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PLANAR PACKINGS AND MAPPINGS RELATED TO CERTAIN MINMAX PROBLEMS

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Abstract. For any integer $N \geqslant 1$, we obtain the extremal values of the minmax problem for exponential sums,

$$f(N) := \min_{a_i \text{ real}} \max \left\{ \left| \sum_{n=1}^{N} e^{ia_n} \right|, \left| \sum_{n=1}^{N} e^{iNa_n} \right| \right\}.$$

In particular, the two extremal problems $f(3)$ and

$$\max_{a,b,c\in[0,2\pi]} \min \left\{ \left| (e^{ia}-e^{ib})(e^{ib}-e^{ic})(e^{ic}-e^{ia}) \right|, \left| (e^{i3a}-e^{i3b})(e^{i3b}-e^{i3c})(e^{i3c}-e^{i3a}) \right| \right\}$$

are reduced to the problems about the packing of certain convex sets in the plane. This packing method also can be used to solve some other extremal problems.

1. Introduction

There is an extensive literature ([3], [4], [5], [6]) concerning planar extremal problems. In this paper, we study certain minmax problems about unit vectors in the plane that are best formulated in terms of complex exponentials. These problems are connected with packings of certain planar convex sets. First, we consider the minimum of sums of three vectors on the unit circle. It is not difficult to determine the value $m(w) := \min_{a,b\in[0,w]} \left|1+e^{ia}+e^{ib}\right|$, where $0 \leqslant w \leqslant 2\pi$. The function $m(w)$ is continuous and we observe that it is obtained when

$$\begin{cases} a = w,\ b = w \quad \text{or} \quad 0, & 0 \leqslant w \leqslant \pi, \\ a = w,\ b = \frac{w}{2}, & \pi < w < \frac{4}{3}\pi, \\ a = \frac{2}{3}\pi,\ b = \frac{4}{3}\pi, & \frac{4}{3}\pi \leqslant w \leqslant 2\pi. \end{cases} \tag{1.1}$$

In particular, $m(\pi) = 1$ and $m(w) = 0$ if $w \geqslant 4\pi/3$. A more interesting problem is the determination of $f(3)$, where, for an integer $N \geqslant 2$,

$$f(N) := \min_{a_i \text{ real}} \max \left\{ \left| \sum_{n=1}^{N} e^{ia_n} \right|, \left| \sum_{n=1}^{N} e^{iNa_n} \right| \right\}.$$

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In fact, in Section 2, we will see that the determination of $f(3)$ can be reduced to a problem about the packing of certain convex sets in the plane. This packing method also can be used to solve some other extremal problems. For example, one can use it to obtain the extremal value of

$$g(3) := \max_{a,b,c\in[0,2\pi]} \min\Big\{\big|(e^{ia}-e^{ib})(e^{ib}-e^{ic})(e^{ic}-e^{ia})\big|, \big|(e^{i3a}-e^{i3b})(e^{i3b}-e^{i3c})(e^{i3c}-e^{i3a})\big|\Big\}$$

and some other related values. In Section 3, we will state some of these results without proof, but will provide the figures of packings to illustrate the method and their proofs.

We note that switching min and max in $f(3)$ and $g(3)$ leads to trivial problems. For $f(3)$ we have

$$\begin{aligned} \left|1+e^{ia}+e^{ib}\right|^2 &= 2(\cos a+\cos b+\cos(a-b))+3,\\ \left|1+e^{i3a}+e^{i3b}\right|^2 &= 2(\cos(3a)+\cos(3b)+\cos 3(a-b))+3. \end{aligned}$$

Thus the original problem is essentially reduced to studying the functions

$$\begin{aligned} F_1(a,b) &:= \cos a+\cos b+\cos(a-b),\\ F_2(a,b) &:= \cos(3a)+\cos(3b)+\cos 3(a-b). \end{aligned} \tag{1.2}$$

Let $F=(F_1,F_2):[0,2\pi]^2\to\mathbb{R}^2$, where $x=F_1(a,b)$ and $y=F_2(a,b)$. By the implicit function theorem, every point except the points where the Jacobian is 0 has a neighborhood in which F is 1 - 1. But this does not imply that any restriction of F to a connected subdomain with Jacobian $\neq 0$ inside and Jacobian $=0$ on boundaries is invertible. But in this particular case it is true. In order to prove this, we use upper envelopes and lower envelopes of the Lissajous figure of (1.2). We can find both envelopes by doing computational algebraic geometry with Groebner bases. The detailed proof of this also will be omitted.

In Section 4, the extremal value $f(N)$, $N\geqslant 4$, is calculated. There has been a lot of work on sums of roots of unity, and in particular vanishing sums of such roots, to which the result in Section 4 is connected. Arising from being of intrinsic interest, vanishing sums of roots of unity arise naturally in a number of algebraic, geometric, and combinatorial contexts (see [1], [2], [9], [11], [12]). For a partial survey up to 1978, see [8]. Lam and Leung [7] showed recently by using technique of group rings that, for any positive integer m (i.e. $m\in\mathbb{N}$), the set of all possible integers n for which there exist mth roots of unity $\alpha_1,\cdots,\alpha_n\in\mathbb{C}$ such that $\alpha_1+\cdots+\alpha_n=0$ is exactly the collection of $\mathbb{N}$-combinations of the prime divisors of m. We will show some related results to the above in Section 4 that, for $N\geqslant 4$,

$$\sum_{n=1}^{N} e^{ia_n} = \sum_{n=1}^{N} e^{iNa_n} = 0$$

for some distinct $a_1,\cdots,a_N$, $0\leqslant a_i<2\pi$, i.e. $f(N)=0$.

2. f(N) for N=2,3

In this section, we study $f(2)$ and $f(3)$. Here is the answer for $N=2$.

PROPOSITION 2.1. *We have that* $f(2)=1$.

Proof. We observe that $f(2)=\min_{a\in[0,2\pi]}\max\left\{\left|1+e^{ia}\right|,\left|1+e^{i2a}\right|\right\}$. We easily see that

$$\begin{cases} |e^{ia}+1|\geqslant|e^{i2a}+1| \text{ and } |e^{ia}+1|>1, & 0\leqslant a<\frac{2}{3}\pi,\\ |e^{ia}+1|=|e^{i2a}+1|=1, & a=\frac{2}{3}\pi,\\ |e^{ia}+1|<|e^{i2a}+1| \text{ and } |e^{ia}+1|>1, & \frac{2}{3}\pi\leqslant a<\frac{4}{3}\pi,\\ |e^{ia}+1|=|e^{i2a}+1|=1, & a=\frac{4}{3}\pi,\\ |e^{ia}+1|\geqslant|e^{i2a}+1| \text{ and } |e^{ia}+1|>1, & \frac{4}{3}\pi<a\leqslant 2\pi. \end{cases}$$

Hence $f(2)=1$. □

In the case of $N=3$, our first assertion is: If

$$f(3)=\min_{a,b\in[0,2\pi]}\max\left\{\left|1+e^{ia}+e^{ib}\right|,\left|1+e^{i3a}+e^{i3b}\right|\right\}$$

is extremal, then the two moduli are equal. We observe that if $a=3\pi/10$ and $b=10\pi/9$, then $\left|1+e^{ia}+e^{ib}\right|=0.798818\cdots$ and $\left|1+e^{i3a}+e^{i3b}\right|=0.716736\cdots$. So we see that

$$f(3)<\frac{\sqrt{3}}{2}=0.866025\cdots.$$

On the other hand, if 1, e^{ia} and e^{ib} (1, e^{i3a} and e^{i3b}) lie on the same closed half complex plane, $\left|1+e^{ia}+e^{ib}\right|\geqslant 1$ ($\left|1+e^{i3a}+e^{i3b}\right|\geqslant 1$). But $f(3)<1$. So we have

$$f(3)=\min_{\substack{0<a<\pi\\ \pi<b<a+\pi}}\max\left\{\left|1+e^{ia}+e^{ib}\right|,\left|1+e^{i3a}+e^{i3b}\right|\right\}.$$

Define two functions $s(a,b)$, $t(a,b)$ on $0<a<\pi$, $\pi<b<a+\pi$ by

$$s(a,b):=\left|1+e^{ia}+e^{ib}\right|^2=2(\cos a+\cos b+\cos(a-b))+3,$$
$$t(a,b):=\left|1+e^{i3a}+e^{i3b}\right|^2=2(\cos(3a)+\cos(3b)+\cos 3(a-b))+3.$$

Now we prove our first assertion.

PROPOSITION 2.2. *If* $f(3)$ *is extremal, then the two moduli are equal.*

Proof. For fixed a, $0<a<\pi$, the extremal value of

$$\min_{\pi<b<a+\pi}\max\{s(a,b),t(a,b)\}$$

is attained at a point (a,b) that satisfies one of the following conditions; (i) $\frac{ds}{db}=0$, (ii) $\frac{dt}{db}=0$, (iii) $(a,b)=(a,\pi)$ or $(a,a+\pi)$, (iv) $s(a,b)=t(a,b)$.

We first consider (i) and (ii). We compute that, on $0 < a < \pi$, $\pi < b < a+\pi$,

$$\frac{\partial}{\partial b} s(a,b) = 4\cos\frac{a}{2}\sin\left(\frac{a}{2}-b\right) = 0 \text{ if and only if } b = \frac{a}{2}+\pi,$$

and

$$\frac{\partial}{\partial b} t(a,b) = 12\cos\frac{a}{2}\sin\left(\frac{a}{2}-b\right)(2\cos a - 1)(2\cos(a-2b)+1) = 0$$

if and only if

$$b = \frac{a}{2}+\frac{2\pi}{3},\ \frac{a}{2}+\pi,\ \frac{a}{2}+\frac{4\pi}{3}. \tag{2.1}$$

Hence (i) and (ii) hold on the lines in (2.1). Now we consider all a, $0 < a < \pi$. By using relative extrema, we can easily show that

$$\begin{aligned} z_1 := &\min_{0<a<\pi}\max\left\{\sqrt{s\left(a,\frac{a}{2}+\frac{2\pi}{3}\right)},\ \sqrt{t\left(a,\frac{a}{2}+\frac{2\pi}{3}\right)}\right\} \\ = &\min_{0<a<\pi}\max\left\{\sqrt{s\left(a,\frac{a}{2}+\frac{4\pi}{3}\right)},\ \sqrt{t\left(a,\frac{a}{2}+\frac{4\pi}{3}\right)}\right\} \\ = &\frac{\sqrt{3}}{2} \end{aligned}$$

and that

$$\min_{0<a<\pi}\max\left\{\sqrt{s\left(a,\frac{a}{2}+\pi\right)},\ \sqrt{t\left(a,\frac{a}{2}+\pi\right)}\right\}$$

is extremal when $s(a,a/2+\pi) = t(a,a/2+\pi) = 1$. Hence the points (a,b) with (i) or (ii) do not have an extremal value $< \sqrt{3}/2$ when the two moduli are not equal. For (iii), we have $s(a,\pi) = t(a,\pi) = s(a,a+\pi) = t(a,a+\pi) = 1$. This proves the proposition. □

By Proposition 2.2, in order that $f(3)$ be extremal, we must have

$$s(a,b) = t(a,b),$$

where $s(a,b) = 2(\cos a + \cos b + \cos(a-b)) + 3$ and $t(a,b) = 2(\cos(3a) + \cos(3b) + \cos 3(a-b)) + 3$. Hence our original problem is reduced essentially to the study of the functions

$$\begin{aligned} F_1(a,b) &= \cos a + \cos b + \cos(a-b), \\ F_2(a,b) &= \cos 3a + \cos 3b + \cos 3(a-b). \end{aligned} \tag{2.2}$$

In fact, we can rephrase the problem on $f(3)$ as : "If $c_0 = -1.204094\cdots$ is the real root of $1 + 2(1+x+x^2+x^3) = 0$, then the sets

$$\begin{aligned} &\{(a,b) \in [0,2\pi]^2 : \cos a + \cos b + \cos(a-b) \leqslant c_0\}, \\ &\{(a,b) \in [0,2\pi]^2 : \cos 3a + \cos 3b + \cos 3(a-b) \leqslant c_0\} \end{aligned} \tag{2.3}$$

intersect in at most a finite number of points (a,b) modulo 2π, at which equality holds."

If the above assertion can be proved and the interior of the curves $F_i(a,b) = c_0$ $(i = 1, 2)$ represent points where $F_i(a,b) < c_0$, then, by Proposition 2.2, $f(3) = \sqrt{2c_0+3}$, where c_0 is the real root of $1 + 2(1 + x + x^2 + x^3) = 0$. This means that $f(3) = 0.769292\cdots$. But we have

$$\cos a + \cos b + \cos(a-b) + 1 = 4\cos\frac{a}{2}\cos\frac{a-b}{2}\cos\frac{b}{2}, \tag{2.4}$$

so the ovals in $F_1(a,b) = -1$ are bounded by the lines $a = \pi, b = \pi, b = a+\pi$ and $b = a - \pi$. Moreover, we will show, in Lemma 2.3 that, for $c < -1$,

$$\{(a,b) \in [0, 2\pi]^2 : \cos a + \cos b + \cos(a-b) \leqslant c\}$$

is a subset of the set that is bounded by the lines $a = \pi, b = \pi, b = a+\pi$ and $b = a-\pi$. So, by continuity of the curves, as c decreases from -1 to -3, the ovals shrink and finally disappear. So the interior of the curves $F_i(a,b) = c_0$ $(i = 1, 2)$ represent points where $F_i(a,b) < c_0$. Thus in order to get $f(3) = \sqrt{2c_0+3}$, it is enough to show the rephrased assertion on $f(3)$ is true. Here, some diagrams are enlightening.

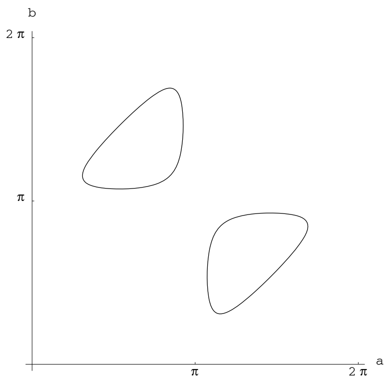

Figure 2.1 a: $F_1(a,b) = c_0$

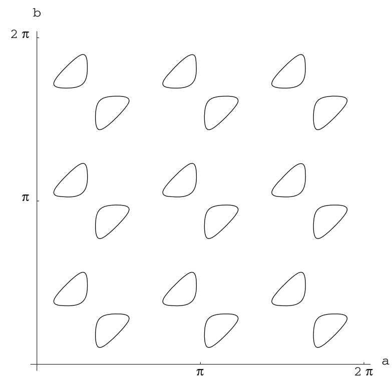

Figure 2.1 b: $F_2(a,b) = c_0$

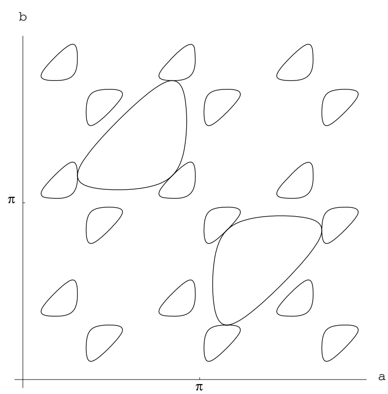

Figure 2.2

We plot the relations in (2.3) with equality on $0 \leqslant a \leqslant 2\pi, 0 \leqslant b \leqslant 2\pi$. The "ovals" in Figure 2.1 a have a very triangular shape and are mirror images of each other in the $a = b$ line. The plot here looks really like a lattice packing of these ovals. In order to plot the second equation, we must shrink both of these ovals by a factor of 3, and then extend the resulting configulation modulo 2π, so we get Figure 2.1 b. We combine these two plots in Figure 2.2. We have a PACKING that is best possible in the sense that any other value of c_0 would either give overlapping interiors or no points of contact. In the language of lattice theory (at least roughly), the set of large bodies here is admissible for the "fat lattice" formed by the small bodies. The above thus transforms local information into global information, and opens the possibility of establishing the result by just checking the six symmetries. We first show that the ovals in Figure 2.1 are in fact convex. It is obviously enough to consider the first case. Define, for $0 \leqslant a, b \leqslant 2\pi$,

$$K_1(a,b) := \cos a + \cos b + \cos(a-b) - c,$$
$$K_2(a,b) := \cos 3a + \cos 3b + \cos 3(a-b) - c,$$

where $c < -1$ is a constant. Then, since $c < -1$, $K_1(a,b) > 0$ if $a = 0, 2\pi$ or $b = 0, 2\pi$. We observe that implicit differentiation with respect to a in $K_1(a,b) = 0$ gives

$$\begin{aligned} \left(\frac{db}{da}\right)_{K_1} &= \cos\frac{b}{2}\csc\left(\frac{a}{2}-b\right)\sec\frac{a}{2}\sin\left(a-\frac{b}{2}\right), \\ \left(\frac{d^2b}{da^2}\right)_{K_1} &= -\frac{1}{4}\cos\left(\frac{b-a}{2}\right)\cos\frac{b}{2}\csc^3\left(\frac{a}{2}-b\right)\sec^2\frac{a}{2} \\ &\quad (-3+\cos(a-2b)+\cos(2a-b))+\cos(a+b)), \end{aligned} \tag{2.5}$$

and implicit differentiation with respect to a in $K_2(a,b) = 0$ gives

$$\begin{aligned} \left(\frac{db}{da}\right)_{K_2} &= \Bigg(\cos\frac{b}{2}\sec\frac{a}{2}\sin\left(a-\frac{b}{2}\right)\csc\left(\frac{a}{2}-b\right) \\ &\quad (2\cos(2a-b)+1)(2\cos b-1)\Bigg)\Bigg/\Bigg((2\cos a-1)(2\cos(a-2b)+1)\Bigg), \\ \left(\frac{d^2b}{da^2}\right)_{K_2} &= \Bigg(-3\cos\frac{b-a}{2}\cos\frac{b}{2}(2\cos b-1)(2\cos(b-a)-1)\csc^3\left(\frac{a}{2}-b\right) \\ &\quad \sec^2\frac{a}{2}(-3+\cos(3a-6b)+\cos(6a-3b)+\cos(3a+3b))\Bigg)\Bigg/ \\ &\quad \left(4(2\cos a-1)^2(2\cos(a-2b)+1)^3\right), \end{aligned} \tag{2.6}$$

where $\left(\frac{db}{da}\right)_{K_i}$ denotes $\frac{db}{da}$ from $K_i(a,b) = 0$.

LEMMA 2.3. *There is no locus of $K_1(a,b) = 0$ in the regions bounded by*
(a) $0 \leqslant a \leqslant \pi,\ a+\pi \leqslant b \leqslant 2\pi$,
(b) $0 \leqslant a \leqslant \pi,\ 0 \leqslant b \leqslant \pi$,
(c) $\pi \leqslant a \leqslant 2\pi,\ 0 \leqslant b \leqslant a-\pi$,
(d) $\pi \leqslant a \leqslant 2\pi,\ \pi \leqslant b \leqslant 2\pi$.

Proof. The cases (c) and (d) are symmetric with (a) and (b) about $b=a$ and $b=-a+2\pi$, respectively. So it suffices to consider the cases (a) and (b). For (a), we have $\cos b \geqslant \cos(a+\pi)$, since $a+\pi \leqslant b \leqslant 2\pi$. So $K_1(a,b) = \cos a + \cos b + \cos(a-b) - c \geqslant \cos a + \cos(a+\pi) + \cos(a-b) - c = \cos(a-b) - c > 0$, since $c < -1$. For (b), we observe that $\cos a + \cos b + \cos(a-b) \geqslant -1$, since $\cos a + \cos b + \cos(a-b) + 1 = 4\cos(a/2)\cos((a-b)/2)\cos(b/2)$ and $-\pi/2 \leqslant (a-b)/2 \leqslant \pi/2$. But $c < -1$. Hence $K_1(a,b) > 0$, which completes the proof. □

Lemma 2.3 implies that, by (2.4), the interior of the curves $F_i(a,b) = c_0$ $(i=1,2)$ represents points where $F_i(a,b) < c$.

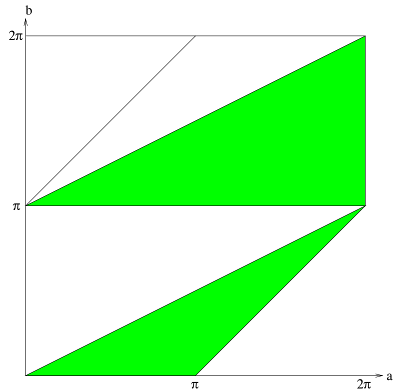

Figure 2.3

PROPOSITION 2.4. *The closed interiors of* $K_i(a,b)=0$ $(i=1,2)$ *are convex.*

Proof. It suffices to consider $K_1(a,b)$. By (2.5),

$$\left(\frac{d^2b}{da^2}\right)_{K_1} < 0 \quad \text{if and only if} \quad \cos\frac{b-a}{2}\cos\frac{b}{2}\csc\left(\frac{a}{2}-b\right) > 0,$$

since $-3+\cos(a-2b)+\cos(2a-b)+\cos(a+b) < 0$ for any a,b. Here we have four cases : $\cos((b-a)/2), \cos(b/2), \csc(a/2-b)$ are $+\,+\,+$, $+\,-\,-$, $-\,+\,-$ and $-\,-\,+$. Then we can get Figure 2.3 by considering each case, where the shadowed open domains are for $\left(\frac{d^2b}{da^2}\right)_{K_1} > 0$, the remaining open domains are for $\left(\frac{d^2b}{da^2}\right)_{K_1} < 0$ and the boundaries are for $\left(\frac{d^2b}{da^2}\right)_{K_1} = 0$. By Lemma 2.3, the locus of $K_1(a,b)=0$ is contained in the region bounded by $a=\pi, b=\pi, b=a+\pi$ and $b=a-\pi$. By the signs of $\left(\frac{d^2b}{da^2}\right)_{K_1}$, the closed interior of $K_1(a,b)=0$ is convex. □

For the main result, we need the following two propositions. Denote the resultant of $p(x)$ and $q(x)$ by

$$res_x\,(p(x),q(x))\,.$$

PROPOSITION 2.5. *Let* $\sigma = 0.970123422\cdots$ *and* $\tau = 5.313061884\cdots$ *be such that* $\cos\sigma$ *and* $\cos\tau$ *satisfy*

$$2x^3 + 2x^2 - 1 = 0.$$

Then, on $b = a + \sigma$ *and* $b = \tau$, *we have*

$$K_1(a,b) \geqslant 0, \quad K_2(a,b) \geqslant 0,$$

where c *is the real root of* $1 + 2(1 + x + x^2 + x^3) = 0$.

Proof. We write out the proof for the case $b = a + \sigma$. The details for the case $b = \tau$ will be written by using [ ] next to the case $b = a + \sigma$ unless they are same. On $b = a + \sigma$ $[b = \tau]$,

$$\begin{aligned} K_1(a,b) &= \cos a + \cos b + \cos(a-b) - c \\ &= \cos a(1 + \cos\sigma) - \sin a \sin\sigma + \cos\sigma - c \\ &\quad [\cos a(1 + \cos\tau) + \sin a \sin\tau + \cos\tau - c]. \end{aligned}$$

So we need to show that

$$\cos a(1+\cos\sigma) - \sin a \sin\sigma + (\cos\sigma - c) \geqslant 0 \ [\cos a(1+\cos\tau) + \sin\tau \sin a + (\cos\tau - c) \geqslant 0].$$

Since $\cos\sigma - c > 0$ ($\cos\tau - c > 0$), it is enough to show that

$$(\cos\sigma - c)^2 = (\cos\sigma + 1)^2 + (\sin\sigma)^2 = 2\cos\sigma + 2, \tag{2.7}$$

$$\left[(\cos\tau - c)^2 = (\cos\tau + 1)^2 + (\sin\tau)^2 = 2\cos\tau + 2\right] \tag{2.8}$$

Since $\cos\sigma = \cos\tau$, we only need to show that (2.7) is true. Now, since

$$2\left(\frac{y-2}{2}\right)^3 + 2\left(\frac{y-2}{2}\right)^2 - 1 = \frac{1}{4}\left(y^3 - 4y^2 + 4y - 4\right),$$

we see that $x^3 - 4x^2 + 4x - 4$ is the minimal polynomial of $2\cos\sigma + 2$. Next, by computer algebra,

$$\begin{aligned} &res_x\left(1 + 2(1 - x + x^2 - x^3),\, 2(y-x)^3 + 2(y-x) - 1\right) \\ &\quad = -16(y^3 - 2y - 2)(4y^6 + 12y^4 - 12y^3 + 28y^2 - 18y + 9), \end{aligned}$$

and one deduces that $x^3 - 2x - 2$ is the minimal polynomial of $\cos\sigma - c$. But if $p(x) = x^3 - 4x^2 + 4x - 4$, then $p(x^2) = (x^3 - 2x + 2)(x^3 - 2x - 2)$. So $x^3 - 4x^2 + 4x - 4$ is the minimal polynomial of $(\cos\sigma - c)^2$. Hence $(\cos\sigma - c)^2$ and $2\cos\sigma + 2$ have the same minimal polynomial, which gives us the desired equality. Now (2.8) has exactly the form of the equality (2.7). Similar arguments show that

$$(\cos(3\sigma) - c)^2 = (\cos(3\sigma) + 1)^2 + (\sin(3\sigma))^2 = 2\cos(3\sigma) + 2\ (= 0.532260178\cdots).$$

So $b = a + \sigma$ also implies $K_2(a,b) \geqslant 0$. This completes the proof of the proposition. $\square$

PROPOSITION 2.6. *Let* $a_0 = 2.656530942\cdots$ *be such that* $\cos a_0$ *satisfies*

$$4x^3 - 2x + 1 = 0.$$

Then

$$K_1(a_0, -a_0 + 2\pi) = K_2(a_0, -a_0 + 2\pi) = 0,$$
$$K_1(a_0, 2a_0) = K_2(a_0, 2a_0) = 0,$$

where c *is the real root of* $1 + 2(1 + x + x^2 + x^3) = 0$.

Proof. We can see that, for $b_0 = 2a_0$ or $b_0 = -a_0 + 2\pi$,

$$K_1(a_0, b_0) = \cos(2a_0) + 2\cos a_0 - c,$$
$$K_2(a_0, b_0) = \cos(6a_0) + 2\cos(3a_0) - c.$$

Now $4\cos^3 a_0 - 2\cos a_0 + 1 = \cos(3a_0) + \cos a_0 + 1$ and

$$\begin{aligned} 0 &= (\cos(3a_0) + \cos a_0 + 1)(\cos(3a_0) - 6\cos(2a_0) + 13\cos a_0 - 7) \\ &= 1 + 2\left(1 + (\cos(2a_0) + 2\cos a_0) + (\cos(2a_0) + 2\cos a_0)^2 + (\cos(2a_0) + 2\cos a_0)^3\right). \end{aligned}$$

Since c is the only real root of $1 + 2(1 + x + x^2 + x^3) = 0$,

$$c = \cos(2a_0) + 2\cos a_0,$$

and this means that $K_1(a_0, -a_0 + 2\pi) = K_1(a_0, 2a_0) = 0$. For K_2,

$$\begin{aligned} 0 = {} & (\cos(3a_0) + \cos(a_0) + 1)(\cos(9a_0) + \cos(3a_0) - 1)) \\ & (2\cos(6a_0) - 2\cos(4a_0) - 8\cos(3a_0) + 4\cos a_0 + 5) \\ = {} & 1 + 2\,(1 + (\cos(6a_0) + 2\cos(3a_0)) + \\ & (\cos(6a_0) + 2\cos(3(a_0))^2 + (\cos(6a_0) + 2\cos(3a_0))^3\big), \end{aligned}$$

which, by the same reasoning, proves the result. □

Now we have

THEOREM 2.7. *If* c_0 *is the real root of* $1 + 2(1 + x + x^2 + x^3) = 0$, *then the sets*

$$\{(a, b) \in [0, 2\pi)^2 : \cos a + \cos b + \cos(a - b) \leqslant c_0\},$$
$$\{(a, b) \in [0, 2\pi)^2 : \cos(3a) + \cos(3b) + \cos 3(a - b) \leqslant c_0\}$$

intersect in at most a finite number of points (a, b) *modulo* 2π, *at which equality holds.*

Proof. Take $c = c_0$ in $K_i(a, b)$ $(i = 1, 2)$ as the real root of $1+2(1+x+x^2+x^3) = 0$. Let

$$A = \{(a, b) \in [0, 2\pi)^2 \mid K_1(a, b) = 0\},$$
$$B = \{(a, b) \in [0, 2\pi)^2 \mid K_2(a, b) = 0\}.$$

Then, by Lemma 2.3, the locus of A consists of at least two closed curves, one of which is bounded by $0 \leqslant a \leqslant \pi$, $\pi \leqslant b \leqslant 2\pi$ and $b \leqslant a + \pi$, and the other of which is bounded by $\pi \leqslant a \leqslant 2\pi$, $0 \leqslant b \leqslant \pi$ and $b \geqslant a-\pi$, since the equation $K_1(a,b)=0$ is invariant if we switch a and b. Moreover, the curves are again symmetric about $b=-a+2\pi$, so it is enough to consider the region, say

$$\triangle = \{(a,b) \in [0,\pi] \times [\pi, 2\pi) : b \geqslant -a+2\pi, b < a+\pi, a < \pi\}.$$

Let

$$D := \{(a,b) \in \triangle \mid \left(\frac{db}{da}\right)_{K_1} = \left(\frac{db}{da}\right)_{K_2} \quad \text{and} \quad K_1(a,b) = K_2(a,b) = 0\}.$$

From (2.5) and (2.6), we can compute that, on $(a,b) \in \triangle$,

$$\left(\frac{db}{da}\right)_{K_1} = \left(\frac{db}{da}\right)_{K_2}$$

if and only if

$$\cos\left(\frac{a-b}{2}\right)\cos\frac{b}{2}\csc\left(\frac{a}{2}-b\right)\sin\left(a-\frac{b}{2}\right)\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)\sin\frac{b}{2}\tan\frac{a}{2} = 0$$

if and only if

$$b = 2a, a+\pi, -a+2\pi.$$

But, for $b=a+\pi$, $K_1(a,b) = K_2(a,b) = -1-c_0$ for $b=a+\pi$. The $K_1(a,b) = K_2(a,b) = 0$ implies that $c_0=-1$, which is not the case. Suppose that $b=2a$ or $b=-a+2\pi$. Then we can see that

$$K_1(a,b) = \cos(2a) + 2\cos a - c_0,$$
$$K_2(a,b) = \cos(6a) + 2\cos(3a) - c_0$$

and equating these two formulae on c_0 gives

$$-32\left(4\cos^3 a - 2\cos a + 1\right)\cos^2(a/2)\sin^2(a/2)\left(\cos^2(a/2) - \sin^2(a/2)\right) = 0.$$

A calculation shows that the possible zeros of $\cos(a/2) = 0$ and $\cos^2(a/2) - \sin^2(a/2) = 0$ are $a=\pi$ and $a=\pi/2$, respectively, and there are no zeros satisfying $\sin(a/2)=0$. But $a=\pi$ and $a=\pi/2$ do not correspond to c_0 which is the real root of $1+2(1+x+x^2+x^3)=0$. Hence

$$\begin{aligned} D &\subset \{(a,b) \in \triangle \mid (b=2a \quad \text{or} \quad b=-a+2\pi) \text{ and } 4\cos^3 a - 2\cos a + 1 = 0\} \\ &= \{(a_0, -a_0+2\pi), (a_0, 2a_0)\}, \end{aligned}$$

where $a_0 = 2.656530942\cdots$ is such that $\cos a_0$ satisfies $4x^3-2x+1=0$. Hence, by Proposition 2.6, we have

$$D = \{(a_0, -a_0+2\pi), (a_0, 2a_0)\}.$$

The lines $b=a+\sigma$, $b=\tau$ that are given in Proposition 2.5 pass through $(a_0, -a_0+2\pi)$ and $(a_0, 2a_0)$ given in Proposition 2.6, respectively. In fact,

$$\begin{aligned} 2\cos^3(2a_0) + 2\cos^2(2a_0) - 1 =& 2\cos^3(2\pi - 2a_0) + 2\cos^2(2\pi - 2a_0) - 1 \\ =& (4\cos^3 a_0 - 2\cos a_0 - 1)(4\cos^3 a_0 - 2\cos a_0 + 1), \end{aligned}$$

and $2x^3+2x^2-1=0$ has only one real root, so it follows from $2\pi-2a_0<2a_0$ that $\sigma=2\pi-2a_0$ and $\tau=2a_0$. On the other hand, by Proposition 2.4, the closed interiors of $K_i(a,b)=0\,(i=1,2)$ are convex. Hence the lines $b=a+\sigma$, $b=\tau$ are tangent at $(a_0,-a_0+2\pi)$ to two closed convex sets and at $(a_0,2a_0)$ to two closed convex sets, respectively, since, on $b=a+\sigma$ and $b=\tau$, we have that

$$K_1(a,b)\geqslant 0 \text{ and } \quad K_2(a,b)\geqslant 0.$$

Therefore, it remains to show that the convex sets lie on opposite sides of the lines. We shall show that

$$\left(\frac{d^2b}{da^2}\right)_{K_1}\left(\frac{d^2b}{da^2}\right)_{K_2}<0$$

at $(a,b)=(a_0,2a_0)$ and $(a,b)=(a_0,-a_0+2\pi)$. From (2.5) and (2.6), we can compute that

$$\left(\frac{d^2b}{da^2}\right)_{K_1}\left(\frac{d^2b}{da^2}\right)_{K_2}=-W(a,b)\frac{(-1+2\cos(a-b))\,(-1+2\cos b)}{1+2\cos(a-2b)},$$

where $-W(a,b)<0$ at $(a,b)=(a_0,2a_0)$ or $(a_0,-a_0+2\pi)$. Set

$$\Phi:=(-1+2\cos(a_0-b_0))\,(-1+2\cos b_0)\,(1+2\cos(a_0-2b_0)).$$

Then, by using $\kappa:=4\cos^3 a_0-2\cos a_0+1=0$,

$$\begin{aligned}\Phi&=\Phi+\kappa\left(4\kappa-8\cos^2 a_0-8\cos a_0+2\right)\\&=3-4\cos a_0-8\cos^2 a_0\\&=-\left(\cos a_0-\frac{-1-\sqrt{7}}{4}\right)\left(\cos a_0-\frac{-1+\sqrt{7}}{4}\right).\end{aligned}$$

Then we can check that

$$\frac{-1-\sqrt{7}}{4}<\cos a_0<\frac{-1+\sqrt{7}}{4}.$$

Hence $\left(\frac{d^2b}{da^2}\right)_{K_1}\left(\frac{d^2b}{da^2}\right)_{K_2}<0$ at $(a,b)=(a_0,2a_0)$ and $(a,b)=(a_0,-a_0+2\pi)$, which proves the theorem. □

By Theorem 2.7, we have the answer for $f(3)$.

COROLLARY 2.8. $f(3)=\sqrt{2c+3}$, *where* c *is the real root of* $1+2(1+x+x^2+x^3)=0$. *Thus* $f(3)=0.769292\cdots$.

REMARK 2.9. Let $F=(F_1,F_2):[0,2\pi]^2\to\mathbb{R}^2$, where

$$\begin{cases} x=F_1(a,b)=\cos a+\cos b+\cos(a-b),\\ y=F_2(a,b)=\cos(3a)+\cos(3b)+\cos(3(a-b)).\end{cases}\tag{2.9}$$

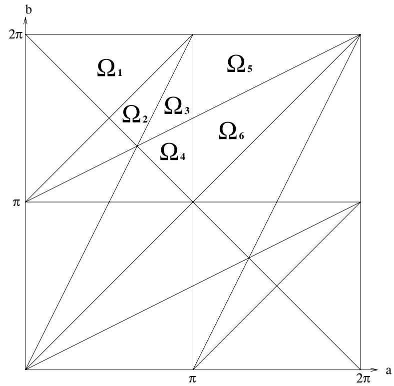

Figure 2.4

Here the Jacobian J of F is not equal to zero at any point of $[0,2\pi]^2$ except the points on the given lines in Figure 2.4 on $[0,2\pi]^2$. Let

$$R_1 = Im\,F \cap \{(x,y) : x > -1\},$$
$$R_2 = Im\,F \cap \{(x,y) : x < -1\}.$$

Then we can prove (details omitted) that

(a) the mapping under F whose domain is Ω_1, Ω_5 or Ω_6 (or the domains symmetric about $b = a$, $b = -a + 2\pi$ and origin) and codomain is R_1 is invertible, and

(b) the mapping under F whose domain is Ω_2, Ω_3 or Ω_4 (or the domains symmetric about $b = a$, $b = -a + 2\pi$ and origin) and codomain is R_2 is invertible.

This is somewhat related to the Jacobian conjecture: given a polynomial mapping $F = (p,q) : \mathbb{C}^2 \to \mathbb{C}^2$, if the Jacobian of F is equal to 1, then F is invertible. This conjecture was posed by O. Keller in 1939 and after 60 years it still remains an open problem. In 1994, Pinchuk [10] constructed a counterexample to the so called real Jacobian conjecture, which is stronger than the classical one. In fact, he found a polynomial mapping $F = (p,q) : \mathbb{R}^2 \to \mathbb{R}^2$ with a nonvanishing Jacobian which is not a global diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. In our example, even though the mapping $F = (F_1, F_2)$ is not a polynomial mapping but a trigonometric polynomial mapping, it satisfies the conclusion of real Jacobian conjecture. Hence it suggests problems on trigonometric polynomial mappings of Jacobian conjecture type.

In Figure 2.5, we have plotted the Lissajous figure of (2.9).

To prove (a) and (b) above we find upper envelopes and lower envelopes of the Lissajous figure of (2.5) by doing computational algebraic geometry with Groebner bases. In fact, for $x \geqslant -1$, the Lissajous figure of (2.5) has upper envelope

$$y = x^3 - 3x^2 + 3,$$

and lower envelope

$$y^2 - 8x^3y - 72x^2y - 126xy - 60y + 16x^6 - 72x^4 - 48x^3 - 27x^2 - 108x - 72 = 0.$$

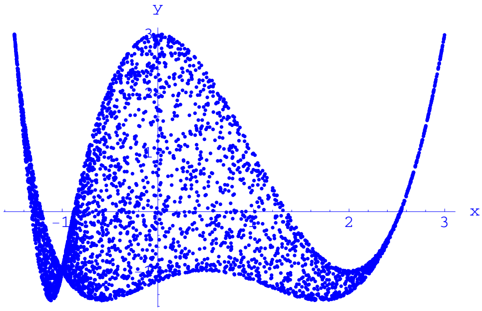

Figure 2.5

For $x \leqslant -1$, the Lissajous figure of (2.5) has

$$y^2 - 8x^3y - 72x^2y - 126xy - 60y + 16x^6 - 72x^4 - 48x^3 - 27x^2 - 108x - 72 = 0.$$

for both its upper envelope and lower envelope.

REMARK 2.10. We can get the value of $f(3)$ by using Lagrange multipliers. In fact, in

$$f(3) = \min_{a,b,c\in[0,2\pi]} \max\left\{\left|e^{ia}+e^{ib}+e^{ic}\right|, \left|e^{i3a}+e^{i3b}+e^{i3c}\right|\right\},$$

we may multiply by $e^{-i(a-b)/2}$ and $e^{-i3(a-b)/2}$ respectively, to obtain, without loss of generality,

$$f(3) = \min_{a,c\in[0,2\pi]} \max\left\{\left|e^{ia}+e^{-ia}+e^{ic}\right|, \left|e^{i3a}+e^{-i3a}+e^{i3c}\right|\right\}.$$

Observe that $\left|e^{ia}+e^{-ia}+e^{ic}\right|^2 = \left|2\cos a + \cos c + i\sin c\right|^2 = 4\cos^2 a + 4\cos a\cos c + 1$. If we let $t = \cos a$, $u = \cos c$, then $f(3)$ becomes

$$\min_{\substack{-1\leqslant t\leqslant 1\\ -1\leqslant u\leqslant 1}} \max\left\{1+4t^2+4tu,\ 1+4(4t^3-3t)^2+4(4t^3-3t)(4u^3-3u)\right\}.$$

By Proposition 2.2, the two quantities are equal at the minimum. Hence one can set the problem as:

$$\min_{\substack{-1\leqslant t\leqslant 1\\ 1\leqslant u\leqslant 1}} \left\{1+4t^2+4tu\right\}$$

subject to

$$1+4t^2+4tu = 1+4(4t^3-3t)^2+4(4t^3-3t)(4u^3-3u).$$

Using Lagrange multipliers, we can prove $f(3) = |4T_0 - 1| = 0.769292\cdots$, where T_0 is the real root of $32T^3 - 48T^2 + 20T - 1 = 0$. However our packing method that we used to first determine $f(3)$ has other applications. In fact by using our packing method one can prove the results in the following section. They do not seem to follow from the Lagrange multiplier method. We state below the results without proof.

3. Products of distances between 3 vectors on the unit circle

In Section 2, we had studied the minimum of sums of three vectors on the unit circle. In this section, we consider products of distances between them. Let

$$g(3) = \max_{a,b,c\in[0,2\pi]} \min\left\{\left|(e^{ia}-e^{ib})(e^{ib}-e^{ic})(e^{ic}-e^{ia})\right|, \left|(e^{i3a}-e^{i3b})(e^{i3b}-e^{i3c})(e^{i3c}-e^{i3a})\right|\right\}$$
$$= \max_{a,b\in[0,2\pi]} \min\left\{\left|(1-e^{ia})(1-e^{ib})(e^{ia}-e^{ib})\right|, \left|(1-e^{i3a})(1-e^{i3b})(e^{i3a}-e^{i3b})\right|\right\}.$$

Notice that

$$\left|(1-e^{ia})(1-e^{ib})(e^{ia}-e^{ib})\right|^2 = 4\left(-\sin a+\sin b+\sin(a-b)\right)^2,$$
$$\left|(1-e^{i3a})(1-e^{i3b})(e^{i3a}-e^{i3b})\right|^2 = 4\left(-\sin(3a)+\sin(3b)+\sin 3(a-b)\right)^2. \tag{3.1}$$

Define, for $0<a,b<2\pi$,

$$\mu(a,b) := -\sin a+\sin b+\sin(a-b), \quad \nu(a,b) := -\sin(3a)+\sin(3b)+\sin 3(a-b).$$

Then

$$g(3) = \max_{a,b\in[0,2\pi]} \min\{2\,|\mu(a,b)|, 2\,|\nu(a,b)|\}$$

and we can compute that

$$\mu(a,b)\nu(a,b) = 16\sin^2(a/2)\sin^2(b/2)\sin^2(a/2-b/2)$$
$$(2\cos a+1)(2\cos(a-b)+1)(2\cos b+1)$$

and $\mu(a,b)\nu(a,b)>0, <0$ in each open domain that is marked as $+, -$ in Figure 3.1. We denote the union of the open domains that are marked as $+, -$ in Figure 3.1 by POS and NEG, respectively.

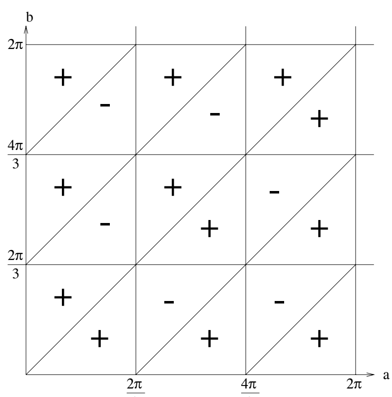

Figure 3.1

Define

$$g_1(3) := \max_{(a,b)\in POS} \min\left\{\left|(1-e^{ia})(1-e^{ib})(e^{ia}-e^{ib})\right|, \left|(1-e^{i3a})(1-e^{i3b})(e^{i3a}-e^{i3b})\right|\right\},$$

$$g_2(3) := \max_{(a,b)\in NEG} \min\left\{\left|(1-e^{ia})(1-e^{ib})(e^{ia}-e^{ib})\right|, \left|(1-e^{i3a})(1-e^{i3b})(e^{i3a}-e^{i3b})\right|\right\},$$

and set $g(3) = \max\{g_1(3), g_2(3)\}$. By using the same packing method that determined $f(3)$ in Section 2, we have the following theorem.

THEOREM 3.1. *We have*

$$\begin{aligned} g_1(3) &= -2c_2 = 3.813955827\cdots, \\ g_2(3) &= 4, \\ g(3) &= 4, \end{aligned}$$

where $c_2 = -1.906977913\cdots$ *is the smallest real root of* $16x^8 - 320x^6 + 2184x^4 - 5084 = 0$.

To prove Theorem 3.1 we use the observations below. By (3.1), our original problems ($g_1(3)$ and $g_2(3)$) are essentially reduced to studying

$$\begin{cases} G_1(a,b) &= -\sin a + \sin b + \sin(a-b), \\ G_2(a,b) &= -\sin(3a) + \sin(3b) + \sin 3(a-b), \end{cases}$$

and

$$\begin{cases} -G_1(a,b) &= \sin a - \sin b - \sin(a-b), \\ G_2(a,b) &= -\sin(3a) + \sin(3b) + \sin 3(a-b). \end{cases}$$

We then follow the procedure of the proof of Theorem 2.7 to prove the four theorems below. The idea is based on the packing method we used in Section 2. The following figures illustrate the method and its proof.

Theorem 3.2 tells us that $g_1(3) = -2c_2 = 3.813955827$, where c_2 is the smallest real root of $16x^8 - 320x^6 + 2184x^4 - 5084x^2 + 2197 = 0$.

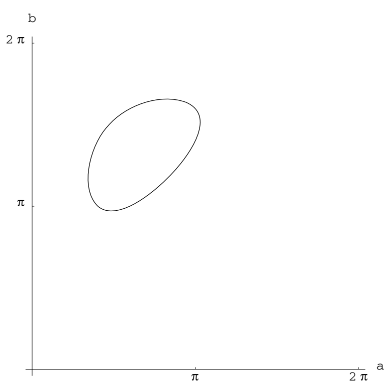

Figure 3.2 a: $G_1(a,b) = c_2$

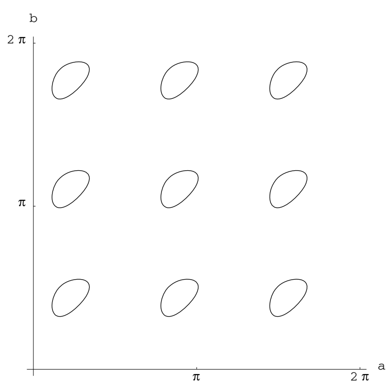

Figure 3.2 b: $G_2(a,b) = c_2$

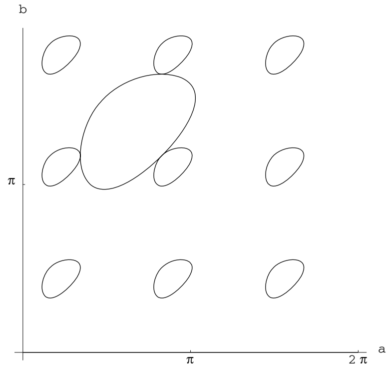

Figure 3.3

THEOREM 3.2. *Let* $c_2 = -1.906977913\cdots$*, where* c_2 *is the smallest real root of* $16x^8 - 320x^6 + 2184x^4 - 5084x^2 + 2197 = 0$. *The sets*

$$\{(a,b) \in [0,2\pi)^2 : G_1(a,b) \leqslant c_2\},$$
$$\{(a,b) \in [0,2\pi)^2 : G_2(a,b) \leqslant c_2\}$$

intersect in at most a finite number of points (a,b) *modulo* 2π*, at which equality holds.*

Theorem 3.3 tells us that $g_2(3) = 4$.

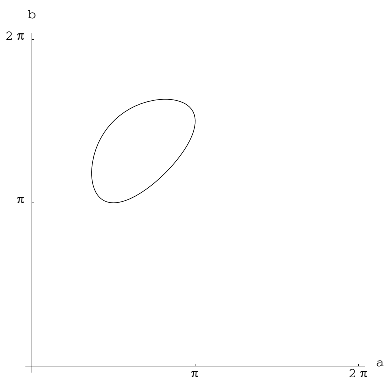

Figure 3.4 a: $-G_1(a,b) = 2$

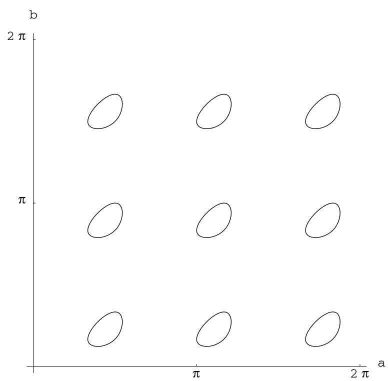

Figure 3.4 b: $G_2(a,b) = 2$

THEOREM 3.3. *The sets*

$$\{(a,b) \in [0,2\pi)^2 : -G_1(a,b) \geqslant 2\},$$
$$\{(a,b) \in [0,2\pi)^2 : G_2(a,b) \geqslant 2\}$$

intersect in at most a finite number of points (a,b) *modulo* 2π*, at which equality holds.*

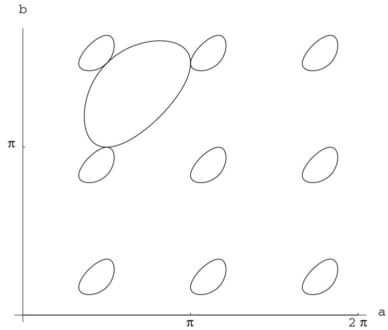

Figure 3.5

We also see that the packing method yields a proof of the following result. In fact, Theorem 3.4 is used in the proof of Theorem 3.2.

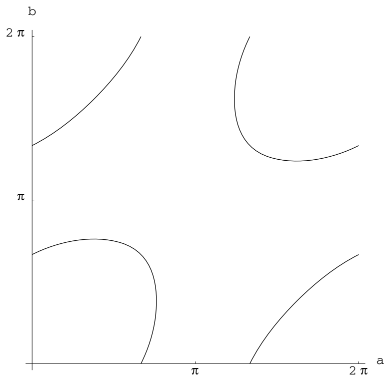

Figure 3.6a: $F_1(a,b) = 0$

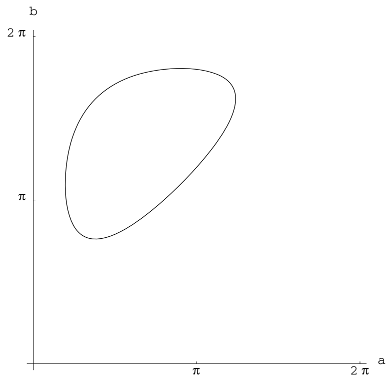

Figure 3.6b: $G_1(a,b) = c_1$

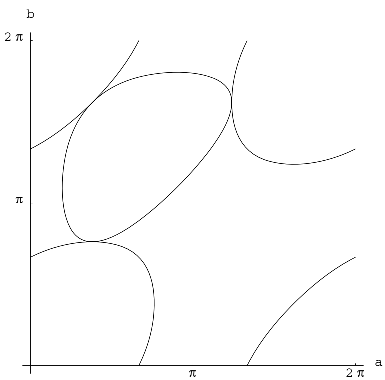

Figure 3.7

THEOREM 3.4. *Let* $c_1 = -\sqrt{6\sqrt{3}-9} = -1.179959679\cdots$ *and* $F_1(a,b) = \cos a + \cos b + \cos(a-b)$. *The sets*

$$\{(a,b) \in [0,2\pi)^2 : F_1(a,b) \geqslant 0\},$$
$$\{(a,b) \in [0,2\pi)^2 : G_1(a,b) \leqslant c_1\}$$

intersect in at most a finite number of points (a,b) *modulo* 2π, *at which equality holds.*

4. f(N) for N ≠ 2,3

In this section, we determine, for an integer $N \geqslant 4$,

$$f(N) = \min_{a_i \text{ real}} \max\left\{\left|\sum_{n=1}^{N} e^{ia_n}\right|, \left|\sum_{n=1}^{N} e^{iNa_n}\right|\right\}. \tag{4.1}$$

First, we prove the following.

THEOREM 4.1. *Let* M *be an integer* $\geqslant 2$. *Then there are infinitely many even numbers* $N(=2^\alpha$, α *a positive integer* $\geqslant 2)$ *such that, for each integer* β, $0 \leqslant \beta \leqslant \frac{\log N}{\log 2} - 2$,

$$\sum_{n=1}^{N} e^{ia_n} = \sum_{n=1}^{N} e^{i(2^\beta M)a_n} = 0$$

for some suitable distinct $a_1, \cdots, a_N$.

Proof. Let α, M be positive integers that are $\geqslant 2$, and set $N = 2^\alpha$. Let

$$\alpha_n = \begin{cases} \frac{2n-1}{2^{\alpha-1}M}\pi, & 1 \leqslant n \leqslant 2^{\alpha-1}, \\ \alpha_{n-2^{\alpha-1}} + \pi, & 2^{\alpha-1}+1 \leqslant n \leqslant 2^\alpha. \end{cases}$$

Thus the sequence $\{e^{i\alpha_n}\}$ consists of $2^{\alpha-1}$ elements together with their negatives, so

$$\sum_{n=1}^{N} e^{i\alpha_n} = 0.$$

Next, for each integer $0 \leqslant \beta \leqslant \alpha - 2 = \frac{\log N}{\log 2} - 2$,

$$\begin{aligned} \sum := \sum_{n=1}^{N} e^{i\alpha_n(2^\beta M)} &= \sum_{n=1}^{2^{\alpha-1}} e^{i\pi\frac{(2n-1)2^\beta}{2^{\alpha-1}}} + \sum_{n=2^{\alpha-1}+1}^{2^\alpha} e^{i\pi 2^\beta M} e^{\frac{i\pi(2n-1)2^\beta}{2^{\alpha-1}}} \\ &= \sum_{n=1}^{2^{\alpha-1}} e^{i\pi\frac{(2n-1)2^\beta}{2^{\alpha-1}}} + e^{i\pi 2^\beta M} \sum_{m=1}^{2^{\alpha-1}} e^{\frac{i\pi(2m-1)2^\beta}{2^{\alpha-1}}} \end{aligned}$$

upon making the change of variable $n = m + 2^{\alpha-1}$. If $2^\beta M$ is odd, this is obviously zero. Otherwise,

$$\sum = 2e^{-i\pi 2^{\beta-\alpha+1}} \sum_{n=1}^{2^{\alpha-1}} e^{i2\pi\left(\frac{n}{2^{\alpha-\beta-1}}\right)} = 0,$$

since $\alpha - \beta - 1 \geqslant 1$. $\square$

The above theorem answers our main question when N in (4.1) is 2^α, where α is a positive integer $\geqslant 2$; one takes $M = 2^\alpha$ and $\beta = 0$. For the cases $N \neq 2, 3$, we divide the positive integers into the four congruence classes modulo 4. Without loss of generality, we assume that one of the e^{ia_n} in our problem is fixed at 1. We start with the following lemma.

LEMMA 4.2. *Let N be an integer $\geqslant 4$.*
(a) If $N = 4k+1$ for some $k \equiv 0$ or $1 \pmod 3$, then

$$\sum_{\substack{0\leqslant n\leqslant N-3\\ n\neq\frac{N-3}{2}}} e^{i\frac{(2n+1)\pi}{N-1}} = \sum_{\substack{0\leqslant n\leqslant N-3\\ n\neq\frac{N-3}{2}}} e^{iN\frac{(2n+1)\pi}{N-1}} = 0,$$

and

$$e^{i\frac{\pi N}{6}} + e^{i\frac{5\pi N}{6}} + e^{i\frac{9\pi N}{6}} = e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{9\pi}{6}} = 0.$$

(b) If $N \equiv 2 \pmod 4$, then

$$\begin{aligned} &e^{i\frac{\pi N}{6}} + e^{i\frac{5\pi N}{6}} + e^{i\frac{9\pi N}{6}} + e^{i\frac{2\pi N}{6}} + e^{i\frac{10\pi N}{6}} + e^{i\pi} \\ &= e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{9\pi}{6}} + e^{i\frac{2\pi}{6}} + e^{i\frac{10\pi}{6}} + e^{i\pi} = 0 \end{aligned} \tag{4.2}$$

and

$$e^{i\frac{\pi N}{4}} + e^{i\frac{3\pi N}{4}} + e^{i\frac{5\pi N}{4}} + e^{i\frac{7\pi N}{4}} = e^{i\frac{\pi}{4}} + e^{i\frac{3\pi}{4}} + e^{i\frac{5\pi}{4}} + e^{i\frac{7\pi}{4}} = 0. \tag{4.3}$$

(c) If $N = 4k+3$ for some $k \equiv 1$ or $2 \pmod 3$, then

$$\sum_{\substack{1\leqslant n\leqslant N-2\\ n\neq\frac{N-1}{2}}} e^{i\frac{2n\pi}{N-1}} = \sum_{\substack{1\leqslant n\leqslant N-2\\ n\neq\frac{N-1}{2}}} e^{iN\frac{2n\pi}{N-1}} = 0,$$

and

$$e^{i\frac{\pi N}{6}} + e^{i\frac{5\pi N}{6}} + e^{i\frac{9\pi N}{6}} = e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{9\pi}{6}} = 0.$$

(d) If $N = 3n$, where n is odd $\geqslant 3$, then

$$\left(0, \frac{2\pi N}{3n-4}, \frac{4\pi N}{3n-4}, \cdots, \frac{(6n-10)\pi N}{3n-4}\right) \equiv \left(0, \frac{2\pi}{3n-4}, \frac{4\pi}{3n-4}, \cdots, \frac{(6n-10)\pi}{3n-4}\right) \pmod{2\pi}$$

as multisets, and hence

$$\begin{aligned} &1 + e^{i\frac{2\pi N}{3n-4}} + e^{i\frac{4\pi N}{3n-4}} + \cdots + e^{i\frac{(6n-10)\pi N}{3n-4}} \\ &= 1 + e^{i\frac{2\pi}{3n-4}} + e^{i\frac{4\pi}{3n-4}} + \cdots + e^{i\frac{(6n-10)\pi}{3n-4}} = 0. \end{aligned}$$

Proof. We can easily prove (b). The result of (d) is obvious from $\gcd(3n, 3n-4) = 1$ (here n is odd $\geqslant 3$). Also the idea of the proof of (c) is essentially same as that of (a). Hence we only prove (a) for this proposition.

(a) The first part follows from

$$\sum_{\substack{0\leqslant n\leqslant N-3\\ n\neq \frac{N-3}{2}}} e^{i\frac{(2n+1)\pi}{N-1}} = \sum_{0\leqslant n\leqslant \frac{N-5}{2}} e^{i\frac{(2n+1)\pi}{N-1}} + \sum_{\frac{N-1}{2}\leqslant n\leqslant N-3} e^{i\frac{(2n+1)\pi}{N-1}}$$

$$= \sum_{0\leqslant n\leqslant \frac{N-5}{2}} e^{i\frac{(2n+1)\pi}{N-1}} + \sum_{0\leqslant n\leqslant \frac{N-5}{2}} e^{i(\frac{2n+1}{N-1}+1)\pi} = 0$$

and $\frac{N(2n+1)\pi}{N-1} \equiv \left(\frac{2n+1}{N-1}+1\right)\pi \pmod{2\pi}$, where n is a nonnegative integer. For the second part, we observe that

$$\left(\frac{\pi N}{6}, \frac{5\pi N}{6}, \frac{9\pi N}{6}\right) \equiv \begin{cases} (\frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}) & \text{if} \quad k\equiv 0 \pmod 3, \\ (\frac{5\pi}{6}, \frac{\pi}{6}, \frac{9\pi}{6}) & \text{if} \quad k\equiv 1 \pmod 3 \end{cases}$$

(mod 2π). This completes the proof. □

By using Lemma 4.2, we shall show the following.

PROPOSITION 4.3. *Let N be an integer $\geqslant 4$. Then*

$$\sum_{n=1}^{N} e^{ia_n} = \sum_{n=1}^{N} e^{iNa_n} = 0$$

for some distinct $a_1, \cdots, a_N$, $0 \leqslant a_i < 2\pi$, i.e.

$$f(N) = \min_{a_i \text{ real}} \max\left\{\left|\sum_{n=1}^{N} e^{ia_n}\right|, \left|\sum_{n=1}^{N} e^{iNa_n}\right|\right\} = 0.$$

Proof. We divide the positive integers $N \geqslant 4$ into the four congruence classes modulo 4.

<u>Case 1</u> $N \equiv 0 \pmod 4$.

Choose

$$a_n = \begin{cases} \frac{(n-1)\pi}{N}, & 1\leqslant n\leqslant \frac{N}{2}, \\ \frac{(2n-N-2)\pi}{2N} + \pi, & \frac{N}{2}+1 \leqslant n \leqslant N, \end{cases}$$

so that $a_{\frac{N}{2}+k+1} = a_{k+1}+\pi$ for $0\leqslant k\leqslant \frac{N}{2}-1$, since $a_{\frac{N}{2}+k+1} = \frac{2(\frac{N}{2}+k+1)-N-2)\pi}{2N}+\pi = \frac{k\pi}{N}+\pi = a_{k+1}+\pi$. Also

$$e^{iNa_n} + e^{iNa_{n+1}} = 0$$

for all odd integers n, $1\leqslant n\leqslant N$. In fact,

$$e^{iNa_n} + e^{iNa_{n+1}} = \begin{cases} e^{i(n-1)\pi} + e^{in\pi} = 0, & 1\leqslant n\leqslant \frac{N}{2}-1, \\ e^{i\left(\frac{(2n-N)\pi}{2}+(N-1)\pi\right)} + e^{i\left(\frac{(2n-N)\pi}{2}+N\pi\right)} = 0, & \frac{N}{2}+1\leqslant n\leqslant N-1. \end{cases}$$

This proves that $f(N) = 0$.
Case 2 $N \equiv 1 \pmod 4$.
Suppose that $N = 4k + 1$ for some $k \equiv 0 \text{ or } 1 \pmod 3$, $k \geqslant 1$. Then, by Lemma 4.2 (a),

$$\begin{aligned} &e^{\frac{i\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{9\pi}{6}} + \sum_{\substack{0\leqslant n\leqslant N-3 \\ n\neq\frac{N-3}{2}}} e^{i\frac{(2n+1)\pi}{N-1}} \\ =&e^{i\frac{\pi N}{6}} + e^{i\frac{5\pi N}{6}} + e^{i\frac{9\pi N}{6}} + \sum_{\substack{0\leqslant n\leqslant N-3 \\ n\neq\frac{N-3}{2}}} e^{iN\frac{(2n+1)\pi}{N-1}} = 0. \end{aligned} \tag{4.4}$$

Here we note that each term $e^{i\frac{(2n+1)\pi}{N-1}}$ of the sum on the left hand side in (4.4) is not any of $e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}$ and $e^{i\frac{9\pi}{6}}$. In fact,

$$\left(\frac{2n+1}{N-1} = \right) \frac{2n+1}{4k} = \frac{h}{6}, \quad (h = 1, 5, 9)$$

implies that $2kh = 3(2n+1)$, which is impossible. So $f(N) = 0$ if $k \equiv 0 \text{ or } 1 \pmod 3$. The case $N = 4k + 1$ with $k \equiv 2 \pmod 3$ will be considered in Case 5.
Case 3 $N \equiv 2 \pmod 4$.
Suppose that $N = 4k + 2$ for some positive integer $k \geqslant 1$. Then, by Lemma 4.2 (b), we immediately get the result for $k = 1$ and $k = 2$ by adding each side of (4.2) and (4.3). Now we assume that $k \geqslant 3$. We first fix 10 numbers

$$e^{i\frac{\pi}{6}},\ e^{i\frac{5\pi}{6}},\ e^{i\frac{9\pi}{6}},\ e^{i\frac{2\pi}{6}},\ e^{i\frac{10\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}},\ e^{i\frac{5\pi}{4}},\ e^{i\frac{7\pi}{4}} \tag{4.5}$$

that were used in $f(10) = 0$, i.e. $k = 2$. Then we only need to find $N_0 := 4k - 8$ more numbers e^{ia_n} so that $\sum_{1\leqslant n\leqslant N_0} e^{ia_n} = \sum_{1\leqslant n\leqslant N_0} e^{iNa_n} = 0$. Let

$$U_0 = \{u_1, u_2, \cdots, u_{2k-4}\}\ (u_1 = 1)$$

be the set of the successive $2k-4$ integers starting from 1 to an integer $< N$ satisfying

$$\frac{u_i}{N} \neq \begin{cases} \frac{h}{6}, & h = 1, 2, 5, \\ \frac{h}{4}, & h = 1, 3, \end{cases}$$

and let

$$U_1 = U_0 + N = \{u_1 + N, u_2 + N, \cdots, u_{2k-4} + N\}$$

and $U = U_0 \cup U_1$. We divide the positive integers k into the three congruence classes modulo 3. If $k = 3j$ or $3j + 2$, then $N = 12j + 2$ or $12j + 10$, respectively, and these are not multiples of 4 and 6. So, in these cases, we have $U_0 = \{1, 2, \cdots, 2k - 4\}$, $U_1 = \{N + 1, N + 2, \cdots, N + 2k - 4\}$. Now choose

$$\{a_i\} = \left\{\frac{u\pi}{N} : u \in U\right\}$$

so that $\sum_{1\leqslant n\leqslant N_0} e^{ia_n} = \sum_{1\leqslant n\leqslant N_0} e^{iNa_n} = 0$. Suppose that $k = 3j + 1$. Then $N = 6(2j + 1)$ is not a multiple of 4 but a multiple of 6. We have

$$2k - 2 < N_0, \quad 2k - 2 < \frac{N}{2} \tag{4.6}$$

and

$$U_0 = \{1, 2, \cdots, 2k-2\} - \left\{u \in \mathbb{Z} : 1 \leqslant u \leqslant 2k-2,\ \frac{u}{N} = \frac{1}{6}, \frac{1}{3}\right\},$$
$$U_1 = U_0 + N.$$

Then, for $h = 6, 9, 10$, we note that, by (4.6),

$$\frac{h}{6} \notin \left\{\frac{u}{N} : u \in U_1\right\}.$$

Now choose

$$\{a_i\} = \left\{\frac{u\pi}{N} : u \in U\right\}$$

so that $\sum_{1\leqslant n\leqslant N_0} e^{ia_n} = \sum_{1\leqslant n\leqslant N_0} e^{iNa_n} = 0$. This gives $f(N) = 0$.

Case 4 $N \equiv 3 \pmod 4$.

Suppose that $N = 4k+3$ for some positive integer k with $k \equiv 1 \text{ or } 2 \pmod 3$. Then by Lemma 4.2 (c),

$$\begin{aligned} &e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{9\pi}{6}} + \sum_{\substack{1\leqslant n\leqslant N-2 \\ n\neq \frac{N-1}{2}}} e^{i\frac{2n\pi}{N-1}} \\ =&e^{i\frac{N\pi}{6}} + e^{i\frac{5N\pi}{6}} + e^{i\frac{9N\pi}{6}} + \sum_{\substack{1\leqslant n\leqslant N-2 \\ n\neq \frac{N-1}{2}}} e^{iN\frac{2n\pi}{N-1}} = 0. \end{aligned} \tag{4.7}$$

Here we note that each term $e^{i\frac{2n\pi}{N-1}}$ of the sum on the left hand side in (4.7) is not any of $e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}$ and $e^{i\frac{9\pi}{6}}$. Thus $f(N) = 0$. The case $N = 4k+3$ with $k \equiv 0 \pmod 3$ will be considered in Case 5.

Case 5 $N = 3n$, where n is an odd integer $\geqslant 3$.

The cases $N = 4k+1$ with $k \equiv 2 \pmod 3$ from Case 2 and $N = 4k+3$ with $k \equiv 0 \pmod 3$ from Case 4 are in fact the case $N = 3n$, where n is an odd integer $\geqslant 3$. For each n, consider the $3n-4$ numbers a_i:

$$0, \frac{2\pi}{3n-4}, \frac{4\pi}{3n-4}, \frac{6\pi}{3n-4}, \cdots, \frac{(6n-10)\pi}{3n-4}. \tag{4.8}$$

Then, by Lemma 4.2 (d),

$$\begin{aligned} &1 + e^{i\frac{2\pi N}{3n-4}} + e^{i\frac{4\pi N}{3n-4}} + \cdots + e^{i\frac{(6n-10)\pi N}{3n-4}} \\ &= 1 + e^{i\frac{2\pi}{3n-4}} + e^{i\frac{4\pi}{3n-4}} + \cdots + e^{i\frac{(6n-10)\pi}{3n-4}} = 0. \end{aligned}$$

So it remains to find four more numbers to get $f(N) = 0$. Obviously,

$$\left(\frac{\pi N}{4}, \frac{3\pi N}{4}, \frac{5\pi N}{4}, \frac{7\pi N}{4}\right) \equiv \left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right) \pmod{2\pi} \tag{4.9}$$

and

$$e^{i\frac{N\pi}{4}} + e^{i\frac{3N\pi}{4}} + e^{i\frac{5N\pi}{4}} + e^{i\frac{N7\pi}{4}} = 0.$$

Here we note that the numbers in (4.9), $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, are not any of the numbers in (4.8). In fact,

$$\frac{2u}{3N-4} = \frac{2v+1}{4}, \quad (0 \leqslant u \leqslant 3n-5,\ 0 \leqslant v \leqslant 3)$$

implies that $8u + 4(2v+1) = 3n(2v+1)$, which is impossible, since the left side is even and the right side is odd. Hence $f(3n) = 0$ for all odd $n \geqslant 3$. This completes the proof. □

By this section, Proposition 2.1 and Corollary 2.8, we have the final answer for $f(N)$.

THEOREM 4.4. *Let N be an integer $\geqslant 2$, and let*

$$f(N) = \min_{a_i \text{ real}} \max \left\{ \left| \sum_{n=1}^{N} e^{ia_n} \right|, \left| \sum_{n=1}^{N} e^{iNa_n} \right| \right\}.$$

Then

$$\begin{cases} f(2) = 1, \\ f(3) = 0.769292\cdots, \\ f(N) = 0, \quad N \geqslant 4. \end{cases}$$

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