Mathematical
Inequalities
& Applications
Volume 6, Number 2 (2003), 303–308

A MONOTONE OPERATOR FUNCTION VIA FURUTA–TYPE INEQUALITY WITH NEGATIVE POWERS

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(*communicated by J. Pečarić*)

Abstract. We strengthen a recent monotonicity theorem on an operator inequality with negative powers, and this is done by using means of positive operators.

1. Introduction

In what follows, H means a complex Hilbert space. A bounded linear operator T on H is said to be positive (in symbol: $T \geqslant 0$), if $(Tx, x) \geqslant 0$ for any $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$), if T is positive and invertible.

As in [7-8], we introduce first a mean of operators [12]. The set containing all positive operators in H is denoted by $\ell^+(H)$. A binary operation m on the class of positive operators $(A, B) \mapsto AmB$ is called a mean if the following requirements are fulfilled:

(a) $A \leqslant C, B \leqslant D$, imply $AmB \leqslant CmD$;
(b) $A \geqslant 0, B \geqslant 0, C \geqslant 0$, imply $C(AmB)C \geqslant (CAC)m(CBC)$;
(c) If $A_n \downarrow A$, $B_n \downarrow B$, and A_n, B_n, A, B are all positive operators, then $(A_n m B_n) \downarrow (AmB)$;
(d) $ImI \leqslant I$, where I is the identity operator on H.

An immediate consequence of the definition is that m is upper semicontinuous and satisfies

$$A \leqslant C \quad \text{and} \quad B \leqslant D \quad \text{imply} \quad AmB \leqslant CmD$$

and the transformer inequality

$$T^*(AmB)T \leqslant (T^*AT)m(T^*BT)$$

for all T. We note that if T is invertible, then this inequality is replaced by the equality as following

$$T^*(AmB)T = (T^*AT)m(T^*BT).$$

Mathematics subject classification (2000): 47A30, 47A63, 47B15.

Key words and phrases: operator, positive operator, Furuta-type inequalities with negative powers.

Foundation item: Natural Science and Education Foundation of Henan Province, 1980.

By the main result in [13] there is a unique mean m corresponding to the operator monotone function x^s for $0 \leqslant s \leqslant 1$,

$$1 m_s x = x^s.$$

In particular, the mean $m = m_{\frac{1}{2}}$ is called the geometric mean as in the case of scalars.

THEOREM F (Furuta Inequality). *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geqslant 0$, $q \geqslant 1$ and $(1+r)q \geqslant p+r$.

We remark that Theorem F yields the following famous *Löwner-Heinz* theorem when we put $r = 0$ in (i) or (ii) stated above.

THEOREM L-H. *$A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$.*

By using Theorem F in [12] Furuta showed that if $A \geqslant B \geqslant 0$, then for each $p \geqslant 1$ and $r \geqslant 0$, $F(p) = (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ is increasing for $p \geqslant 1$, Also, he pointed out that this result didn't remain valid for $0 < p < 1$ and $r \geqslant 0$. In this paper, we give an extension and application of Furuta inequality with negative powers by using means of operators.

THEOREM 1.1. *If $A \geqslant B \geqslant 0, 0 \leqslant p_0 \leqslant 1$ with $-1 \leqslant r \leqslant 0$,*

(a) *if $p_0 \leqslant -r$, then $M(\alpha, r) = A^{-\frac{r}{2}} (A^{\frac{r}{2}} B^\alpha A^{\frac{r}{2}})^{\frac{p_0+r}{\alpha+r}} A^{\frac{-r}{2}}$*

is decreasing for $\alpha \in [0, p_0]$ and increasing for $r \in [-1, -p_0]$.

(b) *if $p_0 \geqslant -r$, then $M(\alpha, r) = A^{-\frac{r}{2}} (A^{\frac{r}{2}} B^\alpha A^{\frac{r}{2}})^{\frac{p_0+r}{\alpha+r}} A^{\frac{-r}{2}}$*

is increasing for $\alpha \in [p_0, 1]$ and decreasing for $r \in [-p_0, 0]$.

2. Proofs of the main results

We need the following lemma and theorem.

LEMMA ([8]). *For invertible positive operator A and invertible operator B, let $A m_s B = A^{1/2}(A^{-1/2} B A^{-1/2})^s A^{1/2}$, then*

$$A m_s B = B m_{1-s} A$$

holds for any real number s.

THEOREM A ([3] [4] [9] [17]). *If $A \geqslant B > 0$, then the following inequalities hold:*

(i) $A^{1-t} \geqslant (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ *for $1 \geqslant p > t \geqslant 0$ with $p \geqslant \frac{1}{2}$,*

(ii) $A^{-t} \geqslant (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ *for $1 \geqslant t > p \geqslant 0$ with $\frac{1}{2} \geqslant p$,*

(iii) $A^{2p-t} \geqslant (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$ *for $\frac{1}{2} \geqslant p > t \geqslant 0$,*

(iv) $A^{2p-1-t} \geqslant (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}}$ *for $1 \geqslant t > p \geqslant \frac{1}{2}$.*

Proof of Theorem 1.1.

Proof of (a). We may assume that A and B are both invertible. First, we prove $M(\alpha+\varepsilon,r)\leqslant M(\alpha,r)$ for $0\leqslant\varepsilon\leqslant p_0-\alpha$

$$\begin{aligned}
M(\alpha+\varepsilon,r) &= A^{-\frac{r}{2}}(A^{\frac{r}{2}}B^{\alpha+\varepsilon}A^{\frac{r}{2}})^{\frac{p_0+r}{\alpha+\varepsilon+r}}A^{\frac{-r}{2}}\\
&= A^{-r}m_{\frac{p_0+r}{\alpha+r}}B^{\alpha+\varepsilon}\\
&= B^{\frac{\alpha}{2}}((B^{-\frac{\alpha}{2}}A^{-r}B^{-\frac{\alpha}{2}})m_{\frac{p_0+r}{\alpha+r}}B^{\varepsilon})B^{\frac{\alpha}{2}}\\
&\leqslant B^{\frac{\alpha}{2}}[(B^{-\frac{\alpha}{2}}A^{-r}B^{-\frac{\alpha}{2}})m_{\frac{p_0+r}{\alpha+r}}(B^{-\frac{\alpha}{2}}A^{-r}B^{-\frac{\alpha}{2}})^{\frac{-\varepsilon}{\alpha+r}}]B^{\frac{\alpha}{2}}\\
&= B^{\frac{\alpha}{2}}(B^{-\frac{\alpha}{2}}A^{-r}B^{-\frac{\alpha}{2}})^{1-\frac{p_0+r}{\alpha+r}}B^{\frac{\alpha}{2}}\\
&= A^{-\frac{r}{2}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})^{\frac{p_0+r}{\alpha+r}}A^{-\frac{r}{2}}\\
&= M(\alpha,r).
\end{aligned}$$

Next we show the monotonicity in r.

Let $\varepsilon>0$ and satisfy $-1<r<r+\varepsilon<-p_0$, then $0\leqslant\frac{p_0+r+\varepsilon}{\alpha+r+\varepsilon}\leqslant 1$.

$$\begin{aligned}
M(\alpha,r+\varepsilon) &= A^{-\frac{r+\varepsilon}{2}}(A^{\frac{r+\varepsilon}{2}}B^{\alpha}A^{\frac{r+\varepsilon}{2}})^{\frac{p_0+r+\varepsilon}{\alpha+r+\varepsilon}}A^{-\frac{r+\varepsilon}{2}}\\
&= A^{-\frac{r+\varepsilon}{2}}(A^{\frac{\varepsilon}{2}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})A^{\frac{\varepsilon}{2}})^{\frac{p_0+r+\varepsilon}{\alpha+r+\varepsilon}}A^{-\frac{r+\varepsilon}{2}}\\
&= A^{-\frac{r}{2}}[A^{-\frac{\varepsilon}{2}}(A^{\frac{\varepsilon}{2}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})A^{\frac{\varepsilon}{2}})^{\frac{p_0+r+\varepsilon}{\alpha+r+\varepsilon}}A^{-\frac{\varepsilon}{2}}]A^{-\frac{r}{2}}\\
&= A^{-\frac{r}{2}}(A^{-\varepsilon}m_{\frac{p_0+r}{\alpha+r}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}}))A^{-\frac{r}{2}}\\
&\geqslant A^{-\frac{r}{2}}[(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})^{\frac{-\varepsilon}{\alpha+r}}m_{\frac{p_0+r}{\alpha+r}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})]A^{-\frac{r}{2}}\\
&= A^{-\frac{r}{2}}(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})^{\frac{p_0+r}{\alpha+r}}A^{\frac{-r}{2}}\\
&= M(\alpha,r).
\end{aligned}$$

Proof of (b). It is easy to show by almost the same way as one in (a), using *L-H* theorem. In fact, for sufficiently small positive number ε,

$$B^{\varepsilon}\geqslant(B^{-\frac{\alpha}{2}}A^{-r}B^{-\frac{\alpha}{2}})^{\frac{-\varepsilon}{\alpha+r}}\quad\text{and}\quad A^{-\varepsilon}\geqslant(A^{\frac{r}{2}}B^{\alpha}A^{\frac{r}{2}})^{\frac{-\varepsilon}{\alpha+r}}$$

because of $-r\in[0,1]$ and $\alpha\in[0,1]$.

3. Some applications

THEOREM 2.1. *If* $A\geqslant B\geqslant 0$ *with* $A>0, 1\geqslant p>t\geqslant 0$ *with* $p\geqslant\frac{1}{2}$, *then for any fixed* $p_0\in[0,1]$,

$$G_{p,t}(A,B,r,s)=A^{-\frac{r}{2}}[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}]^{\frac{(1-t)p_0+r}{(p-t)s+r}}A^{-\frac{r}{2}}$$

is decreasing for $r\in[-(1-t)p_0,\ 0]$ *and increasing for* $s\in[\frac{(1-t)p_0}{p-t},\ \frac{1-t}{p-t}]$, *and the following inequality holds:*

$$A^{(1-t)p_0}\geqslant G_{p,t}(A,B,r,s)\geqslant(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{\frac{(1-t)p_0}{p-t}}.$$

Proof. $A\geqslant B\geqslant 0$ ensures the following (3.1) by theorem A

$$A^{(1-t)}\geqslant(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{\frac{1-t}{p-t}}\tag{3.1}$$

for each $0 \leqslant p \leqslant 1$ and $p \geqslant t \geqslant 0$. Put $A_1 = A^{(1-t)}$ and $B_1 = (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1-t}{p-t}}$. Then

$$A_1 \geqslant B_1 \geqslant 0 \tag{3.2}$$

so that by Theorem 1.1, if $0 \leqslant -r_1 \leqslant 1, -r_1 \leqslant p_0 \leqslant 1$, then

$$G(\alpha, r_1) = A_1^{-\frac{r_1}{2}}(A_1^{\frac{r_1}{2}}B_1^{\alpha}A_1^{\frac{r_1}{2}})^{\frac{p_0+r_1}{\alpha+r_1}}A_1^{-\frac{r_1}{2}} \tag{3.3}$$

is increasing for $p_1 \in [p_0, 1]$ and decreasing for $r_1 \in [-p_0, 0]$.

Put $r_1 = \frac{r}{1-t} \in [-p_0, 0]$ and $\alpha = \frac{(p-t)s}{1-t} \in [p_0, 1]$, then

$$\frac{p_0+r_1}{\alpha+r_1} = \frac{(1-t)p_0+r}{(p-t)s+r}, A_1^{-\frac{r_1}{2}} = A^{-\frac{r}{2}} \quad \text{and} \quad B_1^{p_1} = (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s \tag{3.4}$$

so that by (3.1), (3.2), (3.3) and (3.4), we have that

$$G_{p,t}(A,B,r,s) = A^{-\frac{r}{2}}\left[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\right]^{\frac{(1-t)p_0+r}{(p-t)s+r}}A^{-\frac{r}{2}}$$

is decreasing for $r \in [-(1-t)p_0,\ 0]$ and increasing for $s \in [\frac{(1-t)p_0}{p-t},\ \frac{1-t}{p-t}]$.

THEOREM 2.2. *If $A \geqslant B \geqslant 0$ with $A > 0$, $1 \geqslant q \geqslant p \geqslant \frac{q}{2}$ and $p \geqslant t \geqslant 0$, then for any fixed $p_0 \in [0,1]$,*

$$H_{p,t}(A,B,r,s) = A^{-\frac{r}{2}}\left[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\right]^{\frac{(q-t)p_0+r}{(p-t)s+r}}A^{-\frac{r}{2}}$$

is decreasing for $r \in [-(q-t)p_0,\ 0]$ and increasing for $s \in [\frac{(q-t)p_0}{p-t},\ \frac{q-t}{p-t}]$.

The following inequality holds:

$$A^{(1-t)p_0} \geqslant H_{p,t}(A,B,r,s) \geqslant (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{(q-t)p_0}{p-t}}.$$

Proof. $A \geqslant B \geqslant 0$ with $A > 0$ ensures $A^q \geqslant B^q$ by *Löwner−Heinz* theorem. Put

$$A_1 = A^q \quad \text{and} \quad B_1 = B^q \tag{3.5}$$

then $A_1 \geqslant B_1 \geqslant 0$. Let

$$p_1 = \frac{p}{q},\ t_1 = \frac{t}{q} \quad \text{and} \quad r_1 = \frac{r}{q}. \tag{3.6}$$

Then $1 \geqslant p_1 \geqslant t_1 \geqslant 0$, $p_1 \geqslant \frac{1}{2}$, and $r_1 \in [-(1-t_1)p_0, 0]$, $s_1 \in [\frac{(1-t_1)p_0}{p_1-t_1}, \frac{1-t_1}{p_1-t_1}]$, and by theorem 2.1 we have that

$$G_{p_1,t_1}(A_1,B_1,r_1,s_1) = A_1^{-\frac{r_1}{2}}\left(A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}}\right)^{\frac{(1-t_1)p_0+r_1}{(p_1-t_1)s+r_1}}A_1^{\frac{-r_1}{2}}$$

is decreasing for $r_1 \in [-(1-t_1)p_0,\ 0]$ and increasing for $s \in [\frac{(1-t_1)p_0}{p_1-t_1},\ \frac{1-t_1}{p_1-t_1}]$. Put

$$p_1 = \frac{p}{q},\ t_1 = \frac{t}{q} \quad \text{and} \quad r_1 = \frac{r}{q}. \tag{3.7}$$

Then $p_1 \in [\frac{1}{2}, 1]$, $t_1 \in [0,1]$ and $r_1 \geqslant t_1$. It follows from (3.5), (3.6), and (3.7) that

$$H_{p,t}(A,B,r,s) = A^{-\frac{r}{2}} \left[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}} \right]^{\frac{(q-t)p_0+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

is decreasing for $r \in [-(q-t)p_0, 0]$ and increasing for $s \in \left[\frac{(q-t)p_0}{p-t},\ \frac{q-t}{p-t}\right]$.

THEOREM 2.3. *If $A \geqslant B \geqslant 0$ with $A > 0$, and $\frac{1}{2} > p > t > 0$, then for any fixed $p_0 \in [0,1]$,*

$$F_{p,t}(A,B,r,s) = A^{-\frac{r}{2}} (A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}})^{\frac{(2p-t)p_0+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is decreasing for $r \in [-(2p-t)p_0, 0]$ and increasing for $s \in \left[\frac{(2p-t)p_0}{p-t},\ \frac{2p-t}{p-t}\right]$. Moreover, the following inequality holds:

$$A^{(2p-t)p_0} \geqslant F_{p,t}(A,B,r,s) \geqslant \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^{\frac{(2p-t)p_0}{p-t}}.$$

Proof. We have only to replace '1' in the proof of theorem 2.1 by '2p'.

Acknowledgment. We would like to express our cordial thanks to Professor T. Furuta for his careful reading of the first version of this paper.

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(Received January 15, 2001)

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