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ON p–QUASIHYPONORMAL OPERATORS AND QUASISIMILARITY

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(*communicated by T. Furuta*)

Abstract. In this paper we show some structural properties of a p-quasihyponormal operator via Löwner-Heinz inequality and Hansen inequality. As important applications, it is shown that the normal parts of quasisimilar injective p-quasihyponormal operators are unitarily equivalent and quasisimilar injective p-quasihyponormal operators have same spectra and essential spectra.

1. Introduction

Let H and K be infinite dimensional complex Hilbert spaces and let $L(H,K)$ denote the algebra of bounded linear operators from H to K. If $H = K$, we write $L(H)$ in place of $L(H,K)$. Recall ([1], [6], [10], [14], [16], [20]) that an operator $T \in L(H)$ is called p-*hyponormal* if

$$(T^*T)^p - (TT^*)^p \geqslant 0 \quad \text{for } p \in (0,1].$$

If $p = 1$, T is said to be hyponormal and if $p = 1/2$, T is said to be semi-hyponormal ([19]). Also, recall ([3], [12], [17]) that an operator $T \in L(H)$ is called p-*quasihyponormal* if

$$T^*((T^*T)^p - (TT^*)^p)T \geqslant 0.$$

If $p = 1$, T is quasihyponormal ([3]). An operator T is *paranormal* if

$$||Tx||^2 \leqslant ||T^2x||\,||x|| \quad \text{for all } x \in H.$$

Clearly, a p-hyponormal operator is a p-quasihyponormal operator. It is well-known that a p-hyponormal operator is also q-hyponormal for every $0 < q \leqslant p$. However, p-quasihyponormal operators are not always q-quasihyponormal even if $0 < q < p$ (see [17]). It is known that for $p \in (0,1]$

$$\begin{aligned}\{\text{hyponormal operators}\} &\subseteq \{p - \text{hyponormal operators}\}\\ &\subseteq \{p - \text{quasihyponormal operators}\}\\ &\subseteq \{\text{paranormal operators}\}\end{aligned}$$

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In [17], it is well known that if an operator $T \in L(H)$ is p-quasihyponormal for some $0 < p \leqslant 1$ then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad H = \overline{R(T)} \oplus N(T^*),$$

where $T_1 = T|_{\overline{R(T)}}$ is p-hyponormal which satisfies $(T_1{}^*T_1)^p \geqslant (T_1T_1{}^* + T_2T_2{}^*)^p$. Thus we easily see that if T is p-quasihyponormal with dense range then T is just p-hyponormal (for details, see [17]).

Recall ([5], [6], [10], [18]) that an operator $X \in L(H)$ is called a *quasiaffinity* if X is injective and has dense range. For $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$, if there exist quasiaffinities $X \in L(H_2, H_1)$ and $Y \in L(H_1, H_2)$ such that

$$T_1X = XT_2 \quad \text{and} \quad YT_1 = T_2Y,$$

then we say that T_1 and T_2 are *quasisimilar*.

Conway ([5]) proved that the normal parts of quasisimilar subnormal operators are unitarily equivalent and gave an example showing that the pure parts of quasisimilar subnormal operators need not be quasisimilar. This result was extended by Williams([18]) to the more general case of dominant operators and also, extended by Jeon and Duggal ([11]) to p-hyponormal operators. The major tool in [11] unlike that of Williams comes from *Aluthge transform* (cf. [1], [6], [10], [11]) of a p-hyponormal operator. Since this tool will be used in this paper, we introduce it here for completeness. We decompose a p-hyponormal operator T into its normal and pure parts by $T = T_1 \oplus T_2$ with respect to a decomposition $H = H_1 \oplus H_2$. Then it is well known T_2 is also p-hyponormal. Letting T_2 have the polar decomposition $T_2 = U|T_2|$, we consider its Aluthge transform $\widehat{T_2} = |T_2|^{1/2}U|T_2|^{1/2}$. Again, let $\widehat{T_2} = V|\widehat{T_2}|$, and define $\widetilde{T_2} = |\widehat{T_2}|^{1/2}V|\widehat{T_2}|^{1/2}$. Using the Furuta's inequality([7]), Aluthge ([1, Theorem 1,2]) showed the Aluthge transform $\widehat{T_2}$ of T_2 is semi-hyponormal and the second Aluthge transform $\widetilde{T_2}$ of T_2 is hyponormal. (Though it was proved in the special case in which the partial isometry in the polar decomposition of a p-hyponormal operator is unitary, the proof can be made to work in general case.) Letting $W = |\widehat{T_2}|^{1/2}|T_2|^{1/2}$, by Corollary 4 in [6] we can see that W is a quasiaffinity such that $\widetilde{T_2}W = WT_2$. Since $\widetilde{T_2}$ is hyponormal, if we set $X := I_{H_1} \oplus W$ and $\widetilde{T} := T_1 \oplus \widetilde{T_2}$, then X is a quasiaffinity such that $\widetilde{T}X = XT$ where $\widetilde{T}$ is also hyponormal.

In this paper we show some structural properties of a p-quasihyponormal operator and, as its applications, we extend some results proved in [11] for the class of p-hyponormal operators to the class of p-quasihyponormal operators for $p \in (0,1]$. In addition, it is shown that quasisimilar injective p-quasihyponormal operators have the same spectra and essential spectra, which extends Corollary 12 in [20]. Throughout this paper we only consider the case $p \in (0,1]$.

2. Results

We begin by important structural properties for a p-quasihyponormal operator. These properties are essentially induced by the following famous inequalities.

PROPOSITION A. (Löwner-Heinz inequality [13],[9]). *If* $B \geqslant A \geqslant 0$, *then* $B^{\alpha} \geqslant A^{\alpha} \geqslant 0$ *for* $\alpha \in (0,1]$.

PROPOSITION B. (Hansen's inequality [8]). *If* $A,\ B \in L(H)$ *satisfy* $A \geqslant 0$ *and* $||B|| \leqslant 1$, *then* $(B^*AB)^{\delta} \geqslant B^*A^{\delta}B$ *for all* $\delta \in (0,1]$.

THEOREM 1. *Let* $T \in L(H)$ *be a* p*-quasihyponormal operator and* $\mathcal{M}$ *be its invariant subspace. Then the restriction* $T|_{\mathcal{M}}$ *of* T *to* $\mathcal{M}$ *is also* p*-quasihyponormal.*

Proof. Let E be the orthogonal projection onto $\mathcal{M}$. Put $T' = T|_{\mathcal{M}}$. Then $TE = ETE$ and $T' = (ETE)|_{\mathcal{M}}$. Since T is p-quasihyponormal, we have

$$ET^*(T^*T)^pTE \geqslant ET^*(TT^*)^pTE.$$

Since

$$\begin{aligned} ET^*(T^*T)^pTE &= ET^*E(T^*T)^pETE \\ &\leqslant ET^*E(ET^*TE)^pETE \quad \text{(by Proposition B)} \\ &= ET^*E(ET^*ETE)^pETE \end{aligned}$$

and

$$\begin{aligned} ET^*(TT^*)^pTE &= ET^*E(TT^*)^pETE \\ &\geqslant ET^*E(TET^*)^pETE \quad \text{(by Proposition A)} \\ &= ET^*E(ETET^*E)^pETE, \end{aligned}$$

we have

$$ET^*E(ET^*ETE)^pETE \geqslant ET^*E(ETET^*E)^pETE.$$

From $T' = (ETE)|_{\mathcal{M}}$, we also have

$$T'^*(T'^*T')^pT' \geqslant T'^*(T'T'^*)^pT'.$$

This implies that T' is p-quasihyponormal. $\square$

THEOREM 2. *Let* T *be a* p*-quasihyponormal operator. If the restriction* $T|_{\mathcal{M}}$ *of* T *to an invariant subspace* $\mathcal{M}$ *is an injective normal operator, then* T *is reduced by* $\mathcal{M}$.

Proof. Since T is p-quasihyponormal

$$P\{(T^*T)^p - (TT^*)^p\}P \geqslant 0,$$

where P is the orthogonal projection onto $\overline{R(T)}$. Let E be the orthogonal projection onto $\mathcal{M}$. Put $T' = T|_{\mathcal{M}}$ and

$$T = \begin{pmatrix} T' & A \\ 0 & B \end{pmatrix} \quad \text{on} \quad H = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Then $E \leqslant P$ because T' is injective normal by the assumption and it has dense range(i.e., $\overline{R(T')} = \mathcal{M}$), therefore $\mathcal{M} \subseteq \overline{R(T)}$. Hence we have

$$E\{(T^*T)^p - (TT^*)^p\}E \geqslant 0,$$

and

$$\begin{aligned}\begin{pmatrix} (T'T'^*)^p & 0 \\ 0 & 0 \end{pmatrix} &= E(TET^*)^p E \\ &\leqslant E(TT^*)^p E \qquad \text{(by Proposition A)} \\ &\leqslant E(T^*T)^p E \\ &\leqslant (ET^*TE)^p \\ &= \begin{pmatrix} (T'^*T')^p & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(by Proposition B)}.\end{aligned}$$

Since T' is normal $E(TT^*)^p E = (T'T'^*)^p \oplus 0$. For arbitrary $0 < q \leqslant p$,

$$\begin{aligned}(T'T'^*)^q \oplus 0 &= \{E(TT^*)^p E\}^{\frac{q}{p}} \\ &\geqslant E(TT^*)^q E \qquad \text{(by Proposition B)} \\ &\geqslant E(TET^*)^q E \\ &= (T'T'^*)^q \oplus 0 \qquad \text{(by Proposition A)},\end{aligned}$$

so we have $E(TT^*)^q E = (T'T'^*)^q \oplus 0$. This implies that $(TT^*)^q$ is of the form

$$\begin{pmatrix} (T'T'^*)^q & X_q \\ X_q{}^* & Y_q \end{pmatrix}.$$

Put $q = \frac{p}{2}$. Then

$$\begin{aligned}(T'T'^*)^p \oplus 0 &= E(TT^*)^p E \\ &= E\{(TT^*)^q\}^2 E \\ &= ((T'T'^*)^p + X_q X_q{}^*) \oplus 0\end{aligned}$$

and therefore $X_q = 0$. So we have $TT^* = T'T'^* \oplus (Y_q)^{\frac{1}{q}}$. Since

$$TT^* = \begin{pmatrix} T' & A \\ 0 & B \end{pmatrix} \begin{pmatrix} T'^* & 0 \\ A^* & B^* \end{pmatrix} = \begin{pmatrix} T'T'^* + AA^* & AB^* \\ BA^* & BB^* \end{pmatrix},$$

this shows that $A = 0$, and hence T is reduced by $\mathcal{M}$. □

Theorem 2 does not hold without the injective condition. Because the restriction of a p-quasihyponormal operator to its kernel is trivially normal. But the kernel of p-quasihyponormal operator is not always a reducing subspace for it.

The following result was proved in [11, Lemma 3] for a p-hyponormal operator.

LEMMA 3. *Let $T \in L(H_1)$ be a p-quasihyponormal operator and $N \in L(H_2)$ be a normal operator. If $X \in L(H_2, H_1)$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof.

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with respect to $H_1 = \overline{R(T)} \oplus \ker(T^*)$ and $H_2 = \overline{R(N)} \oplus \ker(N^*)$, respectively. Since $TX = XN$ and X has dense range, we have

$$\overline{X(\overline{R(N)})} = \overline{R(T)}.$$

If we denote the restriction of X to $\overline{R(N)}$ by X_1, then $X_1 : \overline{R(N)} \to \overline{R(T)}$ has dense range and for every $x \in \overline{R(N)}$

$$X_1 N_1 x = XNx = TXx = T_1 X_1 x,$$

so that $X_1 N_1 = T_1 X_1$. Since T_1 is p-hyponormal, there exists a hyponormal operator $\widetilde{T}_1$ corresponding to T_1 and a quasiaffinity Y such that $\widetilde{T}_1 Y = Y T_1$ as mentioned in section 1. Thus we have

$$\widetilde{T}_1 Y X_1 = Y T_1 X_1 = Y X_1 N_1.$$

Since YX_1 has dense range, $\widetilde{T}_1$ is normal by [11, Lemma 3], and so T_1 is normal by [14]. Thus the inequality

$$({T_1}^* T_1)^p \geqslant (T_1 {T_1}^* + T_2 {T_2}^*)^p \geqslant (T_1 {T_1}^*)^p = ({T_1}^* T_1)^p$$

implies that $T_2 = 0$. Hence T is normal. □

We are ready for the next result which was proved in [11] for a p-hyponormal operator and in [18] for a dominant operator.

THEOREM 4. *Let $T_i \in L(H_i) (i = 1, 2)$ be injective p-quasihyponormal operators such that T_1 and T_2 are quasisimilar and let $T_i = N_i \oplus V_i$ on $H_i = H_{i1} \oplus H_{i2}$, where N_i and V_i are the normal and pure parts of T_i, respectively. Then N_1 and N_2 are unitarily equivalent and there exist $X_* \in L(H_{22}, H_{12})$ and $Y_* \in L(H_{12}, H_{22})$ with dense ranges such that $V_1 X_* = X_* V_2$ and $Y_* V_1 = V_2 Y_*$.*

Proof. There exist quasiaffinities $X \in L(H_2, H_1)$ and $Y \in L(H_1, H_2)$ such that $T_1 X = X T_2$ and $Y T_1 = T_2 Y$. Let

$$X := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad Y := \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.$$

Then we show that $X_* = X_4$ and $Y_* = Y_4$. A simple matrix calculation shows that

$$V_1 X_3 = X_3 N_2 \quad \text{and} \quad V_2 Y_3 = Y_3 N_1.$$

We claim that $X_3 = Y_3 = 0$. Indeed, letting $\mathcal{M} = \overline{R(X_3)}$, the subspace $\mathcal{M}$ is invariant under V_1. So let $V_1' = V_1|_{\mathcal{M}}$ and let $X_3' : H_{21} \to \mathcal{M}$ be defined by $X_3' x = X_3 x$ for each $x \in H_{21}$. Since that V_1' is injective p-quasihyponormal by Theorem 1, X_3' has dense range, and $V_1' X_3' = X_3' N_2$. Hence V_1' is normal by Lemma 3. Thus Theorem 2 implies that $\mathcal{M}$ reduces V_1. Since, however, V_1 is pure, we have that $\mathcal{M} = \{0\}$, and hence $X_3 = 0$. Similarly, we have $Y_3 = 0$. Thus it follows that X_1 and Y_1 are injective. Since $N_1 X_1 = X_1 N_2$ and $Y_1 N_1 = N_2 Y_1$, by Lemma 1.1 in [18] we have that N_1 and N_2 are unitarily equivalent. Also, we can notice that X_4 and Y_4 have dense ranges and

$$V_1 X_4 = X_4 V_2 \quad \text{and} \quad Y_4 V_1 = V_2 Y_4.$$

Hence the proof completes. □

We extend [20, Corollary 12] proved for a p-hyponormal operator.

THEOREM 5. *Let $T_1 \in L(H_1)$ and $T_2 \in L(H_2)$ be injective p-quasihyponormal. If T_1 and T_2 are quasisimilar then they have same spectra and essential spectra.*

Proof. Let $T_i = N_i \oplus V_i$ on $H_i = H_{i1} \oplus H_{i2}$ where N_i and V_i are the normal and pure parts of $T_i (i = 1,2)$. Since N_1 and N_2 are unitarily equivalent by Theorem 4, we have

$$\sigma(N_1) = \sigma(N_2) \quad \text{and} \quad \sigma_e(N_1) = \sigma_e(N_2).$$

Also, since there exist operators $X_* \in L(H_{22}, H_{12})$ and $Y_* \in L(H_{12}, H_{22})$ having dense ranges such that

$$V_1 X_* = X_* V_2 \quad \text{and} \quad Y_* V_1 = V_2 Y_*,$$

combining ([20], Theorem 8) and ([15], Theorem 1) we have

$$\sigma(V_1) = \sigma(V_2) \quad \text{and} \quad \sigma_e(V_1) = \sigma_e(V_2).$$

Hence we have

$$\sigma(T_1) = \sigma(T_2) \quad \text{and} \quad \sigma_e(T_1) = \sigma_e(T_2). \qquad \square$$

The next result was proved in [11] for a p-hyponormal operator and in [18] for a dominant operator.

COROLLARY 6. *Let $T_1 \in L(H_1)$ be injective p-quasihyponormal and let $T_2 \in L(H_2)$ be an isometry. Assume that there exist quasiaffinities X and Y such that $T_1 X = X T_2$ and $Y T_1 = T_2 Y$. If either X or Y is compact, then T_1 and T_2 are unitary operators and unitary equivalent.*

Proof. Since T_1 and T_2 have the same spectra by Theorem 5, we have

$$\sigma(T_1) = \sigma(T_2) \subseteq \overline{\mathbf{D}} \quad \text{where } \overline{\mathbf{D}} \text{ is the closed unit disc.}$$

Since T_1 is normaloid, we have that $||T_1|| \leqslant 1$ (i.e., T_1 is a contraction). Assume that Y is compact. Applying Theorem 2 of [2] to $YT_1 = T_2 Y$, it follows that T_2 is unitary. By Theorem 1 of [2], T_1 is unitary. It is obvious that T_1 and T_2 are unitary equivalent. When X is compact, similary the theorem follows. $\square$

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