Mathematical Inequalities & Applications
Volume 6, Number 2 (2003), 241–245

# ANDERSSON'S INEQUALITY

A. M. FINK

(*communicated by R. N. Mohapatra*)

*Abstract.* Andersson's Inequality gives a lower obund for the integral of a product of convex functions in terms of the averages of each factor. We show that this result holds for a wider class of functions and for some signed measures.

## 1. Introduction

Andersson [1] or [2, pages 256] showed that if $F_i$ are convex increasing functions with $F_i(0) = 0$ then

$$\int_0^1 [F_1(x)\dots F_n(x)]dx \geqslant \frac{2n}{n+1}\left(\int_0^1 F_1(x)dx\right)\dots\left(\int_0^1 F_n(x)dx\right). \tag{1}$$

The condition that $F_i(0) = 0$ is crucial since for $F_1(x) = 1 + x^2$ and $F_2(x) = 1 + x^3$, the reverse inequality holds. On the other hand one wonders if the measure needs to be Lebesgue measure or if the functions need to be convex. In fact we will show that neither is required. We will proceed in two steps. The first is to show that the convexity of the functions may be replaced by the condition that $\frac{F_i(x)}{x}$ is increasing. We will modify Andersson's proof and then in the second step we will replace Lebesgue measure by more general measures. This requires a different sort of proof.

## 2. Modifications of Andersson's proof

We suppose that $F_i$ are in the class $M_1$ where $M_1 = \{f \mid f \in C^1, f(0) = 0,$ and $\frac{f(t)}{t}$ is increasing on $[0, 1]\}$. Note that if $f(0) = 0$ and $f$ is increasing and convex (as in Andersson's Theorem) then $f \in M_1$. Indeed, in this case $\frac{f(t)}{t} = \frac{1}{t}\int_0^t f'(s)ds = \int_0^1 f'(tu)du$ is increasing.

*Mathematics subject classification* (2000): 26D15.
*Key words and phrases*: convex function, Andersson's Inequality, Chebyshev's inequality.


Paper MIA-06-23

On the other hand $f(t) = \frac{t^2}{1+t}$ is in $M_1$ but is not convex.
For $f \in M_1$ we define

$$f^*(x) = 2x \int_0^1 f(t)dt = 2xd_f. \tag{1}$$

The basic result is the following lemma.

LEMMA 1. *Let $f \in M_1$ and $g$ be increasing, then*

$$\int_0^1 fg dx \geqslant \int_0^1 f^* g dx \tag{2}$$

*with equality if $g$ is a constant or $f$ is linear.*

*Proof.* Define $\varphi(x)$ by

$$\varphi(x) = \int_0^x [f^*(t) - f(t)]dt.$$

Then $\varphi(0) = \varphi(1) = 0$ by choice of $f^*$ so that $\varphi'$ must have a zero in $(0,1)$. Now $\varphi'(x) = f^*(x) - f(x)$ has a zero at 0 and $\varphi'(x) = x\left[2d - \frac{f(x)}{x}\right]$. The expression in the brackets is decreasing so it changes sign exactly once in $(0,1)$ and is positive near 0. It follows that $\varphi \geqslant 0$ on $(0,1)$. Note that $\varphi \equiv 0$ requires $f$ to be linear.

To complete the proof we write

$$\int_0^1 g(f - f^*)dx = \int_0^1 g(-\varphi') = \int_0^1 \varphi dg \geqslant 0$$

with equality when $\varphi(dg) \equiv 0$.

THEOREM 1. *Let $f_i \in M_1$ for $i = 1, \ldots, n$, then*

$$\int_0^1 \prod_1^n f_i(x)dx \geqslant \int_0^1 \prod_1^n f_i^*(x)dx = \frac{2^n}{n+1} \prod_1^n \int_0^1 f_i(x)dx. \tag{3}$$

*Equality holds if all of the $f_i$ are linear.*

*Proof.* We let $g = \prod_1^{n-1} f_i$ and apply the lemma to get

$$\int_0^1 \prod_1^n f_i(x)dx \geqslant \int_0^1 \prod_1^{n-1} f_i(x) f_n^*(x)dx$$

$$= \int_0^1 \left[\prod_1^{n-2} f_i(x) f_n^*(x)\right] f_{n-1}(x)dx \geqslant \int_0^1 \prod_1^{n-1} f_i(x) f_{n-1}^*(x) f_n^*(x)dx \geqslant \dots$$

$$\geqslant \int_0^1 \prod_1^n f_i^*(x)dx = \prod_1^n d_i \int_0^1 2(x)^n dx.$$

## 3. Extended version

We now replace Lebesgue measure by a (signed) measure $d\sigma$ for which we assume

$$\int_a^x (t-a)d\sigma(t) \geqslant 0, \int_x^b (t-a)d\sigma(t) \geqslant 0 \text{ on } (a,b) \text{ with} \tag{4}$$

$$\int_a^b (t-a)d\sigma(t) > 0. \tag{5}$$

Correspondingly, we define $M_2 = \{f \,|f \in C^1, f(a) = 0, \text{ and } \frac{f(t)}{t-a} \text{ is increasing on } (a,b)\}$. For $f \in M_2$ define

$$d_f = \int_a^b f \; d\sigma / \int_a^b (x-a)d\sigma(x) \tag{6}$$

and

$$f^*(x) = d_f(x-a). \tag{7}$$

We attempt to argue as in Lemma 1. For this purpose we have an introductory lemma.

LEMMA 2. *Let $f \in M_2, f^*$ as in (7) with the measure $d\sigma$ satisfying (4) and (5). Then $\varphi(u) \equiv \int_u^b [f(t) - f^*(t)]d\sigma(t) \geqslant 0$ on $[a,b]$.*

*Proof.* Observe that $\varphi(b) = 0$ and by choice of $f^*, \varphi(a) = 0$. Let $x_+^0$ be the indicator function of $[0,\infty)$ and fix $u \in (a,b)$. Then

$$\varphi(u) = \int_a^b (t-u)_+^0 [f(t) - f^*(t)]d\sigma(t).$$

Using the definition of $f^*$ we have

$$\varphi(u) = \int_a^b (t-u)_+^0 f(t)d\sigma(t) - \int_a^b (t-u)_+^0 (t-a) \left( \frac{\int_a^b f(x)d\sigma(x)}{\int_a^b (x-a)d\sigma(x)} \right) d\sigma(t)$$

$$= \frac{\int_a^b (t-u)_+^0 f(t)d\sigma(t) \int_a^b (x-a)d\sigma(x) - \int_a^b (t-u)_+^0 (t-a)d\sigma(t) \int_a^b f(x)d\sigma(x)}{\int_a^b (x-a)d\sigma(x)}.$$

It is the expression in the numerator that we must show is non-negative. For this purpose we define $g(t) = \frac{f(t)}{t-a}$ and $d\lambda(x) = (x-a)d\sigma(x)$. Note that $g$ is positive and increasing while $\int_a^x d\lambda(t)$ and $\int_x^b d\lambda(t)$ are $\geqslant 0$ by hypothesis. The numerator can be rewritten as

$$\int_a^b (t-u)_+^0 g(t)d\lambda(t) \int_a^b d\lambda(x) - \int_a^b g(x)d\lambda(x) \int_a^b (t-u)_+^0 d\lambda(t) \tag{8}$$

and this expression is what we want to be non-negative, i.e. we want

$$\int_a^b (t-u)_+^0 g(t)d\lambda(t) \int_a^b d\lambda(x) \geqslant \int_a^b g(x)d\lambda(x) \int_a^b (t-u)_+^0 d\lambda(t). \tag{9}$$

This is in fact Chebyshev's Inequality for the two increasng functions $(t-u)_+^0$ and $g(t)$. By Fink and Jodeit's Theorem [2, page 273] or [3] the conditions (4) and (5) are sufficient for (9) to hold.

COROLLARY. *Let $d\sigma$ satisfy (4) and (5), $f \in M_2$ and $g$ be a positive increasing function. Then*

$$\int_a^b g(x)f(x)d\sigma(x) \geqslant \int_a^b g(x)f^*(x)d\sigma(x).$$

*Proof.* We have for the function $\varphi$ of Lemma 2 that

$$\begin{aligned}
\int_a^b g(x)[f(x)-f^*(x)]d\sigma(x) &= \int_a^b [g(a) + \int_a^x dg(t)dt][f(x)-f^*(x)]d\sigma(x) \\
&= g(a)\varphi(a) + \int_a^b [f(x)-f^*(x)] \left( \int_a^x dg(t) \right) d\sigma(x) \\
&= g(a)\varphi(a) + \int_a^b \left( \int_u^b [f(x)-f^*(x)]d\sigma(x) \right) dg(u) \\
&= g(a)\varphi(a) + \int_a^b \varphi(u)dg(u).
\end{aligned}$$

Each term is non-negative by hypothesis and Lemma 2.

THEOREM 2. *Let $f_i \in M_2$ and the measure $d\sigma$ satisfy (4) and (5). Then*

$$\int_0^b \prod_1^n f_i(x)d\sigma(x) \geqslant \frac{\int_a^b (x-a)^n d\sigma(x)}{\left(\int_a^b (x-a)d\sigma(x)\right)^n} \prod_1^n \int_a^b f_i(x)d\sigma(x). \tag{10}$$

*Equality holds if all $f_i(x)$ are linear.*

*Proof.* We merely observe that if $f$ and $g \in M_2$ then $fg$ is positive and increasing on $[a,b]$. We can apply the proof of Theorem 1, this time using the corollary for each step. The final integral is $\int_a^b \prod_1^n f_i^*(x)d\sigma(x)$ which is equal to

$$\int_a^b \prod_1^n d_{f_i}(x-a)^n d\sigma(x) = \frac{\int_a^b (x-a)^n d\sigma(x)}{\left(\int_0^n (x-a)(d\sigma(x)\right)^n} \prod_a^n \int_a^b f_i(x)d\sigma(x).$$

Note that if $\sigma$ is Lebesque measure we get

$$\frac{1}{b-a}\int_a^b \prod_1^n f_i(x)dx \geqslant \frac{2^n}{n+1}\prod_1^n \left(\frac{1}{b-a}\int_a^b f_i(x)dx\right) \tag{11}$$

which is Theorem 1 on the interval $[a,b]$. This is more obviously a result about averages.

## REFERENCES


[1] ANDERSSON, B. X., *An inequality for convex functions*, Nordisk Mat. Tidsk **6** (1958), 25–26.

[2] MITRINOVIĆ, D. S., PEČARIĆ, J., AND FINK A. M., *Classical and New Inequalities in Analysis*, Kluwer, 1993.

[3] FINK, A. M. AND JODEIT, M., JR., *On Chebyshev's other Inequality*, in Inequalities in Statistics and Probability (Lecture notes IMS, no. 5) Inst. Statist. Hayward, CA (1984), 115–120.


(Received January 17, 2002)

*A. M. Fink*
*Mathematics Department*
*Iowa State University*
*Ames, IA 50011, USA*
*e-mail:* fink@math.iastate.edu