Mathematical
Inequalities
& Applications
Volume 6, Number 2 (2003), 223–228

A REMARK ON BARRLUND'S LP METHOD

LAJOS LÁSZLÓ

(*communicated by R. Mathias*)

Abstract. In possession of Friedland's theorem [3] on normal completion, Barrlund's linear programming method [1] can be adopted for sharpening the upper bound for the distance of a triangular matrix from the set of normal matrices.

1. Introduction

Denote by $\mathcal{M}_n$ the set of n-th order complex matrices, and let $\mathcal{T}_n \subset \mathcal{M}_n$ and $\mathcal{N}_n \subset \mathcal{M}_n$ be the sets of triangular and normal matrices, resp. In [5] we guessed that for $A \in \mathcal{T}_n$

$$v_F^2(A) \equiv \|A - \mathcal{N}_n\|_F^2 \leqslant \left(1 - \frac{1}{n}\right) \operatorname{dep}_F^2(A) \tag{1}$$

holds, where $\|.\|_F$ stands for the Frobenius norm, and

$$\operatorname{dep}_F(A) = \{\|A\|_F^2 - \sum_{i=1}^{n} |\lambda_i|^2\}^{1/2}$$

is the departure from normality by Henrici defined for $A \in \mathrm{M}_n$ with eigenvalues $\{\lambda_i\}_{i=1}^n$. By now, (1) is a theorem, due to the positive answer for the *normal completion* (NC) problem, see the next section for a short discussion.

However, A. Barrlund, when investigating [1] inequality (1), found that "for the dimensions $n = 3, 5, 6, 7$ and presumably also other problem dimensions it is possible to derive sharper bounds." With the notation

$$A \in \mathcal{T}_n : \quad v_F^2(A) \leqslant (1 - x_n) \operatorname{dep}_F^2(A) \tag{1'}$$

he proved $x_n = \frac{1}{n}$ and $x_n = \frac{1}{n}\frac{n-7/4}{n-1}$ for n even and odd, resp., and he also proved the sharper bounds $x_3 = \frac{3}{8}$, $x_5 = \frac{71}{342}$, $x_6 = \frac{217}{1184}$ and $x_7 = \frac{9393}{64921}$.

His results raise the problem of finding a formula for x_n with $x_n > \frac{1}{n}$ for $n \geqslant 3$. Our result is a rational formula for $x_n^{(k)}$ with the superscript indicating the number of additional restrictions involved in the linear program. The simplest case is $x_n^{(1)} = n/[n(n-1)+2]$ with $x_n^{(1)} = \frac{1}{n-1} + O\left(\frac{1}{n^3}\right)$. However, more is true: for n large, the ratio $\frac{1}{n}$ can be replaced by $\frac{c_1}{n}$ with $c_1 \approx 1.27$.

Mathematics subject classification (2000): 15A45.
Key words and phrases: normal completion, scaled departure, linear programming.
To the memory of Anders Barrlund.

2. The main result

For $A \in \mathcal{T}_n$ define the partition by diagonals to be the decomposition

$$A = A_0 + A_1 + \cdots + A_{n-1}$$

where the (i,j) element of $A_k \in \mathcal{T}_n$ is zero for all $j \neq i+k$ $(k = 0, 1, \ldots, n-1)$. With this let the "scaled" departure be defined as

$$\mathrm{sdep}_F(A) = \left(\sum_{k=1}^{n-1} \frac{k}{k+1} \, \|A_k\|_F^2 \right)^{1/2},$$

for which an analogous to (1) inequality was guessed [5]:

$$v_F^2(A) \leqslant \mathrm{sdep}_F^2(A). \tag{2}$$

REMARK 1. L. Elsner formulated the conjecture on normal completion: "any $A \in \mathcal{T}_n$ can be completed to $B \in \mathcal{N}_n$ with $B - A$ strict lower triangular." Recently, Sh. Friedland proved it, therefore it became the NC Theorem [3].

Its immediate consequence is that

$$(\mathrm{NC}) \Longrightarrow (2) \Longrightarrow (1),$$

where the first was shown in [6], the second one follows from

$$\mathrm{sdep}_F^2(A) \leqslant \left(1 - \frac{1}{n}\right) \mathrm{dep}_F^2(A), \quad A \in \mathcal{T}_n,$$

and the immediate implication $(\mathrm{NC}) \Longrightarrow (1)$ was observed in [2] (note that all these were established when NC was yet a conjecture).

Hence, both (1) and the stronger, scaled version (2) are theorems by now, whereas (2) will be a fundamental tool to improve (1). First we formulate a preparing lemma to defining the constraints for our LP problem.

LEMMA. *Let $A \in \mathcal{T}_n$. For any $i = 1, \ldots, n-1$ we have*

$$v_F^2(A) \leqslant \mathrm{dep}_F^2(A) - \sum_{j=1}^{[(n-1)/i]} \frac{1}{j+1} \|A_{ji}\|_F^2.$$

(Note that the subscript ji of A is the *product* of j and i.)

Proof. We distinguish between three cases.
Case 1. For $i = 1$ the statement is equivalent to (2).
Case 2. For $i = \left[\frac{n+1}{2}\right], \ldots, n-1$ the statement has the form

$$v_F^2(A) \leqslant \mathrm{dep}_F^2(A) - \frac{1}{2} \|A_i\|_F^2.$$

Since $A_0 + A_i$ is a direct sum of 2×2 matrices, this follows from

$$v_F^2(C) = \frac{1}{2}\mathrm{dep}_F^2(C),\ C \in \mathcal{T}_2.$$

Case 3. For $i = 2, \ldots, \left[\frac{n-1}{2}\right]$, $A_0 + A_i$ is no more irreducible, however, the elements $a_{1,i+1}, \ldots, a_{n-i,n}$ of A_i can be placed into the first superdiagonal by permuting the rows and corresponding columns according to the index vector

$$(1, i+1, 2i+1, \ldots, 2, i+2, 2i+2, \ldots, 3, i+3, 2i+3, \ldots).$$

At the same time, the elements of A_{2i} get into the second superdiagonal (the former A_2), and so on.

Let $\Pi^{-1}A\Pi = B + R$ be the new matrix, where Π is the permutation mentioned, B is blockdiagonal with upper triangular blocks of order $\left(\left[\frac{n-j}{i}\right] + 1\right)_{j=1}^{i}$, and R be the rest. Let $N \in \mathcal{N}_n$ be blockdiagonal with the same structure as B such that $\|B - N\|_F^2 \leqslant \mathrm{sdep}_F^2(B)$, then

$$\begin{aligned} v_F^2(A) &= v_F^2(B+R) \leqslant \|B + R - N\|_F^2 \leqslant \mathrm{sdep}_F^2(B) + \|R\|_F^2 \\ &= \mathrm{dep}_F^2(B) - \sum_{j=1}^{[(n-1)/i]} \frac{1}{j+1}\|B_j\|_F^2 + \|R\|_F^2 = \mathrm{dep}_F^2(A) - \sum_{j=1}^{[(n-1)/i]} \frac{1}{j+1}\|A_{ji}\|_F^2, \end{aligned}$$

where we applied (2) for $B \in \mathcal{T}_n$, used $\|A\|_F^2 = \|B\|_F^2 + \|R\|_F^2$, and observed that the nonzero elements of B_j and A_{ji} coincide for all j. The Lemma is proved.

COROLLARY. *Let $A \in \mathcal{T}_n$. For any $i = 1, \ldots, n-1$ we have*

$$v_F^2(A) \leqslant \mathrm{dep}_F^2(A) - \frac{1}{2}\|A_i\|_F^2.$$

Now we are ready to define the variables for the LP problem. For $n \geqslant 3$ and $A \in \mathcal{T}_n$ with unknown entries let

$$x_i = \frac{\|A_i\|_F^2}{\mathrm{dep}_F^2(A)},\ 1 \leqslant i \leqslant n-1, \quad \text{and} \quad x_n = \frac{\mathrm{dep}_F^2(A) - v_F^2(A)}{\mathrm{dep}_F^2(A)}.$$

We obviously have $x_i \geqslant 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n-1} x_i = 1$. From (2) we get the *fundamental inequality* $x_n \geqslant \sum_{i=1}^{n-1} \frac{1}{i+1} x_i$, herewith the initial set of constraints will be

$$P_{\mathrm{init}} = \left\{ x \in \mathbf{R}^n \;\middle|\; x_i \geqslant 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n-1} x_i = 1,\ \sum_{i=1}^{n-1} \frac{x_i}{i+1} \leqslant x_n \right\}.$$

Barrlund's bright idea was observing that the minimum value of x_n subject to constraints like this satisfies (1').

Additional constraints can be obtained from the above Corollary, which takes in the new variables the form $\frac{1}{2}x_i \leqslant x_n,\ 1 \leqslant i \leqslant n-1$. Thus the following theorem will make use of the set

$$P^{(k)} = \left\{ x \in \mathcal{R}^n \;\middle|\; \frac{1}{2}x_{n-i} \leqslant x_n, 1 \leqslant i \leqslant k \right\},\ 1 \leqslant k \leqslant n-1.$$

Observe that, the subscripts are reversed so that they correspond to the degree of difficulty. Also note that for the sake of simplicity we use here only n (instead of the possible $O(n^2)$) variables, in order to be able to obtain a *formula* – not merely numbers.

THEOREM. *Let n, k be natural numbers with $k \leqslant [cn]$, where*

$$c = 1 - 1/\sqrt{e} \approx 0.3935,$$

then the optimal solution $x_n^{(k)}$ of the linear programming problem

$$\min \{x_n \mid x \in P_{\text{init}} \cap P^{(k)}\}$$

is

$$x_n^{(k)} = \left(n + k - 2(n-k)s_n^{(k)}\right)^{-1},$$

where $s_n^{(k)} = \sum_{i=1}^{k} 1/(n-i+1)$.

Proof. We calculate a feasible solution by assuming

$$x_{n-i} = 2x_n,\ i = 1, \dots, k \ \text{ and } \ x_{n-i} = 0,\ i = k+2, \dots, n-1.$$

The equality constraint gives in turn $x_{n-k-1} = 1 - 2kx_n$. Substituting these values into the fundamental inequality (now assumed to be active, i. e. equality) and denoting x_n by $x_n^{(k)}$ we find the formula of the theorem.

The nonnegativity – in fact: the positivity – of $x_n^{(k)}$ follows from the obvious identity

$$n + k - 2(n-k)s_n^{(k)} = n - k + 2\sum_{i=1}^{k} \frac{k-i+1}{n-i+1}. \tag{3}$$

As regards $x_{n-k-1} \geqslant 0$, this is equivalent with $1/x_n^{(k)} \geqslant 2k$, or $s_n^{(k)} \leqslant \frac{1}{2}$, which is fulfilled e. g. for $k \leqslant [cn]$ with the given c, owing to

$$s_n^{(k)} \leqslant \int_{n-k}^{n} \frac{dx}{x} = \ln \frac{n}{n-k} \leqslant \ln \frac{1}{1-c} = \frac{1}{2}.$$

To show optimality, we determine the dual problem. Denote by y_i the dual variable for the inequality $\frac{1}{2}x_{n-i} \leqslant x_n$, $i = 1, \dots, k$. Further, let y_{k+1} and y_{k+2} denote the dual variables for the fundamental inequality and the equality $\sum_{i=1}^{n-1} x_i = 1$, resp. Recall that y_{k+2} is unrestricted, because it corresponds to an equality; all other dual variables are nonnegative. Note that the primary objective is now rewritten into $\max\{-x_n\}$.

Then the dual constraint set is calculated to be

$$\begin{aligned} D \ = \ & \{y \in \mathcal{R}^{k+2},\ y_i \geqslant 0,\ i = 1, \dots, k+1, \\ & \frac{y_i}{2} + \frac{y_{k+1}}{n-i+1} + y_{k+2} \geqslant 0,\ i = 1, \dots, k, \\ & \frac{y_{k+1}}{i} + y_{k+2} \geqslant 0,\ i = 2, \dots, n-k,\ -\sum_{i=1}^{k+1} y_i \geqslant -1\}, \end{aligned}$$

and the dual problem is

$$\min\{y_{k+2} \mid y \in D\}.$$

We give a feasible solution to this problem, too. Since the strongest of the inequalities $y_{k+1}/i + y_{k+2} \geqslant 0$ is that with $i = n-k$, it is reasonable to require $y_{k+1} = -(n-k)y_{k+2}$. Then (taking again equalities), for $i = 1, \dots, k$ we get $y_i = -2(k-i+1)(n-i+1)^{-1}y_{k+2}$. Finally, substituting the expressions obtained into $\sum_{i=1}^{k+1} y_i = 1$ yields

$$y_{k+2} = -\left\{ n - k + 2\sum_{i=1}^{k} \frac{k-i+1}{n-i+1} \right\}^{-1},$$

i. e. owing to (3), $y_{k+2} = -x_n^{(k)}$. Hence also the dual vector y is feasible. Since the primal and dual objectives coincide, the duality theorem of the linear programming guarantees the optimality of both solutions – the proof is complete.

COROLLARY. *For $A \in \mathcal{M}_n$ and $k \leqslant [cn]$, $c = 1 - 1/\sqrt{e}$ we have*

$$\|A - \mathcal{N}_n\|_F^2 \leqslant \left(1 - x_n^{(k)}\right) \operatorname{dep}_F^2(A).$$

Proof. The inequality is true for $A \in \mathcal{T}_n$ by the above. For arbitrary $A \in \mathcal{M}_n$ apply the Schur decomposition theorem and observe that the functions $\nu_F(A) \equiv \|A - \mathcal{N}_n\|_F$ and $\operatorname{dep}_F(A)$ are unitarily invariant.

REMARK 2. For $n \leqslant 34$ the assumption $k \leqslant [cn]$ can be replaced by the simpler rational bound $k \leqslant \left[\frac{2n}{5}\right]$, owing to $c \approx 2/5$.

Below we list the values $x_n^{(k)}$ for $3 \leqslant n \leqslant 9$, $1 \leqslant k \leqslant n-1$. The arrow means that the subsequent elements in that *row* are equal to the last given value, while the symbol $\lceil\ \rceil$ in a *column* indicates the "validity region" for the theorem. (Observe the doubles and triples in the table, showing the validity of the bound with $\frac{2}{5}$ for n small.)

$n \backslash k$	1	2	3	4
3	$\lceil \frac{3}{8} \rceil$	$\rightarrow$		
4	$\frac{2}{7}$	$\rightarrow$		
5	$\frac{5}{22}$	$\lceil \frac{10}{43} \rceil$	$\rightarrow$	
6	$\frac{3}{16}$	$\frac{15}{76}$	$\rightarrow$	
7	$\frac{7}{44}$	$\frac{21}{124}$	$\rightarrow$	
8	$\frac{4}{29}$	$\frac{14}{95}$	$\lceil \frac{84}{559} \rceil$	$\longrightarrow$
9	$\frac{9}{74}$	$\frac{36}{277}$	$\frac{42}{313}$	$\longrightarrow$

We obviously have $x_n^{(1)} \leqslant x_n^{(2)} \leqslant \cdots \leqslant x_n^{(n-1)}$ (with equality from some index on). All these values are better (i. e. larger) than Barrlund's sharpened bounds, except for $n = 3$, when he also obtained $3/8$. This is not by chance: due to Ikramov [4], the NC theorem for $n = 3$ was already for him available.

It is remarkable that for n large the coefficient of $1/n$ can be increased!

COROLLARY. *For the asymptotic behavior of* $\nu_F^2(A)$ *we have*

$$\nu_F^2(A) \leqslant \left(1 - \frac{1}{2cn}\right) \operatorname{dep}_F^2(A), \; A \in \mathcal{M}_n, \;\; n \;\; large$$

with the precise meaning

$$\liminf_{n\to\infty} \left(n x_n^{([cn])}\right) \geqslant \frac{1}{2c} = \frac{\sqrt{e}}{2(\sqrt{e}-1)} \approx 1.2707.$$

Proof. For arbitrary $\varepsilon > 0$, sufficiently large n and $k = [cn]$ we have

$$\begin{aligned} s_n^{(k)} &= \sum_{i=1}^{k} \frac{1}{n-i+1} \geqslant \int_{n-k+1}^{n+1} \frac{dx}{x} = \ln \frac{n+1}{n+1-k} \\ &\geqslant \ln \frac{n+1}{n+1-(c-\varepsilon)(n+1)} = \ln \frac{1}{1-c+\varepsilon}. \end{aligned}$$

Thus – together with the upper estimate proved in the Theorem – we have

$$\frac{1}{2} - \varepsilon' \leqslant s_n^{([cn])} \leqslant \frac{1}{2},$$

where $\varepsilon' \to 0$ for $\varepsilon \to 0$. Hence we get

$$\frac{1}{n x_n^{([cn])}} = \frac{n+[cn]}{n} - 2\frac{n-[cn]}{n} s_n^{([cn])} \leqslant 1 + c - 2s_n^{([cn])} + 2c\, s_n^{([cn])} \leqslant 2c + 2\varepsilon',$$

and the proof is complete.

Acknowledgements. We wish to express our thanks to Professor Elsner for bringing Friedland's theorem to our knowledge.

REFERENCES

[1] A. BARRLUND, *On a conjecture on the closest normal matrix*, MIA, 1/3 (1998), pp 305–318.
[2] L. ELSNER, private communication.
[3] SH. FRIEDLAND, *Normal matrices and the completion problem*, March 15, 2001.
[4] KH. D. IKRAMOV, *On normal dilations of triangular matrices*, Math. Notes, 60/6 (1996), pp 861–872 (Russian).
[5] L. LÁSZLÓ, *Upper bounds for the best normal approximation*, Sixth ILAS Conference, 1996, Chemnitz.
[6] L. LÁSZLÓ, *Upper bounds for the best normal approximation*, Math. Pannonica, 9/1 (1998), 121–129.

(Received January 2, 2002)

Lajos László
Departement of Numerical Analysis
Eötvös Loránd University
Budapest
Pázmány Péter sétány 1/C
HUNGARY-1117
e-mail: laszlo@numanal.inf.elte.hu