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REVERSION OF HÖLDER TYPE INEQUALITIES FOR SUMS OF WEIGHTED NORMS AND ADDITIVE WEIGHTED ESTIMATES OF INTEGRAL OPERATORS

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Abstract. The additive weighted estimate $\|uKf\|_q \leqslant C(\|\rho f\|_p + \|vHf\|_p)$, $f \geqslant 0$, where $Kf(x) = \int_0^x K(x,s)f(s)ds$, $K(x,s) \geqslant 0$, $Hf(x) = \int_0^x f(s)ds$ is reduced to the two-weighted estimate: $\|uKf\|_q \leqslant C\|\rho^* f\|_p$, $f \geqslant 0$, where the weight function ρ^* is expressed via the weight functions ρ and v.

1. Introduction

Let $R_+ = (0, +\infty)$, $1 < p,\ q < \infty$. Let $u(\cdot)$, $v(\cdot)$ and $\rho(\cdot)$ be weight functions on R_+, i.e. nonnegative measurable functions on R_+. Let K and H be the integral operators of the type:

$$Kf(x) = \int_0^x K(x,s)f(s)ds, \quad Hf(x) = \int_0^x f(s)ds,$$

where $K(x,s) \geqslant 0$ with $x \geqslant s \geqslant 0$.

Consider the weighted inequality:

$$\|uKf\|_q \leqslant C\left(\|\rho f\|_p + \|vHf\|_p\right),\ f \geqslant 0, \tag{1}$$

where $\|\cdot\|_p$ is the norm of the space $L_p(R_+)$, $1 < p < \infty$.

For $q \geqslant p$ the inequalities in the form (1) were considered in [1, 2, 3], and for $p > q$ and $K \equiv H$ they were investigated in [4]. The inequality (1) includes different important inequalities. For example, if $f = y^{(n)} \geqslant 0$, $K(x,s) = (x-s)^{n-k-1}$ and $y^{(i)}(0) = 0$, $k \leqslant i \leqslant n-1$, where $n \geqslant 1$, $0 \leqslant k \leqslant n-1$, then from (1) we have:

$$\|uy^{(k)}\|_q \leqslant C\left(\|\rho y^{(n)}\|_p + \|vy^{(n-1)}\|_p\right). \tag{2}$$

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The inequalities of the type (2) were considered in [5, 6]. If $v \equiv 0$, (1) reduces to the inequality with two weights:

$$\|uKf\|_q \leqslant C\|\rho f\|_p \,, \; f \geqslant 0, \tag{3}$$

which has been intensively investigated during several last years (cf. [7, 8]).

The aim of this paper is to reduce the inequality (1) to the equivalent inequality of the form (3).

To obtain the main result we also consider the problem of evaluation of the quantity:

$$J_p(\rho, v, g, M) \equiv \sup_{f \in M} \frac{\int\limits_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vHf\|_p}. \tag{4}$$

The value (4) is of independent interest and it is the reversion of Hölder type inequality for the norm $\|\rho f\|_p + \|vHf\|_p$ on the class of functions $M = \{f \geqslant 0,\ f$ is measurable on $(0,\infty)\}$.

It is known [9] that

$$J_p(\rho, 0, g, M) \equiv \|\rho^{-1}g\|_{p'}, \;\; \frac{1}{p} + \frac{1}{p'} = 1. \tag{5}$$

For $M\downarrow = \{0 \leqslant f \downarrow\}$, $M\uparrow = \{0 \leqslant f \uparrow\}$ the relations

$$J_p(\rho, 0, g, M\downarrow) \approx \left(\int\limits_0^\infty \left(\int\limits_0^t g \right)^{p'-1} \left(\int\limits_0^t \rho^p \right)^{1-p'} g dt \right)^{\frac{1}{p'}}, \tag{6}$$

$$J_p(\rho, 0, g, M\uparrow) \approx \left(\int\limits_0^\infty \left(\int\limits_t^\infty g \right)^{p'-1} \left(\int\limits_t^\infty \rho^p \right)^{1-p'} g dt \right)^{\frac{1}{p'}} \tag{7}$$

were obtained in [10, 11]. If $M\uparrow\downarrow = \{f \geqslant 0 : f \uparrow, \frac{1}{x}f(x) \downarrow\}$ and $0 \leqslant g \downarrow$, the problem of evaluation of $J_p(\rho, 0, g, M\uparrow\downarrow)$ was considered in [12]. Here and the sequel the notation $0 \leqslant f \downarrow$ $(0 \leqslant f \uparrow)$ means that the function f is nonnegative, non-increasing (non-decreasing). The symbol $A \ll C$ means that there exists a constant $\alpha > 0$ such that the inequality $A \leqslant \alpha C$ holds and the symbol $A \approx C$ means that $A \ll C \ll A$.

The paper consists of 5 sections, including this introduction. The second section contains notation and lemmas, which are necessary to prove the main results of section 3. In the fourth section the inequality (2) is considered for the particular case of power weight function. In the last fifth section the analogues of (1) and (4) are proved in which the operators K and H are replaced by their duals.

2. Auxiliary lemmas

Suppose that the weighted functions u, v and ρ satisfy the following conditions:

$$\rho \in L_p^{loc}(R_+),\ \rho^{-1} \in L_{p'}^{loc}(R_+), \tag{8}$$

$$v \in L_p(t,\infty),\quad u \in L_q(t,\infty),\quad u(\cdot)K(\cdot,t) \in L_q(t,\infty),\quad \forall t > 0, \tag{9}$$

and define the function

$$\varphi(x) = \left\{ \inf_{0<t<x} \left[\left(\int_t^x \rho^{-p'}(s)ds \right)^{-\frac{1}{p'}} + \left(\int_t^\infty v^p(s)ds \right)^{\frac{1}{p}} \right] \right\}^{-1}.$$

LEMMA 1. *Let $1 < p < \infty$ and the weight functions ρ and v satisfy the conditions (8) and (9). Then the function φ is positive, strictly increasing on R_+, i.e. $\varphi(x) > \varphi(\tau) > 0$ if $x > \tau > 0$.*

Proof. Let $x > \tau > 0$, then from (8) we get $\int_\tau^x \rho^{-p'}(s)ds > 0$. Therefore:

$$\varphi(x) \geqslant \left\{ \inf_{0<t<\tau} \left[\left(\int_t^\tau \rho^{-p'}ds + \int_\tau^x \rho^{-p'}ds \right)^{-\frac{1}{p'}} + \left(\int_t^\infty v^p ds \right)^{\frac{1}{p}} \right] \right\}^{-1}$$

$$> \left\{ \inf_{0<t<\tau} \left[\left(\int_t^\tau \rho^{-p'}ds \right)^{-\frac{1}{p'}} + \left(\int_t^\infty v^p ds \right)^{\frac{1}{p}} \right] \right\}^{-1} = \varphi(\tau) > 0.$$

□

LEMMA 2. *Let $1 < p < \infty$ and the weight functions ρ and v satisfy the conditions (8) and (9). Then the function φ is locally absolutely continuous on R_+ and satisfies the condition:*

$$0 \leqslant \rho^p(x)\varphi(x) \left(\frac{d\varphi}{dx} \right)^{p-1} \ll 1 \tag{10}$$

for almost all $x \in R_+$.

Proof. For $x > \tau > 0$ we have:

$$\varphi^{p'}(x) = \sup_{0<t<x} \frac{\int_t^x \rho^{-p'}ds}{\left[1 + \left(\int_t^x \rho^{-p'}ds \right)^{\frac{1}{p'}} \left(\int_t^\infty v^p ds \right)^{\frac{1}{p}} \right]^{p'}}$$

$$< \sup_{0<t<\tau} \frac{\int_t^\tau \rho^{-p'} ds + \int_\tau^x \rho^{-p'} ds}{\left[1 + \left(\int_t^\tau \rho^{-p'} ds\right)^{\frac{1}{p'}} \left(\int_t^\infty v^p ds\right)^{\frac{1}{p}}\right]^{p'}}$$

$$+ \sup_{\tau<t<x} \frac{\int_t^x \rho^{-p'} ds}{\left[1 + \left(\int_t^x \rho^{-p'} ds\right)^{\frac{1}{p'}} \left(\int_t^\infty v^p ds\right)^{\frac{1}{p}}\right]^{p'}} \leqslant \varphi^{p'}(\tau) + 2\int_\tau^x \rho^{-p'} ds.$$

Hence, in view of $\varphi \uparrow$ we obtain:

$$0 < \varphi^{p'}(x) - \varphi^{p'}(\tau) \leqslant 2\int_\tau^x \rho^{-p'} ds. \tag{11}$$

By (8) and absolute continuity of the Lebesque integral we get that the strictly increasing function φ is absolutely continuous on any closed interval in R_+. Therefore, for almost all $x \in R_+$ there exists a positive, locally summable derivative $\frac{d\varphi}{dx}$. Hence, from (11) it follows that:

$$p'\varphi^{p'-1}(x)\varphi'(x) = \lim_{\tau\to x} \frac{\varphi^{p'}(x) - \varphi^{p'}(\tau)}{x-\tau} \leqslant 2\lim_{\tau\to x}\left(\frac{1}{x-\tau}\int_\tau^x \rho^{-p'} ds\right)$$
$$= 2\rho^{-p'}(x)$$

for almost all $x \in R_+$. $\square$

LEMMA 3. *Let $1 < p < \infty$ and the functions ρ and v satisfy the conditions (8) and (9). Then the inequality:*

$$\varphi(x) \leqslant \min\left\{\left(\int_0^x \rho^{-p'} ds\right)^{\frac{1}{p'}}, \left(\int_x^\infty v^p ds\right)^{-\frac{1}{p}}\right\} \tag{12}$$

holds for all $x \in R_+$.

Proof. By the definition of the function φ we have that for all $x \in R_+$:

$$\varphi^{-1}(x) \geqslant \max\left\{\inf_{0<t<x}\left(\int_t^x \rho^{-p'} ds\right)^{-\frac{1}{p'}}, \inf_{0<t<x}\left(\int_t^\infty v^p ds\right)^{\frac{1}{p}}\right\}$$
$$= \max\left\{\left(\int_0^x \rho^{-p'} ds\right)^{-\frac{1}{p'}}, \left(\int_x^\infty v^p ds\right)^{\frac{1}{p}}\right\},$$

and (12) follows. □

COROLLARY 1. *If at least one of the conditions:*

$$\rho^{-1} \in L_{p'}(0,t),\ t > 0 \ \text{ or } \ v \notin L_p(R_+)$$

is satisfied, then $\varphi(0) \equiv \lim\limits_{x\to 0+} \varphi(x) = 0$.

LEMMA 4. *Let* $1 < p < \infty$ *and the weight functions* ρ *and* v *satisfy the conditions (8) and (9). Then:*

$$\left(\int\limits_0^\infty \left(\int\limits_0^t f\,ds\right)^{p-1} f(t)\varphi^{-p}(t)dt\right)^{\frac{1}{p}} \leqslant 8\left(\|\rho f\|_p + \|vHf\|_p\right),\ f \geqslant 0. \tag{13}$$

Proof. Let the right side of (13) be finite for some $f \geqslant 0$. Then there exists a sequence of points x_k, $k \in Z_0$ in R_+ such that

$$2^{k-1} = \int\limits_0^{x_{k-1}} f(s)ds = \int\limits_{x_{k-1}}^{x_k} f(s)ds, \quad k \in Z_0,$$

$$2^k \leqslant \int\limits_0^x f(s)ds \leqslant 2^{k+1} \quad \text{with } x_k \leqslant x \leqslant x_{k+1},\quad k \in Z_0,$$

$$R_+ = \bigcup_k [x_k, x_{k+1}), \quad [x_i, x_{i+1}) \bigcap [x_j, x_{j+1}) = \emptyset \ \text{ for } i \neq j,$$

where $Z_0 \subseteq Z$, Z is the set of integers.

By using these facts and the monotonicity of the function $\varphi > 0$, we have:

$$\left(\int\limits_0^\infty \left(\int\limits_0^t f\,ds\right)^{p-1} f(t)\varphi^{-p}(t)dt\right)^{\frac{1}{p}}$$

$$= \left(\sum_k \int\limits_{x_k}^{x_{k+1}} \left(\int\limits_0^t f\,ds\right)^{p-1} f(t)\varphi^{-p}(t)dt\right)^{\frac{1}{p}}$$

$$\leqslant \left(\sum_k 2^{(k+1)(p-1)}\varphi^{-p}(x_k) \int\limits_{x_k}^{x_{k+1}} f(t)dt\right)^{\frac{1}{p}}$$

$$\ll \left(\sum_k 2^{p(k+1)} \left[\left(\int\limits_{x_{k-1}}^{x_k} \rho^{-p'}ds\right)^{-\frac{1}{p'}} + \left(\int\limits_{x_{k-1}}^{\infty} v^p ds\right)^{\frac{1}{p}}\right]^p\right)^{\frac{1}{p}}$$

$$\ll \left(\sum_k 2^{p(k+1)} \left(\int_{x_{k-1}}^{x_k} \rho^{-p'} ds \right)^{-\frac{p}{p'}} \right)^{\frac{1}{p}}$$

$$+ \left(\sum_k 2^{p(k+1)} \int_{x_{k-1}}^{\infty} v^p ds \right)^{\frac{1}{p}} = I_1 + I_2. \tag{14}$$

We estimate I_1 and I_2 separately. Hölder's inequality yields:

$$I_1 \leqslant 4 \left(\sum_k \int_{x_{k-1}}^{x_k} |\rho f|^p ds \right)^{\frac{1}{p}} \leqslant 4\|\rho f\|_p . \tag{15}$$

By applying Jensen's inequality, we obtain:

$$I_2 = 2^3 \left(\sum_k 2^{p(k-2)} \int_{x_{k-1}}^{\infty} v^p ds \right)^{\frac{1}{p}} = 8 \left(\sum_k \left(\int_{x_{k-2}}^{x_{k-1}} f\, ds \right)^p \sum_{i \geqslant k} \int_{x_{i-1}}^{x_i} v^p ds \right)^{\frac{1}{p}}$$

$$= 8 \left(\sum_i \int_{x_{i-1}}^{x_i} v^p ds \sum_{k \leqslant i} \left(\int_{x_{k-2}}^{x_{k-1}} f\, ds \right)^p \right)^{\frac{1}{p}}$$

$$\leqslant 8 \left(\sum_i \int_{x_{i-1}}^{x_i} v^p ds \left(\sum_{k \leqslant i} \int_{x_{k-2}}^{x_{k-1}} f\, ds \right)^p \right)^{\frac{1}{p}}$$

$$\leqslant 8 \left(\sum_i \int_{x_{i-1}}^{x_i} v^p \left(\int_0^x f(s) ds \right)^p dx \right)^{\frac{1}{p}} \leqslant 8\|vHf\|_p. \tag{16}$$

From (14) – (16) we get (13). □

LEMMA 5. *Let* $1 < p < \infty$ *and the weight functions* ρ *and* v *satisfy the conditions (8) and (9). Then the following estimate holds:*

$$\|\rho f\|_p + \|vHf\|_p \ll \left(\int_0^{\infty} |f(t)|^p \varphi^{-1}(t) \left(\frac{d\varphi}{dt} \right)^{1-p} dt \right)^{\frac{1}{p}}. \tag{17}$$

Proof. From (10), (12) we find that:

$$\rho^p(x) \ll \varphi^{-1}(x) \left(\frac{d\varphi}{dx} \right)^{1-p} \quad \text{for almost all } x \in R_+, \tag{18}$$

$$\sup_{x>0}\left(\int_0^x \varphi^{p'-1}(t)\frac{d\varphi}{dt}dt\right)^{\frac{1}{p'}}\left(\int_x^\infty v^p ds\right)^{\frac{1}{p}} \ll \sup_{x>0}\varphi(x)\left(\int_x^\infty v^p ds\right)^{\frac{1}{p}} \leqslant 1. \tag{19}$$

The inequality (18) shows that:

$$\|\rho f\|_p \ll \left(\int_0^\infty |f|^p \varphi^{-1}\left(\frac{d\varphi}{dt}\right)^{1-p} dt\right)^{\frac{1}{p}},$$

and from (19), by using Hardy inequality [9], we have:

$$\|vHf\|_p \ll \left(\int_0^\infty |f|^p \varphi^{-1}\left(\frac{d\varphi}{dt}\right)^{1-p} dt\right)^{\frac{1}{p}},$$

i.e. (17) holds. □

3. The main results

THEOREM 1. *Let* $1<p<\infty$, $0 \leqslant g \downarrow$ *and the weight functions* ρ *and* v *satisfy the conditions (8) and (9). Let* $\varphi(0)=0$, *then:*

$$J_p(\rho,v,g,M) \equiv \sup_{f\in M}\frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p+\|vHf\|_p} \approx \left(\int_0^\infty g^{p'}(s)d\varphi^{p'}(s)\right)^{\frac{1}{p'}}. \tag{20}$$

Proof. Estimate below. Combing (17) and (5), we have:

$$J_p(\rho,v,g,M) \gg \sup_{f\in M}\frac{\int_0^\infty f(s)g(s)ds}{\left(\int_0^\infty f^p\varphi^{-1}\left(\frac{d\varphi}{dt}\right)^{1-p}dt\right)^{\frac{1}{p}}} = J_p\left(\varphi^{-\frac{1}{p}}\left(\frac{d\varphi}{dt}\right)^{-\frac{1}{p'}},0,g,M\right)$$

$$=\left(\int_0^\infty g^{p'}(s)\varphi^{p'-1}(s)\frac{d\varphi}{ds}ds\right)^{\frac{1}{p'}} \approx \left(\int_0^\infty g^{p'}(s)d\varphi^{p'}(s)\right)^{\frac{1}{p'}}. \tag{21}$$

Estimate above. By using the condition $\varphi(0)=0$ and (6), we find:

$$J_{p'}\left(\left(\varphi^{\frac{p'}{p}}\frac{d\varphi}{dt}\right)^{\frac{1}{p'}},\ 0,f,M\downarrow\right) \approx \left(\int_0^\infty\left(\int_0^t f\,ds\right)^{p-1} f(t)\varphi^{-p}(t)dt\right)^{\frac{1}{p}}.$$

Therefore, for $f \geqslant 0$ and $0 \leqslant g \downarrow$

$$\int_0^\infty f(s)g(s)ds \ll \left(\int_0^\infty g^{p'}(s)d\varphi^{p'}(s)\right)^{\frac{1}{p'}} \left(\int_0^\infty \left(\int_0^t f\,ds\right)^{p-1} f(t)\varphi^{-p}(t)dx\right)^{\frac{1}{p}}. \quad (22)$$

From (13) and (22) we have:

$$J_p(\rho, v, g, M) \ll \left(\int_0^\infty g^{p'}(s)d\varphi^{p'}(s)\right)^{\frac{1}{p'}},$$

which together with (21) gives (20). □

COROLLARY 2. *Let the conditions of Theorem 1 be fulfilled. Then:*

$$Q \equiv \sup_{f\geqslant 0} \frac{\int_0^\infty f\,g ds}{\left(\int_0^\infty \left(\int_0^t f\right)^{p-1} f(t)\varphi^{-p}(t)dx\right)^{\frac{1}{p}}} \approx J_p(\rho, v, g, M) \approx \left(\int_0^\infty g^{p'}d\varphi^{p'}\right)^{\frac{1}{p'}}.$$

Proof. Indeed, from (13), (20) and (22) we have:

$$\left(\int_0^\infty g^{p'}d\varphi^{p'}\right)^{\frac{1}{p'}} \ll Q \ll \left(\int_0^\infty g^{p'}d\varphi^{p'}\right)^{\frac{1}{p'}}.$$

□

REMARK. Let us note that:

$$Q \approx J_p(\rho, v, g, M), \quad (23)$$

although the reverse of the inequality (13) does not hold.

Indeed, let $f_z(\cdot) = \rho^{-p'}(\cdot)\chi_{(t_0,z)}(\cdot)$, $t_0 < z < \infty$, where $t_0 > 0$ is a fixed point and $\chi_{(t_\circ,z)}(\cdot)$ is the characteristic function of (t_0, z), then:

$$\sup_{f\geqslant 0} \frac{\|\rho f\|_p + \|vHf\|_p}{\left(\int_0^\infty \left(\int_0^t f\right)^{p-1} f(t)\varphi^{-p}(t)dx\right)^{\frac{1}{p}}} \geqslant \sup_{t_0<z} \frac{\|\rho f_z\|_p}{\left(\int_0^\infty \left(\int_0^t f_z ds\right)^{p-1} f_z(t)\varphi^{-p}(t)dx\right)^{\frac{1}{p}}}$$

$$\geqslant \sup_{z>t_0} \frac{\left(\int_{t_0}^z \rho^{-p'}ds\right)^{\frac{1}{p}}}{\left(\int_{t_0}^z \rho^{-p'}ds\right)^{\frac{1}{p'}} \left(\int_{t_0}^z \rho^{-p'}\varphi^{-p}ds\right)^{\frac{1}{p}}} \geqslant \sup_{z>t_0} \frac{\varphi(t_0)}{\left(\int_{t_0}^z \rho^{-p'}ds\right)^{\frac{1}{p'}}} = \infty.$$

Next we consider the inequality:

$$\|uTf\|_q \leqslant C\left(\|\rho f\|_p + \|vHf\|_p\right), \; f \geqslant 0, \tag{24}$$

where T is a positive linear operator.

THEOREM 2. *Let $1 < p,q < \infty$ and the weight functions ρ and v satisfy the conditions (8) and (9). Let $\varphi(0) = 0$ and let T^* be dual to T such that $0 \leqslant T^*f \downarrow$ for $f \geqslant 0$. Then the inequality (24) holds if and only if*

$$\left(\int_0^\infty (uTf)^q dx\right)^{\frac{1}{q}} \leqslant C_1 \left(\int_0^\infty f^p \varphi^{-1}\left(\frac{d\varphi}{dt}\right)^{1-p} dt\right)^{\frac{1}{p}}, \; f \geqslant 0. \tag{25}$$

Moreover $C \approx C_1$, where C, C_1 are the smallest constants in the inequalities (24), (25) respectively.

Proof. Let $C > 0$ is the smallest constant of (24). By using (5) and (20), we get:

$$\begin{aligned}
C &= \sup_{f\geqslant 0} \frac{\|uTf\|_q}{\|\rho f\|_p + \|vHf\|_p} = \sup_{f\geqslant 0}\sup_{g\geqslant 0} \frac{\int_0^\infty guTf\,dx}{\|g\|_{q'}\left(\|\rho f\|_p + \|vHf\|_p\right)} \\
&= \sup_{g\geqslant 0} \frac{1}{\|g\|_{q'}} \sup_{f\geqslant 0} \frac{\int_0^\infty guTf\,dx}{\|\rho f\|_p + \|vHf\|_p} = \sup_{g\geqslant 0} \frac{1}{\|g\|_{q'}} \sup_{f\geqslant 0} \frac{\int_0^\infty f\,T^*(gu)dx}{\|\rho f\|_p + \|vHf\|_p} \\
&\approx \sup_{g\geqslant 0} \frac{\left(\int_0^\infty [T^*(gu)]^{p'} d\varphi^{p'}\right)^{\frac{1}{p'}}}{\|g\|_{q'}} = C_1.
\end{aligned}$$

Therefore the inequality (24) is equivalent to the inequality:

$$\left(\int_0^\infty [T^*(gu)]^{p'} d\varphi^{p'}\right)^{\frac{1}{p'}} \leqslant C_1\|g\|_{q'}, \;\; g \geqslant 0,$$

which is dual to the inequality (25). □

The paper [7] (cf.also [8]) contains necessary and sufficient conditions for the validity of the inequalities in the form (3) if the kernel $K(x,s)$ of the operator K satisfies the following property: there exists the constant $d \geqslant 1$ such that

$$\frac{1}{d}\left(K(x,t) + K(t,s)\right) \leqslant K(x,s) \leqslant d\left(K(x,t) + K(t,s)\right) \tag{26}$$

holds for $x \geqslant t \geqslant s \geqslant 0$.

From the left – hand side of the inequality (26) we get that $dK(x,s) \geqslant K(x,t)$ if $x \geqslant t \geqslant s$. Setting $\widetilde{K}(x,s) = \sup_{s\leqslant t\leqslant x} K(x,t)$, we have $dK(x,s) \geqslant \widetilde{K}(x,s) \geqslant K(x,s)$

if $x \geqslant s \geqslant 0$. Also the function $\widetilde{K}(x,s)$ is non-increasing in the second argument. Therefore, if we set $\widetilde{K}f(x) = \int_0^x \widetilde{K}(x,s)f(s)ds$, then $Kf \approx \widetilde{K}f$ for all $f \geqslant 0$. Since $0 \leqslant \widetilde{K}^*f \downarrow$ for all $f \geqslant 0$, where $\widetilde{K}^*f(s) = \int_s^\infty \widetilde{K}(x,s)f(x)dx$, by Theorem 2 and $Kf \approx \widetilde{K}f$ we obtain that the inequality (1) with the condition (26) is equivalent to the inequality (3). This conclusion and results of the paper [7] yield the following two Theorems:

THEOREM 3. *Let $1 < p \leqslant q < \infty$ and the weight functions u, v and ρ satisfy the conditions (8) and (9). Let $\varphi(0) = 0$ and the kernel $K(x,s)$ of the operator K satisfies the condition (26). Then the inequality (1) holds if and only if*

$$A_1 = \sup_{x>0} \varphi(x) \left(\int_x^\infty u^q(s) K^q(s,x) ds \right)^{\frac{1}{q}} < \infty,$$

$$A_2 = \sup_{x>0} \left(\int_0^x K^{p'}(x,s) d\varphi^{p'}(s) \right)^{\frac{1}{p}} \left(\int_x^\infty u^q(s) ds \right)^{\frac{1}{q}} < \infty.$$

Moreover $C \approx \max\{A_1, A_2\}$, where C is the smallest constant in the inequality (1).

THEOREM 4. *Let $1 < q < p < \infty$ and the weight functions u, v and ρ satisfy the conditions (8) and (9). Let $\varphi(0) = 0$ and the kernel $K(x,s)$ of the operator K satisfies the condition (26). Then the inequality (1) holds if and only if*

$$B_1 = \left[\int_0^\infty \left(\int_x^\infty u^q(s) K^q(s,x) ds \right)^{\frac{p}{p-q}} d\varphi^{\frac{pq}{p-q}}(x) \right]^{\frac{p-q}{pq}} < \infty,$$

$$B_2 = \left[\int_0^\infty \left(\int_x^\infty u^q(s) ds \right)^{\frac{p}{p-q}} \left(\int_0^x K^{p'}(x,s) d\varphi^{p'}(s) \right)^{\frac{q(p-1)}{p-q}} u^q(x) dx \right]^{\frac{p-q}{pq}} < \infty.$$

Moreover $C \approx \max\{B_1, B_2\}$, where C is the smallest constant in the inequality (1).

4. Example

To illustrate Theorem 3 and Theorem 4 consider the inequality (2) for $u(x) = x^\gamma$, $\rho(x) = x^\mu$, $v(x) = x^\lambda$. In this case (2) is rewritten in the following form:

$$\left(\int_0^\infty |x^\lambda y^{(k)}(x)|^q dx \right)^{\frac{1}{q}} \leqslant C \left[\left(\int_0^\infty (x^\mu y^{(n)}(x))^p dx \right)^{\frac{1}{p}} + \left(\int_0^\infty (x^\lambda y^{(n-1)}(x))^p dx \right)^{\frac{1}{p}} \right], \tag{27}$$

where

$$y^{(n)}(\cdot) \geqslant 0, \;\; y^{(i)}(0) = 0, \;\; k \leqslant i \leqslant n-1, \;\; n \geqslant 1, \;\; 0 \leqslant k \leqslant n-1. \tag{28}$$

Setting $y^{(n)}(x) = f(x)$ in (27), we have that (27) and (28) are equivalent to the inequality:

$$\left(\int_0^\infty \left(x^\lambda R_k f(x) \right)^q dx \right)^{\frac{1}{q}} \leqslant C \left[\left(\int_0^\infty (x^\mu f(x))^p dx \right)^{\frac{1}{p}} \right. \tag{29}$$

$$\left. + \left(\int_0^\infty \left(x^\lambda \int_0^x f(s) ds \right)^p dx \right)^{\frac{1}{p}} \right], \;\; f \geqslant 0,$$

where

$$R_k f(x) = \int_0^x (x-s)^{n-k-1} f(s) ds.$$

For $\rho(x) = x^\mu$ the condition (8) is satisfied for any μ. For $u(x) = x^\gamma$, $v(x) = x^\lambda$ the condition (9) reduces to the condition:

$$\gamma + n - k - 1 + \frac{1}{q} < 0, \;\; \lambda + \frac{1}{p} < 0. \tag{30}$$

In this case $\varphi(x) = \left[\inf_{0<t<x} g(t,x) \right]^{-1}$, where

$$g(t,x) = \left(\int_t^x x^{-p'\mu} \right)^{-\frac{1}{p'}} + C_0 \, t^{\lambda + \frac{1}{p}}, \;\; C_0 = |\lambda p + 1|^{-\frac{1}{p}}.$$

Due to Lemma 2:

$$\varphi(x) \ll \min \left\{ x^{\frac{1}{p'} - \mu}, x^{|\lambda + \frac{1}{p}|} \right\} \quad \text{for } \mu < \frac{1}{p'}, \tag{31}$$

$$\varphi(x) \ll x^{|\lambda + \frac{1}{p}|} \quad \text{for } \mu \geqslant \frac{1}{p'}. \tag{32}$$

By the definition of the function φ we have that:

$$\varphi^{-1}(x) \leqslant g\left(\frac{1}{2}x, x \right) = x^{\mu - \frac{1}{p'}} \left(\int_{\frac{1}{2}}^1 s^{-p'\mu} \right)^{-\frac{1}{p}} + C_0 \, 2^{|\lambda + \frac{1}{p}|} x^{\lambda + \frac{1}{p}}$$

$$\approx \max \left\{ x^{\mu - \frac{1}{p'}}, x^{\lambda + \frac{1}{p}} \right\},$$

i.e.:

$$\varphi(x) \gg \min\left\{x^{\frac{1}{p'}-\mu}, x^{|\lambda+\frac{1}{p}|}\right\}. \tag{33}$$

If $\mu < \frac{1}{p'}$ from (31) and (33) we get:

$$\varphi(x) \approx \min\left\{x^{\frac{1}{p'}-\mu}, x^{|\lambda+\frac{1}{p}|}\right\}, \quad \forall x \in R_+. \tag{34}$$

From (32) and (33) we have:

$$\varphi(x) \approx x^{|\lambda+\frac{1}{p}|} \quad \text{for} \quad \mu \geqslant \frac{1}{p'} \quad \text{and} \quad x < 1. \tag{35}$$

For $\mu \geqslant \frac{1}{p'}$ and $x > 1$ we obtain:

$$\varphi^{-1}(x) = \min\left\{\inf_{0<t\leqslant 1} g(t,x), \inf_{1\leqslant t<x} g(t,x)\right\} \geqslant \min\left\{1, \left(\int_1^x x^{-p'\mu}\right)^{-\frac{1}{p'}}\right\}.$$

Hence

$$\varphi(x) \leqslant |\ln x|^{\frac{1}{p'}} \quad \text{for} \quad \mu = \frac{1}{p'}, \tag{36}$$

$$\varphi(x) \leqslant 1 \quad \text{for} \quad \mu > \frac{1}{p'}. \tag{37}$$

PROPOSITION 1. *Let $1 < p \leqslant q < \infty$ and the condition (30) be fullfiled. Then the inequality (27) under the condition (28) or the inequality (29) holds if and only if*

$$\gamma - \lambda + n - k - 1 + \frac{1}{q} - \frac{1}{p} \geqslant 0 \quad \text{for} \quad \mu \geqslant \frac{1}{p'}, \tag{38}$$

$$\max\left\{\mu - \frac{1}{p}, \lambda + \frac{1}{p}\right\} \geqslant \gamma + n - k - 1 + \frac{1}{q} \geqslant \min\left\{\mu - \frac{1}{p}, \lambda + \frac{1}{p}\right\} \quad \text{for} \quad \mu < \frac{1}{p'}. \tag{39}$$

Proof. The kernel of the operator R_k satisfies the condition (26) and by (34), (35) we have $\varphi(0) = 0$. Therefore, by Theorem 3 the validity of (29) is equivalent to the validity of the following conditions:

$$\lim_{x\to\infty} \varphi(x) x^{\gamma+n-k-1+\frac{1}{q}} < \infty, \tag{40}$$

$$\lim_{x\to 0} \varphi(x) x^{\gamma+n-k-1+\frac{1}{q}} < \infty, \tag{41}$$

$$\lim_{x\to\infty} x^{\gamma+\frac{1}{q}} \left(\int_0^x (x-s)^{p'(n-k-1)} d\varphi^{p'}(s)\right)^{\frac{1}{p'}} < \infty, \tag{42}$$

$$\lim_{x\to 0} x^{\gamma+\frac{1}{q}} \left(\int_0^x (x-s)^{p'(n-k-1)} d\varphi^{p'}(s)\right)^{\frac{1}{p'}} < \infty. \tag{43}$$

For $\mu \geqslant \frac{1}{p'}$ by (35) and (36) the limits (40) and (42) are equal to zero. From (38) the limits (41) and (43) are finite.

For $\mu < \frac{1}{p'}$ the conditions (40) and (42) are equivalent to the validity of the left – hand side of (39). The conditions (41) and (43) are equivalent to the validity of the right hand – side of (39). □

Similarly, by obtaining the finiteness of integrals in B_1 and B_2 at $x = 0$ and $x = \infty$, we prove the following:

PROPOSITION 2. *Let $1 < q < p < \infty$ and the condition (30) be fullfiled. Then the inequality (27) under the condition (28) or the inequality (29) holds if and only if*

$$\gamma - \lambda + n - k - 1 + \frac{1}{q} - \frac{1}{p} > 0 \quad \text{for} \quad \mu \geqslant \frac{1}{p'},$$

$$\max\left\{\mu - \frac{1}{p}, \lambda + \frac{1}{p}\right\} > \gamma + n - k - 1 + \frac{1}{q} > \min\left\{\mu - \frac{1}{p}, \lambda + \frac{1}{p}\right\} \quad \text{for} \quad \mu < \frac{1}{p'}.$$

5. Appendix

Consider the inequality

$$\|uK^*f\|_q \leqslant C\left(\|\rho f\|_p + \|vH^*\|_p\right), \ f \geqslant 0, \tag{44}$$

where $K^*f(s) = \int\limits_s^\infty K(x,s)f(x)dx$, $H^*f(x) = \int\limits_x^\infty f(s)ds$.

In this case instead of (9) we use the condition:

$$v \in L_p(0,t), u \in L_q(0,t), u(\cdot)K(t,\cdot) \in L_q(0,t), \forall t > 0. \tag{45}$$

Instead of the function φ we use the function:

$$\varphi_*(x) = \left[\inf_{x<t<\infty}\left[\left(\int\limits_x^t \rho^{-p'}ds\right)^{-\frac{1}{p'}} + \left(\int\limits_0^t v^p dt\right)^{\frac{1}{p}}\right]\right]^{-1}.$$

Note that the function φ_* is positive, strictly decreasing, locally absolutely continuous. Moreover:

$$\rho^p(x)\varphi_*(x)\left|\frac{d\varphi_*}{dx}\right|^{p-1} \ll 1,$$

$$\varphi_*(x) \leqslant \min\left\{\left(\int\limits_x^\infty \rho^{-p'}ds\right)^{\frac{1}{p'}}, \left(\int\limits_0^x v^p ds\right)^{-\frac{1}{p}}\right\}.$$

From the last inequality we have $\varphi_*(\infty) \equiv \lim\limits_{x\to\infty}\varphi_*(x) = 0$ if only one of the conditions $\rho^{-1} \in L_p(t,+\infty)$, $t > 0$ or $v \notin L_p(R_+)$ holds.

The inequalities (14) and (18) reduce to the following two inequalities:

$$\left(\int_0^\infty \left(\int_t^\infty f\,ds\right)^{p-1} f(t)\varphi_*^{-p}(t)dx\right)^{\frac{1}{p}} \leqslant 8\left(\|\rho f\|_p + \|vH^*f\|_p\right), \quad f \geqslant 0, \tag{46}$$

$$\|\rho f\|_p + \|vH^*f\|_p \ll \left(\int_0^\infty |f|^p \varphi_*^{-1}\left|\frac{d\varphi_*}{dt}\right|^{1-p} dt\right)^{\frac{1}{p}}. \tag{47}$$

Thus, by (7), (46) and (47) we prove:

THEOREM 5. *Let $1 < p < \infty$, $0 \leqslant q \uparrow$ and the weight functions ρ and v satisfy the conditions (8), (45). Let $\varphi_*(\infty) = 0$, then*

$$\sup_{f \geqslant 0} \frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vH^*f\|_p} \approx \left(\int_0^\infty g^{p'}(s)d(-\varphi_*^{p'}(s))\right)^{\frac{1}{p'}}. \tag{48}$$

From (48) and (5) we get:

THEOREM 6. *Let $1 < p, q < \infty$ and the weight functions ρ and v satisfy the conditions (8), (45). Let $\varphi_*(\infty) = 0$ and let T^* be dual to T such that $0 \leqslant T^*f \uparrow$ for $f \geqslant 0$. Then the inequality*

$$\|uTf\|_q \leqslant C\left(\|\rho f\|_p + \|vH^*f\|_p\right), \quad f \geqslant 0 \tag{49}$$

holds if and only if

$$\left(\int_0^\infty (uTf)^q dx\right)^{\frac{1}{q}} \leqslant C_1 \left(\int_0^\infty f^p \varphi_*^{-1}\left|\frac{d\varphi_*}{dt}\right|^{1-p} dt\right)^{\frac{1}{p}}, \quad f \geqslant 0. \tag{50}$$

Moreover $C \approx C_1$, where C, C_1 are the smallest constants in (49), (50) respectively.

Let $K(x,s)$ satisfies the condition (26). Setting $\overline{K}(x,s) = \sup_{s \leqslant t \leqslant x} K(t,s)$, we have $dK(x,s) \geqslant \overline{K}(x,s) \geqslant K(x,s)$ if $x \geqslant s \geqslant 0$. Therefore, $K^*f \approx \overline{K}^*f$ and $0 \leqslant \overline{K}f \uparrow$ for $f \geqslant 0$, where $\overline{K} \equiv (\overline{K}^*)^*$, $\overline{K}^*f(s) = \int_s^\infty \overline{K}(x,s)f(x)dx$. Thus, by Theorem 6 and results of the paper [7] we have:

THEOREM 7. *Let $1 < p \leqslant q < \infty$ and the weight functions u, v and ρ satisfy the conditions (8), (45). Let $\varphi_*(\infty) = 0$ and $K(x,s)$ satisfies the condition (26). Then the inequality (44) holds if and only if*

$$A_1^* = \sup_{x>0} \varphi_*(x)\left(\int_0^x u^q(s)K^q(x,s)ds\right)^{\frac{1}{q}} < \infty,$$

$$A_2^* = \sup_{x>0} \left(\int_x^\infty K^{p'}(s,x) d(-\varphi_*^{p'}(s)) \right)^{\frac{1}{p'}} \left(\int_0^x u^q(s) ds \right)^{\frac{1}{q}} < \infty.$$

Moreover $C \approx \max\{A_1^*, A_2^*\}$, *where* C *is the smallest constant in (44).*

THEOREM 8. *Let* $1 < q < p < \infty$ *and the weight functions* u, v *and* ρ *satisfy the conditions (8), (45). Let* $\varphi_*(\infty) = 0$ *and* $K(x,s)$ *satisfies the condition (26). Then the inequality (44) holds if and only if*

$$B_1^* = \left[\int_0^\infty \left(\int_0^x u^q(s) K^q(x,s) ds \right)^{\frac{p}{p-q}} d\left(-\varphi_*^{\frac{pq}{p-q}}(x) \right) \right]^{\frac{p-q}{pq}} < \infty,$$

$$B_2^* = \left[\int_0^\infty \left(\int_0^x u^q(s) ds \right)^{\frac{p}{p-q}} \left(\int_x^\infty K^{p'}(s,x) d(\varphi_*^{p'}(s)) \right)^{\frac{q(p-1)}{p-q}} u^q(x) dx \right]^{\frac{p-q}{pq}} < \infty.$$

Moreover $C \approx \max\{B_1^*, B_2^*\}$, *where* C *is the smallest constant in (44).*

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