted μ_f such that*

$$\| p(T)f \|^2_{\mathscr{H}} \leqslant \int_D | p(x) |^2 \mu_f(dx).$$

In the case where T is a measure-preserving transformation, we have an equality in the spectral lemma. It easily follows from Bochner's theorem using the fact that if we define for every $k \in \mathbb{Z}^r$,

$$\gamma_k = \langle T^k f, f \rangle_{\mathscr{H}}$$

then γ is a non negative definite sequence. The extension to contractions can be obtained by a simple inductive argument on the degree of the trigonometric polynomial p.

2. Preliminary results

In this section we consider the dynamical system $(\mathbb{T}^d, \mathscr{B}(\mathbb{T}^d), \lambda, \tau_\alpha)$ where λ is the Lebesgue measure on the torus $\mathbb{T}^d$ and τ_α is the irrational rotation over $\mathbb{T}^d$ defined in the first section. It is well known that under these conditions this dynamical system is ergodic and for every $f \in L^1(\lambda)$, and almost every $x \in \mathbb{T}^d$,

$$M_n = \frac{1}{n}\sum_{l=1}^{n} f(\tau_\alpha^l x) - \int_{\mathbb{T}^d} f(t)dt \underset{n\to\infty}{\longrightarrow} 0.$$

When f is of bounded variation, this result holds for every $x \in \mathbb{T}^d$ and it is possible to determine the speed of convergence of the sequence M_n to 0 in terms of arithmetic properties of the irrational vector α. When $d = 1$, for all irrationals badly approximated by rationals, Denjoy-Koksma's inequality gives us a majorization of M_n uniformly in x for n large enough. But when $d \geqslant 2$, Denjoy-Koksma's inequality does not hold (see Yoccoz [29]) and the method of low discrepancy sequences has to be used.

2.1. Case of one-dimensional torus

Let α be an irrational. We call a rational $\frac{p}{q}$ with p, q relatively prime such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$, a rational approximation of α. When α has the continued fraction expansion $\alpha = [\alpha] + [a_1, \ldots, a_n, \ldots]$, the n-th principal convergent of α is $\frac{p_n}{q_n}$ where, $\forall n \geqslant 2$,

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2};$$

the recurrence is given by defining the values of p_0, p_1 and q_0, q_1.

DENJOY-KOKSMA'S INEQUALITY. *Let $f : \mathbb{R} \to [0,1]$ be a function with bounded variation $V(f)$ and $\frac{p}{q}$ a rational approximation of α. Then, for every $x \in \mathbb{T}^1$,*

$$\left| \sum_{l=1}^{q} f(\tau_\alpha^l x) - q \int_{\mathbb{T}^1} f(t)dt \right| \leqslant V(f).$$

(see [19] where the theorem is given in a more general case).

PROPOSITION 2.1. *Let f be a function with bounded variation $V(f)$. For every irrational α such that the inequality $a_m < m^{1+\epsilon}$, with $\epsilon > 0$, is satisfied for all m large enough,*

$$\sup_{x\in\mathbb{T}^1}\left|\sum_{l=1}^{n} f(\tau_\alpha^l x) - n\int_{\mathbb{T}^1} f(t)dt\right| = \mathcal{O}(\log^{2+\epsilon} n).$$

Proof. The sequence of integers $(q_i)_{i\geqslant 1}$ being strictly increasing, for a given $n \geqslant 1$, there exists $m_n \geqslant 0$ such that

$$q_{m_n} \leqslant n < q_{m_n+1}.$$

By Euclidean division, we have $n = b_{m_n}q_{m_n} + n_{m_n-1}$ with $0 \leqslant n_{m_n-1} < q_{m_n}$. We can use the usual relations

$$q_0 = 1, q_1 = a_1$$

$$q_n = a_n q_{n-1} + q_{n-2}, \quad n \geqslant 2. \tag{1}$$

We obtain that $(a_{m_n+1}+1)q_{m_n} > q_{m_n+1} > n$ and so $b_{m_n} \leqslant a_{m_n+1}$. If $m_n > 0$, we may write $n_{m_n-1} = b_{m_n-1}q_{m_n-1} + n_{m_n-2}$ with $0 \leqslant n_{m_n-2} < q_{m_n-1}$. Again, we find $b_{m_n-1} \leqslant a_{m_n}$. Continuing in this manner, we arrive at a representation for n of the form

$$n = \sum_{i=0}^{m_n} b_i q_i$$

with $0 \leqslant b_i \leqslant a_{i+1}$ for $0 \leqslant i \leqslant m_n$ and $b_{m_n} \geqslant 1$. Using Denjoy-Koksma's inequality, we get

$$\begin{aligned}\left|\sum_{l=1}^{n} f(\tau_\alpha^l x) - n\int_{\mathbb{T}^1} f(x)dx\right| &\leqslant V(f)\sum_{i=0}^{m_n} b_i \\ &\leqslant V(f)\sum_{i=0}^{m_n} a_{i+1}.\end{aligned}$$

By hypothesis, there exists $m_0 \geqslant 1$ such that,

$$a_m < m^{1+\epsilon}, \forall m \geqslant m_0.$$

Let n be such that $m_n > m_0$. Thus,

$$\left|\sum_{l=1}^{n} f(\tau_\alpha^l x) - n\int_{\mathbb{T}^1} f(t)dt\right| \leqslant V(f)\Big(\sum_{i=0}^{m_0-1} a_{i+1} + (m_n+1)^{2+\epsilon}\Big).$$

We need to know the asymptotic behaviour of m_n. When α is the golden ratio, $a_n = 1,\ \forall n \geqslant 1$ and the relation (1) implies that $q_n \sim \frac{1}{\sqrt{5}}\alpha^{n+1}$. Let α' be another irrational; its partial quotients a'_n satisfy necessarily $a'_n \geqslant 1$. Using the relation (1), we see that $q'_n \geqslant q_n, \forall n \geqslant 1$. Therefore, $m_n = \mathcal{O}(\log n)$ and the proposition is proved.

REMARK. Consider ϕ a homeomorphism of the torus $\mathbb{T}^1$ and π the projection from $\mathbb{R}$ onto $\mathbb{T}^1$. We choose $\tilde{\phi}(0)$ such that $\pi\tilde{\phi}(0) = \phi(0)$ and we can gradually define a continuous map $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ such that $\pi\tilde{\phi} = \phi\pi$. The function $\tilde{\phi}$ is called a *lifting* of ϕ; and ϕ *preserves the orientation* of $\mathbb{T}^1$ if its liftings are nondecreasing functions. Then, $(\frac{1}{n}(\tilde{\phi}^n(t)-t))_{n\geqslant 1}$ converges uniformly to a number $\alpha(\tilde{\phi})$ when n goes to infinity. The fractional part $\pi\alpha(\tilde{\phi})$ does not depend on the lifting and is called the *rotation number* of the homeomorphism ϕ. Let μ be a ϕ-invariant probability measure. Using the Denjoy-Koksma's theorem (see [10]), we may generalise the work made in this section to the case where the law of the random variable $X_i^{(j)}, i \geqslant 1, 1 \leqslant j \leqslant r$, is given by

$$\begin{aligned}\mathbb{P}(X_i^{(j)} = +1) &= f_j(\phi^i(x)) \text{ where } x \in \mathbb{T}^1\\ &= 1 - \mathbb{P}(X_i^{(j)} = -1),\end{aligned}$$

where ϕ is an orientation preserving homeomorphism with irrational rotation number α, $f_j, 1 \leqslant j \leqslant r$ and α being defined as in Theorem 1.6. The conclusions of Theorem 1.6 remain valid, that is, the newly defined dynamic random walk $(S_n)_{n\in\mathbb{N}}$ is not universally representative for $L^p, p \geqslant 1$ and has strong sweeping out.

2.2. Generalisation to d-dimensional torus

We recall some definitions and well known results from the method of low discrepancy sequences in dimension $d \geqslant 1$.

Suppose we are given a function $f(x) = f(x^{(1)}, \ldots, x^{(d)})$ with $d \geqslant 1$. By a partition P of $[0,1]^d$, we mean a set of d finite sequences $\eta_0^{(j)}, \eta_1^{(j)}, \ldots, \eta_{m_j}^{(j)}$ $(j = 1, \ldots, d)$, with $0 = \eta_0^{(j)} \leqslant \eta_1^{(j)} \leqslant \ldots \leqslant \eta_{m_j}^{(j)} = 1$ for $j = 1, \ldots, d$. In connection with such a partition, we define, for $j = 1, \ldots, d$ an operator Δ_j by

$$\begin{aligned}\Delta_j f(x^{(1)}, \ldots, x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, \ldots, x^{(d)}) &= f(x^{(1)}, \ldots, x^{(j-1)}, \eta_{i+1}^{(j)}, x^{(j+1)}, \ldots, x^{(d)})\\ &\quad - f(x^{(1)}, \ldots, x^{(j-1)}, \eta_i^{(j)}, x^{(j+1)}, \ldots, x^{(d)}),\end{aligned}$$

for $0 \leqslant i < m_j$.

DEFINITION 2.2.

1. For a function f on $[0,1]^d$, we set

$$V^{(d)}(f) = \sup_P \sum_{i_1=0}^{m_1-1} \ldots \sum_{i_d=0}^{m_d-1} |\Delta_{1,\ldots,d} f(\eta_{i_1}^{(1)}, \ldots, \eta_{i_d}^{(d)})|,$$

   where the supremum is extended over all partitions P of $[0,1]^d$. If $V^{(d)}(f)$ is finite, then f is said to be *of bounded variation on* $[0,1]^d$ *in the sense of Vitali*.
2. For $1 \leqslant p \leqslant d$ and $1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant d$, we denote by $V^{(p)}(f; i_1, \ldots, i_p)$ the p-dimensional variation in the sense of Vitali of the restriction of f to $E^d_{i_1\ldots i_p} = \{(t_1, \ldots, t_d) \in [0,1]^d; t_j = 1$ whenever j is none of the $i_r, 1 \leqslant r \leqslant p\}$.

If all the variations $V^{(p)}(f;i_1,\dots,i_p)$ are finite, the function f is said to be *of bounded variation on* $[0,1]^d$ *in the sense of Hardy and Krause*.

Let $x_1,\dots,x_n$ be a finite sequence of points in $[0,1]^d$ with $x_l=(x_{l_1},\dots,x_{l_d})$ for $1\leqslant l\leqslant n$. We introduce the function

$$R_n(t_1,\dots,t_d)=\frac{A(t_1,\dots,t_d;n)}{n}-t_1\dots t_d$$

for $(t_1,\dots,t_d)\in[0,1]^d$, where $A(t_1,\dots,t_d;n)$ denotes the number of elements x_l, $1\leqslant l\leqslant n$, for which $x_{l_i}<t_i$ for $1\leqslant i\leqslant d$.

DEFINITION 2.3. The *discrepancy* D_n^* of the sequence $x_1,\dots,x_n$ in $[0,1]^d$ is defined to be

$$D_n^*=\sup_{(t_1,\dots,t_d)\in[0,1]^d}|R_n(t_1,\dots,t_d)|.$$

For a real number t, let $\|t\|$ denote its distance to the nearest integer, namely,

$$\begin{aligned}\|t\| &= \inf_{n\in\mathbb{Z}} \mid t-n\mid\\ &= \inf(\{t\},1-\{t\})\end{aligned}$$

where $\{t\}$ is the fractional part of t.

DEFINITION 2.4. For a real number η, a d-tuple $\alpha=(\alpha_1,\dots,\alpha_d)$ of irrationals is said to be of *type* η if η is the infimum of all numbers σ for which there exists a positive constant $c=c(\sigma;\alpha_1,\dots,\alpha_d)$ such that

$$r^\sigma(h)\|\langle h,\alpha\rangle\|\geqslant c$$

holds for all $h\neq 0$ in $\mathbb{Z}^d$, where $r(h)=\prod_{i=1}^d\max(1,|h_i|)$ and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^d$.

REMARK. The type η of α is also equal to

$$\sup\{\gamma:\inf_{h\in(\mathbb{Z}^d)^*}r^\gamma(h)\|\langle h,\alpha\rangle\|=0\}.$$

We always have $\eta\geqslant 1$ (see [20]). We now give a result (see [17]) which gives us the asymptotic behaviour of the discrepancy of the sequence $w=(x_1+l\alpha_1,\dots,x_d+l\alpha_d), l=1,2,\dots$ in function of the mutual irrationality of the components of α.

PROPOSITION 2.5. *Let* $\alpha=(\alpha_1,\dots,\alpha_d)$ *be an irrational vector. Suppose there exists* $\eta\geqslant 1$ *and* $c>0$ *such that*

$$r^\eta(h)\|\langle h,\alpha\rangle\|\geqslant c$$

for all $h \neq 0$ *in* $\mathbb{Z}^d$. *Then, for every* $x \in [0,1]^d$, *the discrepancy of the sequence* $w = (x_1 + l\alpha_1, \ldots, x_d + l\alpha_d), l = 1, 2, \ldots$ *satisfies* $D_n^*(w) = \mathscr{O}(n^{-1} \log^{d+1} n)$ *for* $\eta = 1$ *and* $D_n^*(w) = \mathscr{O}(n^{-\frac{1}{((\eta-1)d+1)}} \log n)$ *for* $\eta > 1$.

The proof is based on the Erdős-Turán-Koksma's theorem: For $h \in \mathbb{Z}^d$, define $p(h) = \max_{1 \leqslant j \leqslant d} |h_j|$, and let $x_1, \ldots, x_n$ be a finite sequence of points in $\mathbb{R}^d$. Then, for any positive integer m, we have

$$D_n^* \leqslant C_d \left(\frac{1}{m} + \sum_{0 \leqslant p(h) \leqslant m} \frac{1}{r(h)} \left| \frac{1}{n} \sum_{l=1}^{n} e^{2\pi i \langle h, x_l \rangle} \right| \right)$$

where C_d only depends on the dimension d. This theorem combined with the results of [17] (p. 131) gives us the result.

THEOREM 2.6 (Hlawka, Zaremba). *Let* f *be of bounded variation on* $[0,1]^d$ *in the sense of Hardy and Krause, and let* ω *be a finite sequence of points* $x_1, \ldots, x_n$ *in* $[0,1]^d$. *Then, we have*

$$\left| \frac{1}{n} \sum_{l=1}^{n} f(x_l) - \int_{\mathbb{T}^d} f(t) dt \right| \leqslant \sum_{p=1}^{d} \sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant d} V^{(p)}(f; i_1, \ldots, i_p) D_n^*(\omega_{i_1 \ldots i_p}),$$

where $D_n^*(\omega_{i_1 \ldots i_p})$ *is the discrepancy in* $E^d_{i_1 \ldots i_p}$ *of the sequence* $\omega_{i_1 \ldots i_p}$ *obtained by projecting* ω *onto* $E^d_{i_1 \ldots i_p}$.

PROPOSITION 2.7. *Let* f *be a function with bounded variation in the sense of Hardy and Krause and* α *an irrational vector of type* η, *then*

$$\sup_{x \in \mathbb{T}^d} \left| \sum_{l=1}^{n} f(\tau_\alpha^l x) - n \int_{\mathbb{T}^d} f(t) dt \right| = \begin{cases} \mathscr{O}(\log^{d+1} n) \text{ if } \eta = 1 \\ \mathscr{O}(n^{1-\frac{1}{((\eta-1)d+1)}} \log n) \text{ if } \eta > 1. \end{cases}$$

Proof. Let η' be such that $\eta \leqslant \eta' < 1 + \frac{1}{d}$. Then there exists $c > 0$ such that

$$r^{\eta'}(h) \| \langle h, \alpha \rangle \| \geqslant c$$

holds for all $h \neq 0$ in $\mathbb{Z}^d$. If we are given a p-tuple $\alpha_p = (\alpha_{i_1}, \ldots, \alpha_{i_p}), 1 \leqslant p \leqslant d$, of α, then

$$r^{\eta'}(h) \| \langle h, \alpha_p \rangle \| \geqslant c$$

holds for all $h \neq 0$ in $\mathbb{Z}^p, 1 \leqslant p \leqslant d$. Thus, every p-tuple, $1 \leqslant p \leqslant d$, is of type δ such that $1 \leqslant \delta \leqslant \eta$ and $(\alpha_{i_1}, \ldots, \alpha_{i_p})$ is an irrational vector. For every $p, 1 \leqslant p \leqslant d$, we define $w_{i_1 \ldots i_p}$ by the projection of w on $E^d_{i_1 \ldots i_p}$. By Proposition 2.5, we have for every $p, 1 \leqslant p \leqslant d$,

$$\begin{cases} nD_n^*(w_{i_1 \ldots i_p}) = \mathscr{O}(\log^{p+1} n) \text{ if } \delta = 1 \\ nD_n^*(w_{i_1 \ldots i_p}) = \mathscr{O}(n^{1-\frac{1}{((\delta-1)p+1)}} \log n) \text{ if } 1 < \delta \leqslant \eta. \end{cases}$$

Now, $\forall p = 1, \dots, d$,

$$0 \leqslant 1 - \frac{1}{(\delta - 1)p + 1} \leqslant 1 - \frac{1}{(\eta - 1)d + 1} \leqslant 1.$$

Therefore, using Hlawka-Zaremba's theorem, we obtain the proposition.

3. Proof of Theorems 1.6 and 1.7

In order to prove that a standard random walk $(S_n)_{n\in\mathbb{N}}$ (sum of i.i.d. random variables with Bernoulli distribution) is not universally representative for $L^p, p \geqslant 1$, Lacey and al. [18] used the Strassen's Functional Law of the Iterated Logarithm [27] implying that the walk is constant for long stretches with some fluctuations that they are able to control by choosing a suitable function g. In our situation we could not use the classical Strassen's Functional Law of t ted Logarithm which only concerns sums of identically distributed and independent variables satisfying suitable conditions. So we will follow the ideas developped by Szüsz and Volkmann [28]. Their aim was to give a proof of Strassen's result without using functional tools but the probabilistic concepts used by Kolmogorov [14] to establish the classical Law of the Iterated Logarithm. In this way they extend the Strassen's result to sums of random variables not necessarily identically distributed and even weaken Kolmogorov's hypotheses.

3.1. A Strassen's Functional Law of the Iterated Logarithm for the dynamic random walk

The dynamic $\mathbb{Z}$-random walk $(S_n)_{n\in\mathbb{N}}$ is defined as in section 1. We assume in this section that f is a Riemann integrable function such that $c = \int_{\mathbb{T}^d} 4f(t)(1-f(t))dt > 0$. Let us define for every $i \geqslant 1$, the random variables $Y_i = X_i - (2f(\tau_\alpha^i x) - 1)$ and the sum $\tilde{S}_n = \sum_{i=1}^{n} Y_i$. Then we investigate the behaviour of the functions $\psi_n(t)$ obtained by linear interpolation of the values

$$\psi_n\left(\frac{i}{n}\right) = T_n(i) \text{ where } T_n(i) = \frac{\tilde{S}_i}{\sqrt{2nc\log\log n}},\ i = 1, \dots, n;\ \psi_n(0) = 0.$$

The set Σ will denote the set of absolutely continuous functions F defined on $[0,1]$ with $F(0) = 0$ whose derivative satisfies

$$\int_0^1 F'(t)^2 dt \leqslant 1.$$

PROPOSITION 3.1. *For every $x \in \mathbb{T}^d$ and every irrational vector α, the set of limit functions of the sequence $\psi_3, \psi_4, \psi_5, \dots$ under uniform convergence is almost certainly the set Σ.*

The proof will be deduced from the following lemmas.

LEMMA 3.1. *For any* v *with* $0 < v < C_2 . \sqrt{\dfrac{\log\log \operatorname{Var}(\tilde{S}_n)}{\operatorname{Var}(\tilde{S}_n)}}$, *one has*

$$\mathbb{E}\left(e^{v\tilde{S}_n}\right) < e^{\frac{v^2}{2}\operatorname{Var}(\tilde{S}_n)(1+\delta_n)}, n \geqslant 1$$

where $\delta_n \to 0$ *as* $n \to \infty$.

Proof. One has for every $i \geqslant 1$,

$$\mathbb{E}\left(e^{vY_i}\right) = \sum_{l=0}^{\infty} \frac{v^l}{l!}\mathbb{E}\left(Y_i^l\right) = 1 + \frac{v^2}{2}\operatorname{Var}(Y_i) + \sum_{l=3}^{\infty} \frac{v^l}{l!}\mathbb{E}\left(Y_i^l\right).$$

Therefore,

$$\mathbb{E}\left(e^{v\tilde{S}_n}\right) = \prod_{i=1}^{n} \mathbb{E}\left(e^{vY_i}\right) < \exp\left(\frac{v^2}{2}\operatorname{Var}(\tilde{S}_n) + \sum_{l=3}^{\infty} \frac{v^l}{l!} \sum_{i=1}^{n} \mathbb{E}\left(|Y_i|^l\right)\right).$$

We have

$$\frac{1}{n}\operatorname{Var}(\tilde{S}_n) = \frac{1}{n}\sum_{i=1}^{n} 4f\left(\tau_\alpha^i x\right)\left(1 - f\left(\tau_\alpha^i x\right)\right) \underset{n\to+\infty}{\longrightarrow} c, \text{ uniformly in } x \in \mathbb{T}^d$$

(see [21] p. 156). This result and the fact that the random variables Y_i are bounded by 2 imply that the sequence

$$\sqrt{\frac{\log\log \operatorname{Var}(\tilde{S}_n)}{\operatorname{Var}(\tilde{S}_n)}} . \left(\frac{\sum_{i=1}^{n} \mathbb{E}\left(|Y_i|^l\right)}{\log\log \operatorname{Var}(\tilde{S}_n)}\right)^{1/l}$$

converges uniformly with respect to l with $l \geqslant 3$ to zero when n tends to $+\infty$. Thus the lemma follows.

LEMMA 3.2 (Bernstein-Kolmogorov Inequality). *For any* $t < C_3 \cdot \log\log \operatorname{Var}(\tilde{S}_n)$, *one has*

$$\mathbb{P}\left(\tilde{S}_n > \sqrt{2t\operatorname{Var}(\tilde{S}_n)}\right) < e^{-t(1+o(1))}.$$

Proof. We have the classical inequality, for any $v > 0$,

$$\mathbb{P}\left(\tilde{S}_n > \frac{1}{v}\left(t + \log \mathbb{E}\left(e^{v\tilde{S}_n}\right)\right)\right) < e^{-t}.$$

The lemma 3.1 with

$$v = \sqrt{\frac{2t}{\operatorname{Var}(\tilde{S}_n)}}$$

gives us

$$\log \mathbb{E}\left(e^{v\tilde{S}_n}\right) < \frac{v^2}{2}\operatorname{Var}(\tilde{S}_n)(1 + o(1)),$$

which proves the inequality.

The following inequality can be found in Kolmogorov [14].

LEMMA 3.3. *For any t satisfying $C_4 < t < C_3 \cdot \log\log \mathrm{Var}(\tilde{S}_n)$, one has*

$$\mathbb{P}(\tilde{S}_n > \sqrt{2t\mathrm{Var}(\tilde{S}_n)}) > e^{-t(1+o(1))}.$$

By combining lemmas 3.2 and 3.3, we deduce

LEMMA 3.4. *For any $\beta > 1$ and $\epsilon > 0$ there exists an $n_o = n_o(\beta, \epsilon)$ such that, for all y satisfying $C_4 < |y| < \sqrt{C_5 \cdot \log\log \mathrm{Var}(\tilde{S}_n)}$, we have*

$$e^{-\frac{y^2}{2}(1+\epsilon)} < \mathbb{P}\left(\tilde{S}_n \in \left(y\sqrt{\mathrm{Var}(\tilde{S}_n)}, \beta y\sqrt{\mathrm{Var}(\tilde{S}_n)}\right)\right), n \geqslant n_o.$$

Proof of Proposition 3.1. Let k be a sufficiently large natural number. We restrict our attention to indices n divisible by k. Now we can split the sum $\tilde{S}_n$ into k sums of $\frac{n}{k}$ terms each,

$$\xi_1 = Y_1 + \ldots + Y_{\frac{n}{k}},\ \xi_2 = Y_{\frac{n}{k}+1} + \ldots + Y_{\frac{2n}{k}}, \ldots$$

Let $t_l \in [0,1]$ and $\beta > 1$ be given. If we apply Lemma 3.4 with $\frac{n}{k}, \xi_l, \sqrt{\frac{nc}{k}} + o(1)$ instead of $n, \tilde{S}_n, \sqrt{\mathrm{Var}(\tilde{S}_n)}$ and with $y = t_l\sqrt{2k\log\log n}$, then

$$\mathbb{P}\left(\xi_l \in \left(y\sqrt{\frac{nc}{k}}, \beta y\sqrt{\frac{nc}{k}}\right)\right) > e^{-k\log\log n(1+\epsilon)t_l^2}.$$

Using that the random variables ξ_i are independent, we get

$$\mathbb{P}\left(\frac{\xi_l}{\sqrt{2nc\log\log n}} \in (t_l, \beta t_l) \text{ for } l = 1,\ldots,k\right) > (\log n)^{-(1+\epsilon)k\sum_{l=1}^{k} t_l^2}.$$

Now we can choose

$$t_l = F\left(\frac{l+1}{k}\right) - F\left(\frac{l}{k}\right),\ l = 0, 1, \ldots, k-1.$$

Let $\epsilon' > 0$ be given and set $I = \int_0^1 F'(t)^2 dt$. Then for k large enough, we get

$$\left|k\sum_{l=1}^{k} t_l^2 - I\right| < \epsilon'.$$

If $I < 1$, the series

$$\sum_{m=1}^{\infty}(\log q^m)^{-(1+\epsilon)I}, q > 1,$$

diverges for sufficiently small ϵ and the proof is finished by applying the Borel-Cantelli Lemma and using the argument of [14]. If $I = 1$, we apply all this to a function $F^* \in \Sigma$ whose corresponding integral $I^* < 1$ and which is close to F under the metric of uniform convergence.

3.2. Dynamic random walk is not universally representative

By Theorem 1.3 in Del Junco and Rosenblatt [2], it is enough to find for a.e. ω, for each $\epsilon > 0$ and $N \in \mathbb{N}$, a set E of measure less than ϵ for which $\mu(\sup_{n \geqslant N} A_n^{\omega} 1_E > 1-\epsilon) > 1-\epsilon$. Given $\epsilon > 0$, choose $q \in \mathbb{N}$ with $\frac{4}{q} < \epsilon$ and fix n large. Take $\alpha, \beta \in]0,1[$ such that $\beta(q+1)^{q-1} < 1$. Let $\beta_{-1} = 0$ and for $i = 0, \ldots, q-1$, let $\beta_i = (q+1)^i \beta$. We define a continuous function F on $[0,1]$ by $F(0) = 0$ and

$$F = (i+1)\alpha \ \text{ on } \ [\beta_{i-1} + \beta(q+1)^{i-1}, \beta_i] \text{ for } i = 0, \ldots, q-1,$$

linearly in between and constant elsewhere. We choose α small enough so that $\int_0^1 F'(t)^2 dt \leqslant 1$. Under the hypotheses of Theorem 1.6 (resp. Theorem 1.7), by Proposition 2.1 (resp. Proposition 2.7), if $w_n = \sqrt{2nc \log\log n}$, for any $\epsilon > 0$ given, for n large enough, we have

$$\frac{1}{w_n} \left| \sum_{i=1}^{k} (2f(\tau_\alpha^i x) - 1) \right| < \epsilon \ \text{ for } \ k = 1, \ldots, n.$$

Then, by Proposition 3.1, for a.e. ω and n large enough,

$$\left| \frac{S_k(\omega)}{w_n} - F\left(\frac{k}{n}\right) \right| < \alpha \text{ for } \ k = 1, \ldots, n.$$

We fix such n, ω. By standards methods for transferring counterexamples by use of Rokhlin towers, we may work with the translation action of $\mathbb{Z}$ on itself and find $g : \mathbb{Z} \to \{0,1\}$ taking value 1 on an infrequently visited set but giving large values of the averages for most initial points. For each $i = 1, \ldots, q$, let $I_i = [(i-1)\alpha w_n, i\alpha w_n[$. Let us define the function g by $g(x) = 1$ if $x (\operatorname{mod} q\alpha w_n) \in I_1 \cup I_2 \cup I_q$, 0 otherwise. If g is transferred by means of the Rokhlin lemma to any aperiodic transformation of $\mathbb{Z}$, we have $\mu(g=1) < \frac{4}{q} < \epsilon$. For each $x \in \mathbb{Z}$, we can choose $i \in \{1, \ldots, q\}$ such that $x (\operatorname{mod} q\alpha w_n) \in I_{q-i+1}$. Then,

$$\begin{aligned}
A^{\omega}_{n\beta_{i-1}} g(x) &\geqslant \frac{1}{n\beta_{i-1}} \sum_{n\beta_{i-2} \leqslant k \leqslant n\beta_{i-1}} g(x + S_k(\omega)) \\
&= \frac{1}{n\beta_{i-1}} \sum_{n\beta_{i-2} \leqslant k \leqslant n\beta_{i-1}} g\left(x + w_n F\left(\frac{k}{n}\right) + \delta_k w_n\right) \text{ where } \delta_k \in]-\alpha, \alpha[. \\
&\geqslant \frac{q-1}{qn\beta_{i-1}} \sum_{n\beta_{i-2} + n\beta(q+1)^{i-2} \leqslant k \leqslant n\beta_{i-1}} g(x + i\alpha w_n + \delta_k w_n) \\
&\geqslant \frac{q-1}{q} \cdot \frac{n\beta_{i-1} - n\beta_{i-2} - n\beta(q+1)^{i-2}}{n\beta_{i-1}} \geqslant 1 - \frac{3}{q} > 1 - \epsilon
\end{aligned}$$

since $(x + i\alpha w_n + \delta_k w_n)(\operatorname{mod} q\alpha w_n) \in [\delta_k w_n (\operatorname{mod} q\alpha w_n), (\alpha + \delta_k) w_n (\operatorname{mod} q\alpha w_n)[\subset I_1 \cup I_2 \cup I_q$ (since $\delta_k \in]-\alpha, \alpha[$).

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5 assuming that Theorem 1.8 is satisfied and using Van der Corput's inequality. The proof of Theorem 1.8 will be given in the next section. Van der Corput's inequality permits us to determine sufficient conditions for a sequence of points in a Hilbert space to go to 0 in the Cesàro sense for the Hilbertian norm. Given a finite family of n points $(u_k)_{1\leqslant k\leqslant n}$ in a Hilbert space $\mathscr{H}$, a sufficient condition to get

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}u_k\right\|_{\mathscr{H}}=0\,,$$

is:

$$(a)\qquad \forall l\geqslant 0,\quad \left|\frac{1}{n}\sum_{k=1}^{n}\langle\, u_{k+l},u_k\rangle_{\mathscr{H}}\right|\leqslant\psi_{l,n}\text{ and }\lim_{n\to\infty}\psi_{l,n}=\gamma_l\text{ exists}$$

and

$$(b)\qquad \lim_{L\to\infty}\frac{1}{L}\sum_{l=0}^{L-1}\gamma_l=0.$$

For all $l\geqslant 1$, we denote $S_k^l=X_{k+1}+\ldots+X_{k+l},\forall k\geqslant 1$ and we define $\phi_k^{(l)}(\theta),\theta\in[0,1[^r$ the characteristic function of the random variable S_k^l,

$$\phi_k^{(l)}(\theta)=\mathbb{E}\,(e^{2\pi i\langle\,\theta,S_k^l\rangle}\,)$$

The random vectors X_i are independent with independent directions and the law of $X_i^{(j)}$ is known, so

$$\phi_k^{(l)}(\theta)=\prod_{i=1}^{r}\prod_{j=1}^{l}(\cos(2\pi\theta_i)+i(2f_i(\tau_\alpha^{k+j}x)-1)\sin(2\pi\theta_i)),\theta\in[0,1[^r.\qquad(2)$$

The $\mathbb{Z}^r$-action T can be rewritten as a composition of r commuting automorphisms $T_1,\ldots,T_r$ of the space Y i.e.

$$T=T_1\circ\ldots\circ T_r.$$

We denote by H the closed space of $L^2(\mu)$ spanned by the functions f such that for every $i=1,\ldots,r$, $T_if=\pm f$. Then, the Hilbert space $L^2(\mu)$ can be decomposed as a direct sum of the space H and the orthogonal complement $H^\perp$ of the space H.

LEMMA 4.1. *Let $g\in H^\perp$ and let μ_g be the spectral measure of T at the point g. Then $\mu_g(\theta)=0$ for every $\theta\in\{0,\frac{1}{2}\}^r$.*

Proof. Let $\theta=(\theta_1,\ldots,\theta_r)$ with $\theta_i\in\{0,\frac{1}{2}\}$ for every $i=1,\ldots,r$. Let us define the new $\mathbb{Z}^r$-action

$$\tilde{T}=\tilde{T}_1\circ\ldots\circ\tilde{T}_r$$

where $\tilde{T}_i = \exp(-2\pi i\theta_i)T_i, i = 1, \dots, r$. The $\mathbb{Z}^r$-action $\tilde{T}$ is a contraction of $L^2(\mu)$. Then, by Von Neumann's theorem for the $\mathbb{Z}^r$-action $\tilde{T}$ (see for instance [21] p. 24), the average

$$\frac{1}{n^r} \sum_{k_1,\dots,k_r=1}^{n} \tilde{T}^k g$$

converges in $L^2(\mu)$ to a function $h \in L^2(\mu)$. Moreover, the function h is fixed for the $\mathbb{Z}^r$-action $\tilde{T}$: for every $i = 1, \dots, r$, $\tilde{T}_i h = h$ that is to say for every $i = 1, \dots, r$, $T_i h = \pm h$ so $h \in H$. Using the spectral lemma, we now have

$$\begin{aligned}
\langle h, g\rangle_{2,\mu} &= \lim_{n\to\infty} \frac{1}{n^r} \sum_{k_1,\dots,k_r=1}^{n} \exp(-2\pi i\langle \theta, k\rangle)\langle T^k g, g\rangle_{2,\mu} \\
&= \int_{[0,1[^r} \lim_{n\to\infty} \frac{1}{n^r} \sum_{k_1,\dots,k_r=1}^{n} \exp(2\pi i\langle t-\theta, k\rangle)\mu_g(dt) \\
&= \mu_g(\theta) = 0.
\end{aligned}$$

Let us come back to the proof of Theorem 1.5. Let $g \in H^\perp$. We use the spectral lemma and get the equality

$$\left\| \frac{1}{n}\sum_{k=1}^{n} g \circ T^{S_k} \right\|_{2,\mu}^2 = \int_{[0,1[^r} \left| \frac{1}{n}\sum_{k=1}^{n} \exp(2\pi i\langle \theta, S_k\rangle) \right|^2 \mu_g(d\theta).$$

In order to prove that this term goes to 0 as $n \to \infty$, we apply the Van der Corput's inequality to the Hilbert space $L^2([0,1[^r, \mu_g)$ and to the sequence $u_k = \exp(2\pi i\langle \theta, S_k\rangle)$, $k \geqslant 1$. Let us begin by verifying the point (a). For every $l \geqslant 1$,

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \langle u_{k+l}, u_k\rangle_{2,\mu_g} &= \frac{1}{n}\sum_{k=1}^{n} \int_{[0,1[^r} \exp(2\pi i\langle \theta, S_k^l\rangle)\mu_g(d\theta) \\
&= \int_{[0,1[^r} \frac{1}{n}\sum_{k=1}^{n} (\exp(2\pi i\langle \theta, S_k^l\rangle) - \phi_k^{(l)}(\theta))\mu_g(d\theta) \\
&+ \int_{[0,1[^r} \frac{1}{n}\sum_{k=1}^{n} \phi_k^{(l)}(\theta)\mu_g(d\theta).
\end{aligned}$$

By applying Theorem 1.8 to the independent random variables $Y_n = S_{ln+m}^l$, the first integral goes to 0 as $n \to \infty$. We have to estimate the second one. Using (2) and the fact that $\log(1-x) \leqslant -x$ for every $x \in [0,1[$, we get

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n}\phi_k^{(l)}(\theta)\right| &\leqslant \frac{1}{n}\sum_{k=1}^{n}|\phi_k^{(l)}(\theta)| \\ &= \frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{r}\exp\left(\frac{1}{2}\sum_{j=1}^{l}\log(1-4f_i(\tau_\alpha^{k+j}x)(1-f_i(\tau_\alpha^{k+j}x))\sin^2(2\pi\theta_i))\right) \\ &\leqslant \max_{k\geqslant 1}\prod_{i=1}^{r}\exp\left(-\frac{\sin^2(2\pi\theta_i)}{2}\sum_{j=1}^{l}4f_i(\tau_\alpha^{k+j}x)(1-f_i(\tau_\alpha^{k+j}x))\right) = \psi_l(\theta).\end{aligned}$$

We denote

$$\gamma_l = \int_{[0,1[^r}\psi_l(\theta)\mu_g(d\theta).$$

Let us now verify that the condition (b) is satisfied so that we need to study the behaviour as $L \to \infty$ of the sequence

$$\frac{1}{L}\sum_{l=1}^{L}\gamma_l = \frac{1}{L}\sum_{l=1}^{L}\int_{[0,1[^r}\psi_l(\theta)\mu_g(d\theta).$$

The sequence $\frac{1}{L}\sum_{l=1}^{L}\psi_l(\theta)$ is bounded by 1 for every $\theta \in [0,1[^r$ and we have

$$\begin{aligned}\frac{1}{L}\sum_{l=1}^{L}\psi_l(\theta) &\leqslant \frac{1}{L}\sum_{l=1}^{L}\max_{k\geqslant 1}\prod_{i=1}^{r}\exp\left(-\frac{\sin^2(2\pi\theta_i)}{2}\Big(\sum_{j=1}^{l}4f_i(\tau_\alpha^{k+j}x)(1-f_i(\tau_\alpha^{k+j}x))-lc_i\Big)\right) \\ &\times \exp\left(-\frac{lc_i\sin^2(2\pi\theta_i)}{2}\right)\end{aligned}$$

where $c_i = \int_{\mathbb{T}^d} f_i(t)(1-f_i(t))dt > 0$. Under the hypotheses of Theorem 1.5, we have for every $i \in \{1,\dots,r\}$,

$$\sup_{x\in\mathbb{T}^d}\left|\frac{1}{n}\sum_{j=1}^{n}4f_i(\tau_\alpha^{j}x)(1-f_i(\tau_\alpha^{j}x)) - c_i\right| \underset{n\to\infty}{\longrightarrow} 0$$

(see [21] p. 156). Then, $\frac{1}{L}\sum_{l=1}^{L}\gamma_l$ goes to 0 as $L \to \infty$, the spectral measure μ_g having no mass point at the points $(\theta_1,\dots,\theta_r)$, with $\theta_i \in \{0, \frac{1}{2}\}$ from the previous lemma so that the condition (b) is verified.

We now have to treat the functions belonging to the space H. As these functions are limits of finite linear combinations of the functions spanning the space H, it is enough to treat the latter ones. If g is a function such that for every $i = 1, \dots, r$, $T_i g = \pm g$, then the sampled ergodic average of g is given by

$$\frac{1}{n}\sum_{k=1}^{n}(-1)^{\sum_{i\in I} S_k^{(i)}} g$$

where $I = \{i \in \{1, \dots, r\}; T_i g = -g\}$. It is easy to see from the definition of the dynamic random walk that when $\operatorname{card}(I)$ is even, this average is g. Otherwise, it is either 0 when n is even or $-\frac{g}{n}$ for n odd, so that its norm in $L^2(\mu)$ converges to 0 as $n \to \infty$.

5. Proof of Theorem 1.8

Since we assumed that for some positive ϵ, we have $\mathbb{E}\left(|Y_n|^\epsilon\right) \leqslant c < \infty$, there exists $\delta > 0$ so that for almost every ω, the absolute value of the derivative of

$$\sum_{k\leqslant n} Z_k(\theta, \omega)$$

with respect to θ is less than n^δ if n is large enough. It follows that for each n, for a suitably large $\sigma > 0$, the sup over all θ can be replaced by a sup over the set

$$\Theta_n = \{kn^{-\sigma}; k = 1, \dots, n^\sigma\}.$$

By Borel-Cantelli lemma, we need to show that for a large enough constant K,

$$\mathbb{P}\left(\sup_{\theta\in\Theta_n} (\log n/n)^{1/2}\left|\sum_{k\leqslant n} Z_k(\theta, \omega)\right| > K\log n\right) < cn^{-2}.$$

But then we just need to show that for every positive σ, there is a K so that for each θ,

$$\mathbb{P}\left((\log n/n)^{1/2}\left|\sum_{k\leqslant n} Z_k(\theta, \omega)\right| > K\log n\right) < cn^{-\sigma}.$$

Denoting by R_k the real part of Z_k, we only show

$$\mathbb{P}\left((\log n/n)^{1/2}\sum_{k\leqslant n} R_k > K\log n\right) < cn^{-\sigma},$$

for large enough K, the remaining cases (imaginary part, $< -K\log n$) being entirely similar. Because of the estimate

$$\mathbb{P}\left((\log n/n)^{1/2}\sum_{k\leqslant n} R_k > K\log n\right) \leqslant \mathbb{E}\exp\left((\log n/n)^{1/2}\sum_{k\leqslant n} R_k\right)\cdot e^{-K\log n},$$

we only need to prove that

$$\mathbb{E}\,\exp\left((\log n/n)^{1/2}\sum_{k\leqslant n}R_k\right)\leqslant e^{c\log n}.$$

Denote $s=(\log n/n)^{1/2}$. Since the random variables R_k are uniformly bounded by 2, have zero mean and $\mathbb{E}\,(R_k^2)\leqslant 2$, using the power series expansion of the function $\exp$, we obtain the bound

$$\mathbb{E}\,(e^{sR_k})\leqslant e^{4s^2}.$$

But then since the R_k are independent,

$$\mathbb{E}\,\exp\left((\log n/n)^{1/2}\sum_{k\leqslant n}R_k\right)\leqslant e^{4ns^2}=e^{4\log n}.$$

6. Open problems

- The assumption on the partial quotients of the angle α in Theorem 1.6 is sufficient to obtain the result. In fact, this hypothesis means that the dynamic random walk we look at is not so far away from the standard Bernoulli random walk but we don't know if it is necessary.
- In [18], it is shown that in case of a random walk in $\mathbb{Z}$, one can get almost everywhere convergence in case of a transient random walk. We think that under the hypotheses $\int f\ dt\neq\frac{1}{2}$ but still $\int f(1-f)\ dt>0$, the almost everywhere convergence should hold for the dynamic random walk.
- In [18], it is also shown that in case of a random walk in dimension strictly greater than one, there is no almost everywhere convergence result. It should be the same for the random walks considered in this paper. Technical difficulties appear due to the temporal inhomogeneity of the dynamic random walk.

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