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INEQUALITIES DESCRIBING THE GROWTH OF POLYNOMIALS NOT VANISHING IN A DISK OF PRESCRIBED RADIUS

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(*communicated by J. Pečarić*)

Abstract. In this paper we study the growth of polynomials of degree n having no zeros in $|z| < \kappa$, where κ is an arbitrary positive number. Using the notation $M(p;t) = \max_{|z|=t} |p(z)|$ we measure the growth of p by estimating $M(p;t)/M(p;1)$ from above for any $t > 1$, and from below for any $t < 1$.

1. Introduction

For any entire function f, and $r \geqslant 0$, let $M(f;r) := \max_{|z|=r} |f(z)|$. It is well known that if p is a polynomial of degree at most n, then

$$M(p;R) < M(p;1)R^n \qquad (R > 1), \tag{1}$$

$$M(p';1) < M(p;1)n, \tag{2}$$

and

$$M(p;\rho) > M(p;1)\rho^n \qquad (0 < \rho < 1), \tag{3}$$

unless $p(z) \equiv M(p,1)e^{i\gamma}z^n$, $\gamma \in \mathbb{R}$.

The first inequality is a simple deduction from the maximum modulus principle (see [12, p. 158, Problem 269]). The second inequality is better known as S. Bernstein's inequality, although it first appeared in a paper of M. Riesz [15, p. 357]. Varga [18, p. 44] attributes (3) to E. H. Zarantonello.

Inequalities (1) and (3) are equivalent. For this it suffices to observe that p is a polynomial of degree at most n if and only if $q(z) := z^n\overline{p(1/\overline{z})}$ is, and that $M(q;r) = r^nM(p;1/r)$ for $0 < r < \infty$. It was observed by Bernstein [2] that (2) can be deduced from (1), making use of the Gauss–Lucas Theorem which says that the critical points of a polynomial lie in the closed convex hull of its zeros. Few people, *if any*, know that (1) can be deduced from (2). So, we shall explain how. This is interesting since there are proofs of (2) which do not use (1) at all (see [14, p. 34]). Let $p(z) \not\equiv M(p;1)e^{i\gamma}z^n$ for all $\gamma \in \mathbb{R}$. Applying (2) to the polynomial $p(\rho z)$, we

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conclude that $\rho|p'(\rho z)| < M(p;\rho)n$. For any given $R > 1$, let $M(p;R) = |p(Re^{i\varphi})|$. Then

$$M(p;R) = \left|p(e^{i\varphi}) + \int_1^R p'(\rho e^{i\varphi})e^{i\varphi}\,d\rho\right| < M(p;1) + \int_1^R \frac{n}{\rho}M(p;\rho)\,d\rho.$$

Denoting the right-hand side of the preceding inequality by $\Phi(R)$ we see that

$$\frac{d}{dR}\{R^{-n}\Phi(R)\} = R^{-n}\Phi'(R) - nR^{-n-1}\Phi(R) < 0 \qquad (R > 1).$$

Hence, $R^{-n}\Phi(R)$ is a decreasing function of R for $R > 1$. In particular,

$$M(p;R) < \Phi(R) < \Phi(1)R^n = M(p;1)R^n.$$

Thus, all the three inequalities are equivalent.

When $p(z) \neq 0$ in $|z| < 1$, inequalities (1), (2), and (3) can be replaced by

$$M(p;R) \leqslant M(p;1)\frac{R^n+1}{2} \qquad (R > 1), \tag{4}$$

$$M(p';1) \leqslant M(p;1)\frac{n}{2}, \tag{5}$$

and

$$M(p;\rho) \geqslant M(p;1)\left(\frac{1+\rho}{2}\right)^n \qquad (0 \leqslant \rho < 1), \tag{6}$$

respectively.

Inequality (4) is due to Ankeny and Rivlin [1, Theorem 1]. It becomes an equality for polynomials of the form $p(z) := c(z^n + e^{i\gamma})$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$. Inequality (5) was conjectured by P. Erdős. It was proved independently by G. Pólya, and G. Szegö in the special case where all the zeros of p lie on the unit circle. The two proofs appear in a paper of Lax [9], who showed how the inequality conjectured by Erdős could be deduced from the special case settled by Pólya, and Szegö. Inequality (5) becomes an equality if p is a polynomial of degree n having all its zeros on the unit circle. Inequality (6) is a result of Rivlin [16]. The bound in (6) is attained for polynomials of the form $p(z) := c(z + e^{i\gamma})^n$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$. We wish to emphasize that (6) and (4) are not equivalent.

An entire function f is said to be of exponential type τ if for any $\varepsilon > 0$,

$$|f(z)| = O(e^{(\tau+\varepsilon)|z|}) \qquad (|z| \to \infty).$$

If p is a polynomial of degree at most n, then $f(z) := p(e^{iz})$ is an entire function of exponential type n. Therefore, the preceding inequalities suggest generalizations to such functions. Let f be an entire function of exponential type τ, bounded on the real axis, and let $\mathscr{M}(f;y) := \sup_{x\in\mathbb{R}} |f(x+iy)|$. Then

$$\mathscr{M}(f;y) \leqslant \mathscr{M}(f;0)e^{\tau|y|} \qquad (y \in \mathbb{R}), \tag{7}$$

$$\mathscr{M}(f';0) \leqslant \mathscr{M}(f;0)\tau, \tag{8}$$

and

$$\mathscr{M}(f;y) \geqslant \mathscr{M}(f;0)\mathrm{e}^{-\tau|y|} \qquad (y \in \mathbb{R}). \tag{9}$$

Inequality (7), which is a generalization of (1), is a consequence of the Phragmén–Lindelöf principle (for references see [3, p. 82]). Inequality (8), which extends (2), is Bernstein's generalization [3, Chapter 11] of (2). Inequality (9) is equivalent to (7).

Looking for analogous generalizations of (4) and (5), Boas [4] observed that if $p(z) \neq 0$ for $|z| < 1$, then $f(z) := p(\mathrm{e}^{\mathrm{i}z}) \neq 0$ in the open upper half-plane, and

$$h_f\left(\frac{\pi}{2}\right) := \limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y} = 0.$$

He proved that if f is an entire function of exponential type τ, bounded on the real axis, not vanishing in the open upper half-plane with $h_f(\pi/2) = 0$, then (see [4, Theorems 1 and 2])

$$\mathscr{M}(f;y) \leqslant \mathscr{M}(f;0)\frac{\mathrm{e}^{\tau|y|}+1}{2} \qquad (y < 0), \tag{10}$$

$$\mathscr{M}(f';0) \leqslant \mathscr{M}(f;0)\frac{\tau}{2}. \tag{11}$$

Inequalities (4) and (5) are contained in (10) and (11), respectively. It is no wonder that [4] does not contain any reference to (6), since it was not known at the time. For another generalization of (4) and an $L^p(\mathbb{R})$ analogue of (10), the reader may look up [5] and [6].

In 1958, the late Prof. R. P. Boas, Jr. proposed to one of us to extend (4) and (5) by obtaining the sharp upper bounds for $M(p;R)/M(p;1)$, $R > 1$, and $M(p';1)/M(p;1)$ under the assumption that $p(z) \neq 0$ in $|z| < \kappa$, where κ is a given positive number; and then obtain the corresponding extensions (see [7, p. 502, lines 1–5]) of his inequalities (10) and (11). It was independently shown in [7] and [10] that if p is a polynomial of degree at most n such that $p(z) \neq 0$ for $|z| < \kappa$, where $\kappa \geqslant 1$, then

$$M(p';1) \leqslant M(p;1)\frac{n}{1+\kappa}, \tag{12}$$

with equality for polynomials of the form $c(z+\kappa\mathrm{e}^{\mathrm{i}\gamma})^n$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$. An extension of this inequality to entire functions of exponential type not vanishing in the half-plane $\Im z > \eta$, for some $\eta \leqslant 0$, was also obtained in [7]. Thus, the proposed problem concerning (5) has been solved *in the case where* $\kappa \geqslant 1$. However, to the best of our knowledge, little of any value is known as to how large $M(p;R)/M(p;1)$ can be for $R > 1$, and how small $M(p;\rho)/M(p;1)$ can be for $\rho < 1$, if p is a polynomial of degree at most n not vanishing in $|z| < \kappa$, where κ is a given positive number. The purpose of this paper is to present certain observations concerning this problem.

2. Statement of the main results

Our first result is a partial extension of (4) for polynomials not vanishing in $D(0;\kappa) := \{z \in \mathbb{C} : |z| < \kappa\}$ for some $\kappa \geqslant 1$.

THEOREM 1. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < \kappa$, *where* $\kappa \geqslant 1$, *and let* $\lambda = \lambda(\kappa) := \kappa a_1/(na_0)$. *Then*

$$M(p;R) \leqslant \left(\frac{R^2+2|\lambda|R\kappa+\kappa^2}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} M(p;1) \qquad (1 \leqslant R \leqslant \kappa^2). \tag{13}$$

It is easily seen that $|\lambda| \leqslant 1$, and so for any n, inequality (13) may be replaced by the known estimate ([7, Theorem 1], [14, Theorem 4.23])

$$M(p;R) \leqslant \left(\frac{R+\kappa}{\kappa+1}\right)^{n} M(p;1) \qquad (1 \leqslant R \leqslant \kappa^2), \tag{14}$$

where the bound is attained if $p(z) := c(ze^{i\beta}+\kappa)^n$, $c \in \mathbb{C}$, $c \neq 0$, $\beta \in \mathbb{R}$. In the case where n is even, (13) becomes an equality for polynomials of the form $c(z^2e^{2i\beta}+2\kappa z e^{i\beta}\cos\alpha+\kappa^2)^{n/2}$, $c \in \mathbb{C}$, $c \neq 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.

Theorem 1 does not say anything about $M(p;R)/M(p;1)$ for $R > \kappa^2$. From (13) it follows that

$$M(p;\kappa) \leqslant \kappa^n \left(\frac{2+2|\lambda|}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} M(p;1) \leqslant \left(\frac{2\kappa}{\kappa+1}\right)^{n} M(p;1)$$

since $|\lambda| = \kappa|a_1/(na_0)| \leqslant 1$. Now let $p_\kappa(z) := p(\kappa z)$. Then $p_\kappa(z) \neq 0$ for $|z| < 1$, and $M(p_\kappa;1) = M(p;\kappa)$. Hence, if $R > \kappa$, then writing $R = S\kappa$, where $S := R/\kappa > 1$, we may apply (4) to p_κ and use the above estimate for $M(p;\kappa)$ to conclude that for any $R > \kappa$, we have

$$M(p;R) = M(p_\kappa;S) \leqslant \frac{S^n+1}{2}M(p;\kappa) \leqslant 2^{n-1}\frac{R^n+\kappa^n}{(1+\kappa)^n}M(p;1).$$

Although, this inequality can be seen as a generalization of (4) to which it reduces if we put $\kappa = 1$, it is not satisfactory for large values of κ, since

$$2^{n-1}\frac{R^n+\kappa^n}{(1+\kappa)^n}M(p;1) \sim 2^{n-1}\frac{R^n+\kappa^n}{1+\kappa^n}M(p;1) \qquad \text{as } \kappa \to \infty.$$

The following complement to Theorem 1 shows that the factor 2^{n-1} in this last (asymptotic) estimate is out of place.

THEOREM 2. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < \kappa$, *where* $\kappa > 1$. *Then,*

$$M(p;R) \leqslant \frac{R^n}{\kappa^n}\left(\frac{\kappa^n}{\kappa^n+1}\right)^{(R-\kappa^2)/(R+\kappa^2)} M(p;1) \qquad (R \geqslant \kappa^2). \tag{15}$$

REMARK 1. Note that the right-hand side of (15) agrees with that of (13) for $R = \kappa^2$, and for $R > \kappa^2$ it is strictly less than $(R^n/\kappa^n)M(p;1)$. More precisely, if $R > \kappa^2$, then

$$\left(\frac{\kappa^n}{\kappa^n+1}\right)^{(R-\kappa^2)/(R+\kappa^2)} < 1 - \frac{R-\kappa^2}{R+\kappa^2}\cdot\frac{1}{\kappa^n+1},$$

and so for $R > \kappa^2$, we have

$$M(p;R) < \frac{R^n + \kappa^n}{\kappa^n + 1} M(p;1) + \frac{1}{\kappa^n + 1} \left\{ \frac{2}{\kappa^{n-2}} \frac{R^n}{R + \kappa^2} - \kappa^n \right\} M(p;1).$$

The example $p(z) := z^n + \kappa^n$ shows that $M(p;R)/M(p;1)$ can be at least as large as $(R^n + \kappa^n)/(\kappa^n + 1)$ if not larger.

Our next result is an extension of (6) to polynomials not vanishing in $|z| < \kappa$, for some $\kappa \geqslant 1$.

THEOREM 3. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < \kappa$, *where* $\kappa \geqslant 1$, *and let* $\lambda = \lambda(\kappa) := \kappa a_1/(na_0)$. *Then*

$$M(p;\rho) \geqslant \left(\frac{\kappa^2 + 2\kappa|\lambda|\rho + \rho^2}{\kappa^2 + 2\kappa|\lambda| + 1} \right)^{n/2} M(p;1) \qquad (0 \leqslant \rho < 1). \tag{16}$$

In the case where n is even, (16) becomes an equality for polynomials of the form $c\left(\kappa^2 + 2\kappa z e^{i\beta} \cos\alpha + z^2 e^{2i\beta}\right)^{n/2}$, $c \in \mathbb{C}$, $c \neq 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.

For any n, inequality (16) may be replaced by

$$M(p;\rho) \geqslant \left(\frac{\rho + \kappa}{\kappa + 1} \right)^n M(p;1) \qquad (0 \leqslant \rho < 1), \tag{17}$$

where the bound is attained if $p(z) := c\left(z e^{i\beta} + \kappa\right)^n$, $c \in \mathbb{C}$, $c \neq 0$, $\beta \in \mathbb{R}$. It may be noted that even (17) is a generalization of (6).

Assuming that $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ in $D(0;\kappa) := \{z \in \mathbb{C} : |z| < \kappa\}$ for some $\kappa \leqslant 1$, we prove the following complement to Theorem 3.

THEOREM 4. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < \kappa$, *where* $\kappa \in (0,1]$, *and let* $\lambda = \lambda(\kappa) := \kappa a_1/(na_0)$. *Then*

$$M(p;\rho) \geqslant \left(\frac{\kappa^2 + 2|\lambda|\kappa\rho + \rho^2}{\kappa^2 + 2|\lambda|\kappa + 1} \right)^{n/2} M(p;1) \qquad (0 \leqslant \rho \leqslant \kappa^2). \tag{18}$$

In the case where n is even, (18) becomes an equality for polynomials of the form $c\left(z^2 e^{2i\beta} + 2\kappa z e^{i\beta} \cos\alpha + \kappa^2\right)^{n/2}$, $c \in \mathbb{C}$, $c \neq 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$.

For any n, inequality (18) may be replaced by

$$M(p;\rho) \geqslant \left(\frac{\rho + \kappa}{\kappa + 1} \right)^n M(p;1) \qquad (0 \leqslant \rho \leqslant \kappa^2), \tag{19}$$

where the bound is attained if $p(z) := c(z e^{i\beta} + \kappa)^n$, $c \in \mathbb{C}$, $c \neq 0$, $\beta \in \mathbb{R}$.

Inequality (19) extends and refines a result of Jain [8, inequality (1.4)], who had obtained it under the assumption that all the zeros of p lie on the circle $|z| = \kappa$.

3. Auxiliary results

The following lemma (see [13, p. 340, Corollary 1]; also see [11, p. 444, Theorem 1.7.6) is in fact an extension of (6). We shall not prove it here for obvious reasons.

LEMMA 1. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *in* $D(0;1) := \{z \in \mathbb{C} : |z| < 1\}$, *and let* $\lambda := a_1/(na_0)$. *Then, we have*

$$M(p;\rho_1) \geqslant \left(\frac{1+2|\lambda|\rho_1+\rho_1^2}{1+2|\lambda|\rho_2+\rho_2^2}\right)^{n/2} M(p;\rho_2) \qquad (0 \leqslant \rho_1 < \rho_2 \leqslant 1). \tag{20}$$

The next lemma is also an extension of (6).

LEMMA 2. *Let* p *be a polynomial of degree at most* n *such that* $p(z) \neq 0$ *in* $D(0;\ell) := \{z \in \mathbb{C} : |z| < \ell\}$ *for some* $\ell > 0$. *Then*

$$M(p;\rho) \geqslant \left(\frac{\rho+\ell}{1+\ell}\right)^{n} M(p;1) \qquad \big(0 \leqslant \rho \leqslant \min\{1,\ell^2\}\big).$$

Proof. Let $z_\nu = r_\nu e^{i\theta_\nu}$, and $z = \rho e^{i\theta}$. Then,

$$\left|\frac{z-z_\nu}{e^{i\theta}-z_\nu}\right|^2 = \frac{(\rho+r_\nu)^2 - 2\rho r_\nu\big(1+\cos(\theta-\theta_\nu)\big)}{(1+r_\nu)^2 - 2r_\nu\big(1+\cos(\theta-\theta_\nu)\big)} \geqslant \left(\frac{\rho+r_\nu}{1+r_\nu}\right)^2,$$

where the inequality holds only if $(1-\rho)(r_\nu^2-\rho) \geqslant 0$. Thus, if $r_\nu \geqslant \ell$, then

$$\left|\frac{z-z_\nu}{e^{i\theta}-z_\nu}\right| \geqslant \frac{\rho+r_\nu}{1+r_\nu} \geqslant \frac{\rho+\ell}{1+\ell} \quad \text{if} \quad 0 \leqslant \rho \leqslant \min\{1,\ell^2\}.$$

Hence, if the polynomial $p(z) := a_m \prod_{\nu=1}^{m}(z-z_\nu), a_m \neq 0$, has no zeros in $|z| < \ell$, then

$$\left|\frac{p(\rho e^{i\theta})}{p(e^{i\theta})}\right| \geqslant \left(\frac{\rho+\ell}{1+\ell}\right)^m \quad \text{for} \quad -\pi \leqslant \theta \leqslant \pi \ \text{ if } \ 0 \leqslant \rho \leqslant \min\{1,\ell^2\}.$$

Consequently, if θ_0 is such that $|p(e^{i\theta_0})| = M(p;1)$, then

$$M(p;\rho) \geqslant |p(e^{i\theta_0})| \geqslant \left(\frac{\rho+\ell}{1+\ell}\right)^m M(p;1) \quad \text{if } \ 0 \leqslant \rho \leqslant \min\{1,\ell^2\}. \qquad \square$$

4. Proofs of the main results

We shall first prove Theorem 4, since it is used in the proof of Theorem 1. After that we shall present the proofs of Theorems 1, 2, and 3, respectively.

Proof of Theorem 4. Let $p_\kappa(z) := p(\kappa z) = a_0 + a_1\kappa z + \cdots + a_n\kappa^n z^n$. Then $p_\kappa(z) \neq 0$ for $|z| < 1$. Hence, Lemma 1 may be applied to p_κ taking $\rho_1 = \rho/\kappa$ and $\rho_2 = \kappa$ to obtain

$$\begin{aligned} M(p;\rho) = M\left(p_\kappa;\frac{\rho}{\kappa}\right) &\geqslant \left(\frac{1+2|\lambda|\rho\kappa^{-1}+\rho^2\kappa^{-2}}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} M(p_\kappa;\kappa) \\ &= \left(\frac{\kappa^2+2|\lambda|\kappa\rho+\rho^2}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} \kappa^{-n}M(p;\kappa^2) \\ &\geqslant \left(\frac{\kappa^2+2|\lambda|\kappa\rho+\rho^2}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} M(p;1), \end{aligned}$$

since $M(p;\kappa^2) \geqslant \kappa^n M(p;1)$ by Lemma 2. □

Proof of Theorem 1. First, let $1 \leqslant R \leqslant \kappa$. Then

$$p_\kappa(z) := p(\kappa z) = \sum_{\nu=0}^{n} a_\nu \kappa^\nu z^\nu \neq 0 \quad \text{for } |z| < 1.$$

Besides,

$$M(p_\kappa;\rho_2) = M(p;R) \text{ and } M(p_\kappa;\rho_1) = M(p;1),$$

where $\rho_2 = R/\kappa$ and $\rho_1 = 1/\kappa$. Since $R \in (1,\kappa]$, we see that $0 < \rho_1 < \rho_2 \leqslant 1$ and so from (20) we obtain

$$M(p;1) \geqslant \left(\frac{1+2|\lambda|\kappa^{-1}+\kappa^{-2}}{1+2|\lambda|R\kappa^{-1}+R^2\kappa^{-2}}\right)^{n/2} M(p;R) \qquad (1 < R \leqslant \kappa),$$

which is the same as (13) for $1 < R \leqslant \kappa$.

Next let $\kappa \leqslant R \leqslant \kappa^2$. Then $p_R(z) := p(Rz) \neq 0$ for $|z| < \kappa/R$. Since $\kappa/R \leqslant 1$ and $1/R \leqslant \kappa^2/R^2$ we may apply Theorem 4 to the polynomial p_R with κ/R instead of κ and $\rho = 1/R$ to obtain

$$\begin{aligned} M(p;R) &= \max_{|z|=1} |p(Rz)| \\ &\leqslant \left(\frac{\kappa^2+2|\lambda|\kappa+1}{\kappa^2+2|\lambda|\kappa\rho+\rho^2}\right)^{n/2} \max_{|z|=1/R} |p(Rz)| \\ &= \left(\frac{R^2+2|\lambda|R\kappa+\kappa^2}{1+2|\lambda|\kappa+\kappa^2}\right)^{n/2} M(p;1) \qquad (\kappa \leqslant R \leqslant \kappa^2), \end{aligned}$$

which gives us the desired inequality for $\kappa \leqslant R \leqslant \kappa^2$. □

Proof of Theorem 2. Without loss of generality we may assume that p is of degree n, and that $M(p;1)=1$. From (13) it follows that

$$M(p;\kappa^2) \leqslant \kappa^n M(p;1) = \kappa^n. \tag{21}$$

Hence, if $g(z) := p(\kappa^2 z) = a_0 + \kappa^2 a_1 z + \cdots + \kappa^{2n} a_n z^n$, then $|g(z)| \leqslant \kappa^n$ for $|z|=1$. Besides, $g(z) \neq 0$ for $|z| < 1/\kappa$. Setting

$$G(z) := \kappa^{-n} z^n \overline{g(1/\overline{z})} = \kappa^{-n}\overline{a}_0 z^n + \kappa^{-n+2}\overline{a}_1 z^{n-1} + \cdots + \kappa^n \overline{a}_n,$$

we see that $|G(z)| \leqslant 1$ for $|z|=1$ and that G has all its zeros in the closed disk $|z| \leqslant \kappa$.

Since $|p(z)| \leqslant 1$ for $|z|=1$ it follows from an inequality of Visser [19] that

$$|a_0| + |a_n| \leqslant 1.$$

Hence, writing $p(z) := a_n \prod_{\nu=1}^{n}(z - z_\nu)$, where $|z_\nu| \geqslant \kappa$ for $1 \leqslant \nu \leqslant n$ we see that $|a_o| \geqslant \kappa^n |a_n|$, and so

$$|a_n| \leqslant \frac{1}{\kappa^n + 1},$$

which implies that

$$|G(0)| = |\kappa^n \overline{a}_n| \leqslant \frac{\kappa^n}{\kappa^n + 1}. \tag{22}$$

Now, let us suppose that $G(z) \neq 0$ for $|z| \leqslant 1$. Then applying Poisson's integral formula [17, p. 124] to $\operatorname{Log}|G(z)|$, we obtain

$$\operatorname{Log}\left|G(re^{i\theta})\right| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} \operatorname{Log}|G(e^{i\varphi})|\,d\varphi \qquad (0 \leqslant r < 1).$$

Since $\operatorname{Log}|G(e^{i\varphi})| \leqslant 0$ we conclude that for $0 \leqslant r < 1$, we have

$$\operatorname{Log}\left|G(re^{i\theta})\right| \leqslant \frac{1-r}{1+r}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{Log}|G(e^{i\varphi})|\,d\varphi = \frac{1-r}{1+r}\operatorname{Log}|G(0)|,$$

that is

$$|G(z)| \leqslant |G(0)|^{(1-|z|)/(1+|z|)} \qquad (0 \leqslant |z| \leqslant 1),$$

which when combined with (22) gives

$$|G(z)| \leqslant \left(\frac{\kappa^n}{\kappa^n+1}\right)^{(1-|z|)/(1+|z|)} \qquad (0 \leqslant |z| \leqslant 1). \tag{23}$$

Next, we shall show that (23) remains true even if G has some zeros in $|z|<1$, say $\kappa^2/\overline{z}_1, \ldots, \kappa^2/\overline{z}_m$. In such a case

$$|a_0| \geqslant |a_n|\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|,$$

and from Visser's inequality [19] it follows that

$$|a_n| \leqslant \frac{1}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|},$$

and that

$$|G(0)| \leqslant \frac{\kappa^n}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|}.$$

Now, let

$$G^*(z) := G(z)\prod_{\mu=1}^{m}\frac{\kappa^2 z/z_\mu - 1}{z-\kappa^2/\overline{z}_\mu} = G(z)\prod_{\mu=1}^{m}\left(\frac{\overline{z}_\mu}{z_\mu}\cdot\frac{\kappa^2 z - z_\mu}{\overline{z}_\mu z-\kappa^2}\right).$$

Then

$$|G^*(0)| \leqslant \frac{\kappa^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|}.$$

Since $|G^*(z)| \leqslant 1$ for $|z|=1$ and $G^*(z)\neq 0$ for $|z|<1$, we may again use Poisson's formula to conclude that

$$|G^*(z)| \leqslant \left(\frac{\kappa^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|}\right)^{(1-|z|)/(1+|z|)} \qquad (|z|<1).$$

Hence,

$$\begin{aligned}|G(z)| &\leqslant \left(\frac{\kappa^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|}\right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m}\left|\frac{z-\frac{\kappa^2}{\overline{z}_\mu}}{\frac{\kappa^2}{z_\mu}z-1}\right| \\ &\leqslant \left(\frac{\kappa^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+\kappa^{n-m}\prod_{\mu=1}^{m}|z_\mu|}\right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m}\frac{|z|+\frac{\kappa^2}{|z_\mu|}}{\frac{\kappa^2}{|z_\mu|}|z|+1} \qquad (|z|<1).\end{aligned}$$

Setting $t_\mu := \kappa^2/|z_\mu|$ for $1\leqslant\mu\leqslant m$ we see that for $|z|<1$, we have

$$|G(z)| \leqslant \psi(t_1,\dots,t_m) := \left(\frac{\kappa^n}{t_1\cdots t_m+\kappa^{n+m}}\right)^{(1-|z|)/(1+|z|)}\prod_{\mu=1}^{m}\frac{|z|+t_\mu}{t_\mu|z|+1}.$$

Setting

$$\Lambda := \frac{1}{\kappa^{n+m} + t_1 \cdots t_m}$$

and

$$A_\nu := (\kappa^n)^{\frac{1-|z|}{1+|z|}} \prod_{\mu=1,\mu\neq\nu}^{m} \frac{|z| + t_\mu}{t_\mu|z| + 1} \qquad (1 \leqslant \nu \leqslant m),$$

we see that for any $\nu \in \{1, \ldots, m\}$, the partial derivatives

$$\frac{\partial\psi}{\partial t_\nu} = A_\nu \left\{ \frac{1 - |z|^2}{(t_\nu|z| + 1)^2} \Lambda^{\frac{1-|z|}{1+|z|}} - \frac{|z| + t_\nu}{t_\nu|z| + 1} \frac{1 - |z|}{1 + |z|} \Lambda^{\frac{2}{1+|z|}} \frac{t_1 \cdots t_m}{t_\nu} \right\}$$

are positive if and only if

$$(1 + |z|)^2 > (|z| + t_\nu)(t_\nu|z| + 1) \left(\frac{1}{\kappa^{n+m} + t_1 \cdots t_m} \right) \frac{t_1 \cdots t_m}{t_\nu},$$

which is indeed true since $t_\mu < 1$ for $1 \leqslant \mu \leqslant m$ and $\kappa \geqslant 1$. Hence,

$$\psi(t_1, \ldots, t_m) \leqslant \psi(1, \ldots, 1) = \left(\frac{\kappa^n}{\kappa^{n+m} + 1} \right)^{(1-|z|)/(1+|z|)} \qquad (|z| < 1),$$

and so (23) holds even if G has some zeros in the open disk $|z| < 1$.

From (23) we conclude that

$$\left| p\left(\frac{\kappa^2}{\bar{z}} \right) \right| \leqslant \frac{\kappa^n}{|z|^n} \left(\frac{\kappa^n}{\kappa^n + 1} \right)^{(1-|z|)/(1+|z|)} \qquad (0 < |z| \leqslant 1).$$

This implies that

$$|p(\zeta)| \leqslant \frac{|\zeta|^n}{\kappa^n} \left(\frac{\kappa^n}{\kappa^n + 1} \right)^{(|\zeta|-\kappa^2)/(|\zeta|+\kappa^2)} \qquad (|\zeta| > \kappa^2),$$

which is equivalent to (15). □

Proof of Theorem 3. Let

$$p_\kappa(z) = p(\kappa z) := a_0 + \kappa a_1 z + \cdots + \kappa^n a_n z^n.$$

Then $p_\kappa(z) \neq 0$ for $|z| < 1$. Applying Lemma 1 to p_κ taking $\rho_1 := \rho/\kappa$, and $\rho_2 := 1/\kappa$, we obtain

$$\begin{aligned} M(p;\rho) &= M(p_\kappa;\rho_1) \\ &\geqslant \left(\frac{1 + 2|\kappa a_1/(na_0)|\rho_1 + \rho_1^2}{1 + 2|\kappa a_1/(na_0)|\rho_2 + \rho_2^2} \right)^{n/2} M(p_\kappa;\rho_2) \\ &= \left(\frac{1 + 2|\lambda|\rho/\kappa + \rho^2/\kappa^2}{1 + 2|\lambda|/\kappa + 1/\kappa^2} \right)^{n/2} M(p;1) \qquad (0 \leqslant \rho < 1), \end{aligned}$$

which is equivalent to (16). □

5. Some additional results

Looking at Theorem 4, one might wonder how small $M(p;\rho)/M(p;1)$ can be if $\rho \in (\kappa^2, 1)$. The next result contains a lower bound for this quantity in the case where $\rho \in (\kappa^2, \kappa)$.

PROPOSITION 1. *Let p be a polynomial of degree n such that $p(z) \neq 0$ in $D(0;\kappa) := \{z \in \mathbb{C} : |z| < \kappa\}$ for some $\kappa \in (0,1)$. Then*

$$M(p;\rho) \geqslant \kappa^n \left(\frac{\kappa^n + \rho^n}{\kappa^n}\right)^{(\rho-\kappa^2)/(\rho+\kappa^2)} M(p;1) \qquad (\kappa^2 \leqslant \rho \leqslant \kappa). \tag{24}$$

Proof. Let $p_\rho(z) := p(\rho z)$. Then $p_\rho(z) \neq 0$ for $|z| < \kappa/\rho$. Observe that $\kappa/\rho \geqslant 1$ since $\rho \leqslant \kappa$ and that $R := 1/\rho \geqslant \kappa^2/\rho^2$ since $\rho \geqslant \kappa^2$. Hence, applying (15) to p_ρ with κ/ρ instead of κ and $R := 1/\rho$, we obtain

$$\begin{aligned} M(p;1) = M\left(p_\rho; \frac{1}{\rho}\right) \ &\leqslant \ \frac{1}{\kappa^n}\left(\frac{(\kappa/\rho)^n}{(\kappa/\rho)^n + 1}\right)^{\frac{(1/\rho)-(\kappa^2/\rho^2)}{(1/\rho)+(\kappa^2/\rho^2)}} M(p_\rho;1) \\ &= \ \frac{1}{\kappa^n}\left(\frac{\kappa^n}{\kappa^n+\rho^n}\right)^{\frac{\rho-\kappa^2}{\rho+\kappa^2}} M(p;\rho) \qquad (\kappa^2 \leqslant \rho \leqslant \kappa), \end{aligned}$$

which is equivalent to (24). □

REMARK 2. The right-hand side of (24) agrees with that of (18) for $\rho = \kappa^2$. For $\rho = \kappa$, inequality (24) reduces to

$$M(p;\kappa) \geqslant \frac{2\kappa^n}{2^{2\kappa/(1+\kappa)}} M(p;1),$$

which is not precise. We can replace it by the sharp estimate

$$M(p;\kappa) \geqslant \frac{2\kappa^n}{1+\kappa^n} M(p;1). \tag{25}$$

Indeed, $p_\kappa(z) := p(\kappa z) \neq 0$ for $|z| < 1$, and hence, applying (4) to p_κ taking $R = 1/\kappa$, we obtain

$$M(p;1) = M\left(p_\kappa; \frac{1}{\kappa}\right) \leqslant \frac{\kappa^{-n}+1}{2} M(p_\kappa;1) = \frac{1+\kappa^n}{2\kappa^n} M(p;\kappa).$$

In (25), equality holds for $p(z) := c(z^n + \kappa^n)$, $c \neq 0$.

The next result can be seen as a supplement to Proposition 1.

PROPOSITION 2. *Let p be a polynomial of degree n such that $p(z) \neq 0$ in $D(0;\kappa) := \{z \in \mathbb{C} : |z| < \kappa\}$ for some $\kappa \in (0,1)$. Then*

$$M(p;\rho) \geqslant \left(\frac{2\kappa^n}{1+\kappa^n}\right)^{(\log\rho)/(\log\kappa)} M(p;1) \qquad (\kappa^2 \leqslant \rho \leqslant \kappa). \tag{26}$$

Proof. Recall that $\log M(p;r)$ is a convex function of $\log r$. Hence, by the definition of convexity, if $0 < \rho < \kappa < 1$, then

$$\frac{\log M(p;1) - \log M(p;\kappa)}{\log 1 - \log \kappa} \geqslant \frac{\log M(p;\kappa) - \log M(p;\rho)}{\log \kappa - \log \rho},$$

that is

$$\{M(p;\kappa)\}^{\log(1/\rho)} \leqslant \{M(p;\rho)\}^{\log(1/\kappa)}\{M(p;1)\}^{\log(\kappa/\rho)}.$$

Thus, in view of (25), we have

$$\left(\frac{2\kappa^n}{1+\kappa^n}\right)^{\log(1/\rho)}\{M(p;1)\}^{\log(1/\rho)} \leqslant \{M(p;\rho)\}^{\log(1/\kappa)}\{M(p;1)\}^{\log(\kappa/\rho)},$$

and so

$$M(p;\rho) \geqslant \left(\frac{2\kappa^n}{1+\kappa^n}\right)^{(\log\rho)/(\log\kappa)} M(p;1) \qquad (0<\rho<\kappa<1). \qquad \square$$

REMARK 3. Note that (26) agrees with (25) for $\rho = \kappa$. However, neither of the two preceding results contains the other. In fact, comparing them we see that Proposition 1 gives a better lower bound for $M(p;\rho)$ near κ^2 whereas Proposition 2 does the same near κ. Together, they say that if p is as in Proposition 1, then for $\kappa^2 \leqslant \rho \leqslant \kappa$, we have

$$M(p;\rho) \geqslant \max\left\{\kappa^n\left(\frac{\kappa^n+\rho^n}{\kappa^n}\right)^{(\rho-\kappa^2)/(\rho+\kappa^2)}, \left(\frac{2\kappa^n}{1+\kappa^n}\right)^{(\log\rho)/(\log\kappa)}\right\} M(p;1).$$

The following result gives a lower bound for $M(p;\rho)/M(p;1)$ in the remaining case where $\kappa < \rho < 1$.

PROPOSITION 3. *Let* $p(z) := a_n \prod_{\nu=1}^{n}(z-z_\nu)$ *be a polynomial of degree* n *such that* $\left(\prod_{\nu=1}^{n}|z_\nu|\right)^{1/n} \geqslant \kappa$ *for some* $\kappa \in [0,1]$. *Then*

$$M(p;\rho) \geqslant \frac{k^n + \sqrt{\kappa^{2n} + 4(1+\kappa^n)\rho^2}}{2(1+\kappa^n)}\rho^{n-1}M(p;1) \qquad (0 \leqslant \rho \leqslant 1). \tag{27}$$

In particular, (27) holds if $p(z) \neq 0$ *for* $|z| < \kappa$ *for some* $\kappa \in (0,1)$.

Proof. Let $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$. Then $|a_0|/|a_n| \geqslant \kappa^n$, so that by Visser's inequality [19], we have

$$|a_n|\kappa^n + |a_n| \leqslant |a_o| + |a_n| \leqslant M(p;1),$$

that is

$$|a_n| \leqslant \frac{M(p;1)}{1+\kappa^n}. \tag{28}$$

Let $q(z) := z^n \overline{p(1/\bar{z})}$, and

$$\Phi_R(z) := \frac{q(Rz)}{M(q;R)} \qquad (R > 1).$$

Then by (28),

$$|\Phi_R(0)| = \frac{|a_n|}{M(q;R)} \leqslant \frac{M(p;1)}{(1+\kappa^n)M(q;R)}.$$

Since $|\Phi_R(z)| \leqslant 1$, we may apply Schwarz's lemma [17, p. 212] to conclude that for any $\theta \in (-\pi, \pi]$, we have

$$\begin{aligned} |\Phi_R(R^{-1}\mathrm{e}^{\mathrm{i}\theta})| &\leqslant \frac{R^{-1} + \{(1+\kappa^n)M(q;R)\}^{-1}M(p;1)}{\{(1+\kappa^n)M(q;R)\}^{-1}M(p;1)R^{-1} + 1} \\ &= \frac{(1+\kappa^n)M(q;R) + RM(p;1)}{M(p;1) + (1+\kappa^n)M(q;R)R} \qquad (R > 1). \end{aligned}$$

Hence, if θ is such that $|q(R \cdot R^{-1}\mathrm{e}^{\mathrm{i}\theta})| = |p(\mathrm{e}^{\mathrm{i}\theta})| = M(p;1)$, then

$$\frac{M(p;1)}{M(q;R)} \leqslant \frac{(1+\kappa^n)M(q;R) + RM(p;1)}{M(p;1) + (1+\kappa^n)M(q;R)R}.$$

Setting $\eta := (1+\kappa^n)M(q;R)/M(p;1)$, we conclude that

$$1 + \kappa^n \leqslant \eta \frac{\eta + R}{1 + R\eta}.$$

Hence, if

$$\eta_1 := \frac{\kappa^n R - \sqrt{\kappa^{2n}R^2 + 4(1+\kappa^n)}}{2} \quad \text{and} \quad \eta_2 := \frac{\kappa^n R + \sqrt{\kappa^{2n}R^2 + 4(1+\kappa^n)}}{2},$$

then

$$(\eta - \eta_1)(\eta - \eta_2) \geqslant 0,$$

and so, η must belong to $(-\infty, \eta_1] \cup [\eta_2, \infty)$. Since $\eta_1 < 0$ whereas $\eta > 0$, we conclude that $\eta \in [\eta_2, \infty)$. Thus

$$M(q;R) \geqslant \frac{\kappa^n R + \sqrt{\kappa^{2n}R^2 + 4(1+\kappa^n)}}{2(1+\kappa^n)} M(p;1) \qquad (R > 1). \tag{29}$$

Now note that if $0 < \rho < 1$ and $R := 1/\rho$, then

$$|q(R\mathrm{e}^{\mathrm{i}\theta})| = R^n \left|\overline{p(R^{-1}\mathrm{e}^{\mathrm{i}\theta})}\right| = \rho^{-n} \left|p(\rho\mathrm{e}^{\mathrm{i}\theta})\right|,$$

which implies that

$$M(q;R) = \rho^{-n} M(p;\rho).$$

Hence (29) says that

$$M(p;\rho) \geqslant \frac{\kappa^n + \sqrt{\kappa^{2n} + 4(1+\kappa^n)\rho^2}}{2(1+\kappa^n)} \rho^{n-1} M(p;1) \qquad (0 < \rho < 1). \qquad \square$$

REMARK 4. Proposition 3, which always applies with $\kappa = 0$, may be seen as extension of (3). In fact, (27) reduces to (3) for $\kappa = 0$. It is easily checked that the right-hand side of (27) is larger than $\rho^n M(p;1)$ for all $\rho \in (0,1)$ if $\kappa \in (0,1)$.

Inequality (4), Theorem 1 and Theorem 2 deal with the upper bound for the ratio $M(p;R)/$ $M(p;1)$, $R > 1$, when p is a polynomial of degree at most n not vanishing in $|z| < \kappa \in [1,\infty)$. What can we say in the case where $p(z) \neq 0$ in $|z| < \kappa$ for some $\kappa < 1$? An answer to this question is contained in the following result.

PROPOSITION 4. *Let* $p(z) := a_n \prod_{\nu=1}^{n} (z - z_\nu)$ *be a polynomial of degree* n *such that* $\left(\prod_{\nu=1}^{n} |z_\nu|\right)^{1/n} \geqslant \kappa$ *for some positive* κ. *Then,*

$$M(p;R) \leqslant R^n \frac{R + (1+\kappa^n)}{(1+\kappa^n)R + 1} M(p;1) \qquad (R > 1). \tag{30}$$

Proof. Let $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$. Then

$$q(z) := z^n \overline{p(1/\overline{z})} = \overline{a}_n + \overline{a}_{n-1} z + \cdots + \overline{a}_0 z^n,$$

and by (28),

$$|q(0)| = |a_n| \leqslant \frac{M(p;1)}{1+\kappa^n}.$$

Since $M(q;1) = M(p;1)$, we may apply Schwarz's lemma to the polynomial $q(z)/M(p;1)$ to conclude that

$$\left| z^n \overline{p(1/\overline{z})} \right| = |q(z)| \leqslant M(p;1) \frac{|z| + 1/(1+\kappa^n)}{|z|/(1+\kappa^n) + 1} \qquad (|z| < 1),$$

and so

$$|p(z)| \leqslant M(p;1) \frac{1 + \kappa^n + |z|}{1 + (1+\kappa^n)|z|} |z|^n \qquad (|z| > 1),$$

which is equivalent to (30). $\square$

REMARK 5. Note that the right-hand side of (30) is *strictly* less than $R^n M(p;1)$. It may also be mentioned that

$$R^n \frac{R + (1+\kappa^n)}{(1+\kappa^n)R + 1} < \frac{R^n}{1+\kappa^n} + 2\frac{\kappa^n}{1+\kappa^n} R^{n-1}.$$

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