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REVERSE INEQUALITIES ON CHAOTICALLY GEOMETRIC MEAN VIA SPECHT RATIO

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(*communicated by T. Furuta*)

Abstract. As an application of Mond-Pečarić method, we shall estimate bounds of operator convexity for convex functions. Consequently, we obtain some order relations between the arithmetic mean and the chaotically geometric one $A\diamondsuit_\alpha B$ of positive operators A and B, i.e., $A\diamondsuit_\alpha B = e^{(1-\alpha)\log A+\alpha\log B}$ for $\alpha \in [0,1]$. Among others, we show that if $0 < m \leqslant A, B \leqslant M$ for some scalars $m < M$ and $h = \frac{M}{m}$, then

$$M_h(1)^{-1}A \diamondsuit_\alpha B \leqslant A \nabla_\alpha B \leqslant M_h(1)A \diamondsuit_\alpha B$$

holds for all $\alpha \in [0,1]$, where the Specht ratio $M_h(1)$ is defined as

$$M_h(1) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}} \quad (h>1) \quad \text{and} \quad M_1(1) = 1.$$

1. Introduction

A continuous function f on an interval I is called operator concave on I if

$$f((1-\alpha)A+\alpha B) \geqslant (1-\alpha)f(A)+\alpha f(B) \tag{1}$$

holds for $\alpha \in [0,1]$ and selfadjoint operators A and B whose spectra are contained in I. Also it is called operator convex on I if the reverse inequality of (1) holds. Typical examples of such functions are as follows: $\log t$ is operator concave on $(0,\infty)$, t^r is operator concave for $0 \leqslant r \leqslant 1$ and t^r is operator convex for $1 \leqslant r \leqslant 2$ or $-1 \leqslant r \leqslant 0$ ([12, 3]).

Very recently, the following inequality is shown by Tominaga [14] as a reverse of the arithmetic-geometric mean inequality:

$$(1-\alpha)a+\alpha b \leqslant M_h(1)a^{1-\alpha}b^\alpha \tag{2}$$

for $\alpha \in [0,1]$ and $0 < a < b$, where $h = \frac{b}{a}$ and $M_h(1)$ is the Specht ratio ([13, 2]);

$$M_h(1) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}} \quad (h>1) \quad \text{and} \quad M_1(1) = 1. \tag{3}$$

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Let us recall that the geometric mean and arithmetic mean of positive operators A and B are defined as

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},$$

and

$$A \nabla_\alpha B = (1-\alpha)A + \alpha B,$$

where $\alpha \in [0,1]$. Tominaga pointed out that (2) holds for positive operators A and B satisfying $0 < m \leqslant A$, $B \leqslant M$ for some scalars $m < M$, i.e.,

$$A \nabla_\alpha B \leqslant M_h(1) A \sharp_\alpha B, \tag{4}$$

where $h = \frac{M}{m}$. It is a reverse inequality of the noncommutative arithmetic-geometric mean inequality

$$A \sharp_\alpha B \leqslant A \nabla_\alpha B \tag{5}$$

for $A,\ B \geqslant 0$ and $\alpha \in [0,1]$.

On the other hand, Nakamoto and one of the authors discussed the monotonicity of a family of power means in [4] recently. For fixed $A,\ B > 0$, we put

$$F(r) = \begin{cases} (A^r \nabla_\alpha B^r)^{\frac{1}{r}} & (r \neq 0), \\ e^{\log A \ \nabla_\alpha \ \log B} & (r = 0). \end{cases} \tag{6}$$

Then $F(r)$ is monotone increasing under the chaotic order, $X \gg Y$, i.e., $\log X \geqslant \log Y$ for $X,\ Y > 0$ [7]. In particular, $A \diamondsuit_\alpha B = e^{\log A \ \nabla_\alpha \ \log B}$ is called the chaotically α-geometric mean. In general, it does not coincide with $A \sharp_\alpha B$.

The main purpose in this note is to consider some order relations between the arithmetic mean and the chaotically geometric mean. As a matter of fact, we show that if $0 < m \leqslant A,\ B \leqslant M$ and $h = \frac{M}{m}$, then

$$M_h(1)^{-1} A \diamondsuit_\alpha B \leqslant A \nabla_\alpha B \leqslant M_h(1) A \diamondsuit_\alpha B$$

holds for all $\alpha \in [0,1]$. Equivalently, we have

$$M_h(1)^{-1} A \nabla_\alpha B \leqslant A \diamondsuit_\alpha B \leqslant M_h(1) A \nabla_\alpha B.$$

This says that (4) holds for the chaotically geometric mean.

Next we discuss reverse inequalities of ratio type among the family $\{F(r); r \in \mathbb{R}\}$. For this, we need the reverse inequality for the Hölder-McCarthy inequality, due to Furuta [6]. As a consequence, we have some relations between $(A \sharp_\alpha B)^r$ and $A^r \sharp_\alpha B^r$ for $r > 0$.

Concluding this section, we have to mention that almost all results in this note are based on our privious result [9, Corollary 4] coming from the Mond-Pečarić method [10]. Namely this note might be understood as an application of the Mond-Pečarić method.

2. Preliminary on Mond-Pečarić method

Let A be a positive operator on a Hilbert space H satisfying $0 < m \leqslant A \leqslant M$ for some scalars $m < M$, and let $f(t)$ be a real valued continuous convex function on $[m,M]$. Mond and Pečarić [10] proved that

$$f((Ax,x)) \leqslant (f(A)x,x) \leqslant \lambda(m,M,f)f((Ax,x)) \tag{1}$$

holds for every unit vector $x \in H$, where

$$\lambda(m,M,f) = \max\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right);\ t\in[m,M]\right\}. \tag{2}$$

The following result is a generalization of (1) and based on the idea due to Furuta's work [5, 6]. We here cite it for convenience:

THEOREM A . ([9]) *Let A_j $(j = 1,2,\dots,k)$ be positive operators on a Hilbert space H satisfying $0 < m \leqslant A_j \leqslant M$ for some scalars $m < M$. Let $f(t)$ be a real valued continuous convex function on $[m,M]$, and let $x_1,\dots,x_k$ be vectors in H with $\sum_{j=1}^{k}\|x_j\|^2 = 1$. If $f(t)$ satisfies either (i) $f(t) > 0$ or (ii) $f(t) < 0$ on $[m,M]$, then*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leqslant \lambda f\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big) \tag{3}$$

holds for some $\lambda \geqslant 1$ in case (i), or $0 < \lambda \leqslant 1$ in case (ii), where a value of λ is, in each case, $\lambda(m,M,f)$ defined as (2).

We note that (3) in Theorem A gives a reverse inequality of the following known inequality, eg. [11]: With notations as in Theorem A

$$f\Big(\sum_{j=1}^{k}(A_jx_j,x_j)\Big) \leqslant \sum_{j=1}^{k}(f(A_j)x_j,x_j). \tag{4}$$

For the function $f(t) = t^p$, we know the following fact by Furuta [6], which gives a reverse inequality of the Hölder-McCarthy inequality:

THEOREM B . *Let A be a positive operator on a Hilbert space H satisfying $0 < m \leqslant A \leqslant M$ for some scalars $m < M$ and put $h = \frac{M}{m}$. Then for each $p > 1$*

$$(A^px,x) \leqslant K_+(h,p)(Ax,x)^p \tag{5}$$

holds for every unit vector $x \in H$ where the Ky Fan-Furuta constant $K_+(h,p)$ ([8, 6]) is defined as

$$K_+(h,p) = \frac{(p-1)^{p-1}}{p^p}\cdot\frac{(h^p-1)^p}{(h-1)(h^p-h)^{p-1}}. \tag{6}$$

We obtain a complement of Theorem B by itself.

PROPOSITION 2.1. *Assume that the conditions of Theorem B hold. If* $0 < p < 1$*, then*

$$K_+\left(h^p, \frac{1}{p}\right)^{-p} (Ax,x)^p \leqslant (A^p x,x) \leqslant (Ax,x)^p$$

holds for every unit vector $x \in H$*.*

Proof. Since $0 < p < 1$, we have $1 < \frac{1}{p}$ and so Theorem B implies that

$$(A^{1/p}x,x) \leqslant K_+(h,1/p)(Ax,x)^{1/p}.$$

Replacing A by A^p, we have $(Ax,x) \leqslant K_+(h^p,1/p)(A^p x,x)^{1/p}$ and by raising all terms to the power p we obtain the desired result. □

Moreover, by Theorem B, Furuta [6] showed the following Kantorovich type order preserving inequality.

THEOREM C. *Let* A *and* B *be positive operators on a Hilbert space* H *satisfying* $0 < m \leqslant A \leqslant M$ *or* $0 < m \leqslant A \leqslant M$ *for some scalars* $m < M$*. If* $0 \leqslant A \leqslant B$*, then*

$$A^p \leqslant K_+(h,p)B^p \quad \textit{for all} \quad p \geqslant 1,$$

where $h = \frac{M}{m}$*.*

3. Reverse inequalities on operator convexity

In this section, by virtue of Theorem A, we shall estimate the bounds of the operator convexity for convex functions.

THEOREM 3.1. *Let* A *and* B *be positive operators on a Hilbert space* H *satisfying* $0 < m \leqslant A,\ B \leqslant M$ *for some scalars* $m < M$*. If* $f(t)$ *is a positive real valued continuous convex function on* $[m,M]$*, then for each* $0 \leqslant \alpha \leqslant 1$

$$\frac{1}{\lambda(m,M,f)} f(A\nabla_\alpha B) \leqslant f(A)\nabla_\alpha f(B) \leqslant \lambda(m,M,f) f(A\nabla_\alpha B), \tag{1}$$

where $\lambda(m,M,f)$ *is defined as (2).*

Proof. For each $0 < \alpha < 1$ and unit vector $x \in H$, put $A_1 = A$, $A_2 = B$, $x_1 = \sqrt{1-\alpha}x$ and $x_2 = \sqrt{\alpha}x$ in Theorem A. Then we have

$$(1-\alpha)(f(A)x,x) + \alpha(f(B)x,x) \leqslant \lambda(m,M,f) f((1-\alpha)(Ax,x) + \alpha(Bx,x)).$$

Hence it follows that

$$\begin{aligned}(((1-\alpha)f(A) + \alpha f(B))x,x) &\leqslant \lambda(m,M,f) f(((1-\alpha)A + \alpha B)x,x)) \\ &\leqslant \lambda(m,M,f)\,(f((1-\alpha)A + \alpha B)x,x)\end{aligned}$$

and the last inequality holds by the convexity of $f(t)$. Therefore we have

$$f(A)\nabla_\alpha f(B) \leqslant \lambda(m,M,f)f(A\ \nabla_\alpha\ B).$$

Next, since $f(t)$ is convex, it follows from (4) that

$$(1-\alpha)(f(A)x,x)+\alpha(f(B)x,x) \geqslant f((1-\alpha)(Ax,x)+\alpha(Bx,x)).$$

Since $0<m\leqslant(1-\alpha)A+\alpha B\leqslant M$, it follows from (1) that

$$\begin{aligned} f((1-\alpha)(Ax,x)+\alpha(Bx,x)) &= f(((A\ \nabla_\alpha\ B)x,x)) \\ &\geqslant \frac{1}{\lambda(m,M,f)}(f(A\ \nabla_\alpha\ B)x,x). \end{aligned}$$

Therefore we have

$$\frac{1}{\lambda(m,M,f)}f(A\nabla_\alpha B) \leqslant f(A)\nabla_\alpha f(B). \qquad \square$$

We have the following complementary result of Theorem 3.1 for concave functions.

THEOREM 3.2. *Let A and B be positive operators on a Hilbert space H satisfying $0<m\leqslant A,\ B\leqslant M$ for some scalars $m<M$. If $f(t)$ is a real valued continuous concave function on $[m,M]$ such that $f(t)>0$ on $[m,M]$, then for each $0\leqslant\alpha\leqslant 1$*

$$\frac{1}{\mu(m,M,f)}f(A\nabla_\alpha B) \geqslant f(A)\nabla_\alpha f(B) \geqslant \mu(m,M,f)f(A\nabla_\alpha B), \tag{2}$$

where

$$\mu(m,M,f)=\min\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right);\ t\in[m,M]\right\}.$$

Next, consider the function $f(t)=t^r$ on $[0,\infty)$. Then $f(t)$ is operator concave if $0\leqslant r\leqslant 1$, operator convex if $1\leqslant r\leqslant 2$ and $f(t)$ is not operator convex though $f(t)$ is convex if $r\geqslant 2$ [3]. By Theorems 3.1 and 3.2, we obtain reverse inequalities on operator convexity and operator concavity for $f(t)=t^r$.

THEOREM 3.3. *Let A and B be positive operators on a Hilbert space H satisfying $0<m\leqslant A,\ B\leqslant M$ for some scalars $m<M$. Let $0\leqslant\alpha\leqslant 1$.*

(i) If $0<r\leqslant 1$, then

$$(A\nabla_\alpha B)^r \geqslant A^r\nabla_\alpha B^r \geqslant K_+\left(h^r,\frac{1}{r}\right)^{-r}(A\nabla_\alpha B)^r.$$

(ii) If $1\leqslant r\leqslant 2$, then

$$(A\nabla_\alpha B)^r \leqslant A^r\nabla_\alpha B^r \leqslant K_+(h,r)(A\nabla_\alpha B)^r.$$

(iii) If $r>2$, then

$$\frac{1}{K_+(h,r)}(A\nabla_\alpha B)^r \leqslant A^r\nabla_\alpha B^r \leqslant K_+(h,r)(A\nabla_\alpha B)^r,$$

where $h=\frac{M}{m}$ and $K_+(h,r)$ is defined as (6).

Proof. Put $f(t) = t^r$ for $r > 1$ in Theorem 3.1, then we obtain $\lambda(m,M,f) = K_+(h,r)$. Also, in the case of $0 < r \leqslant 1$, we have $\mu(m,M,f) = K_+(h^r, 1/r)^{-r}$ in Theorem 3.2. □

Though $F(r)$ defined in (6) is monotone increasing under the chaotic order, $F(r)$ is not monotone increasing for $0 < r < 1$ under the usual order. By virtue of Theorem 3.3, we see that $F(r)$ is monotone increasing for $r > 0$ in the following sense:

COROLLARY 3.4. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A,\ B \leqslant M$ for some scalars $m < M$. Let $0 < r \leqslant s$ and $0 \leqslant \alpha \leqslant 1$.*

(i) If $0 < r \leqslant 1$, then

$$K_+\left(h^r, \frac{1}{r}\right)^{-1} K_+(h^r, \frac{s}{r})^{-1/s} F(s) \leqslant F(r) \leqslant K_+\left(h^r, \frac{1}{r}\right) F(s).$$

(ii) If $r \geqslant 1$, then

$$K_+\left(h^r, \frac{s}{r}\right)^{-1/s} F(s) \leqslant F(r) \leqslant F(s),$$

where $h = \frac{M}{m}$ and $K_+(h,r)$ is defined as (6).

Proof. Since $0 < \frac{r}{s} \leqslant 1$ and $m^s \leqslant A^s,\ B^s \leqslant M^s$, we apply Theorem 3.3 (i) to obtain the following inequality

$$(A^s \nabla_\alpha B^s)^{\frac{r}{s}} \geqslant A^r \nabla_\alpha B^r \geqslant K_+\left(h^r, \frac{s}{r}\right)^{-\frac{r}{s}} (A^s \nabla_\alpha B^s)^{\frac{r}{s}}. \tag{3}$$

If $r \geqslant 1$, then $1 \geqslant \frac{1}{r} > 0$ and by raising all terms of (3) to the power $\frac{1}{r}$ it follows from Löwner-Heinz Theorem that

$$(A^s \nabla_\alpha B^s)^{\frac{1}{s}} \geqslant (A^r \nabla_\alpha B^r)^{1/r} \geqslant K_+\left(h^r, \frac{s}{r}\right)^{-\frac{1}{s}} (A^s \nabla_\alpha B^s)^{\frac{1}{s}}.$$

Also if $0 < r \leqslant 1$, then $\frac{1}{r} \geqslant 1$ and by raising all terms of (3) to the power $\frac{1}{r}$ it follows from Theorem C that

$$K_+\left(h^r, \frac{1}{r}\right) (A^s \nabla_\alpha B^s)^{\frac{1}{s}} \geqslant (A^r \nabla_\alpha B^r)^{1/r} \geqslant K_+\left(h^r, \frac{1}{r}\right)^{-1} K_+\left(h^r, \frac{s}{r}\right)^{-\frac{1}{s}} (A^s \nabla_\alpha B^s)^{\frac{1}{s}}.$$

□

Though $f(t) = e^t$ is not operator convex, we have the following result.

COROLLARY 3.5. *Let A and B be selfadjoint operators on a Hilbert space H satisfying $m \leqslant A,\ B \leqslant M$ for some scalars $0 < m < M$. Then for each $0 \leqslant \alpha \leqslant 1$*

$$\frac{1}{M_h(1)} e^{(1-\alpha)A+\alpha B} \leqslant (1-\alpha)e^A + \alpha e^B \leqslant M_h(1) e^{(1-\alpha)A+\alpha B},$$

where $h = e^{M-m}$ and $M_h(1)$ is defined as (3).

Proof. Applying Theorem 3.1 to $f(t) = e^t$, we have this corollary. In fact, we have

$$\lambda(m,M,f) = \frac{1}{e^{t_0}}\left(\frac{e^M - e^m}{M-m}(t_0 - m) + e^m\right)$$

where

$$t_0 = \frac{(m+1)e^M - (M+1)e^m}{e^M - e^m}.$$

Therefore, if we put $h = e^{M-m}$, then

$$\lambda(m,M,f) = e^{-\frac{(m+1)e^M-(M+1)e^m}{e^M-e^m}}\left(\frac{e^M - e^m}{M-m}\right) = e^{-1+\frac{M-m}{h-1}}\left(\frac{h-1}{\log h}\right)$$
$$= \frac{h^{\frac{1}{h-1}}(h-1)}{e\log h} = M_h(1). \qquad \square$$

4. Comparison between $(A \natural_\alpha B)^r$ and $A^r \natural_\alpha B^r$

In this section, we consider some order relations on the geometric mean, which will be used in the next section (Theorem 5.3) to give a comparison between the geometric mean and the chaotically geometric one. Ando and Hiai [1] showed the following result in terms of the log-majorization.

THEOREM D. *For every $A,B \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$*

$$(A\natural_\alpha B)^r \succ_{(\log)} A^r\natural_\alpha B^r \quad \textit{for} \quad r \geqslant 1$$

or equivalently

$$(A^q\natural_\alpha B^q)^{1/q} \succ_{(\log)} (A^p\natural_\alpha B^p)^{1/p} \quad \textit{for} \quad 0 < q \leqslant p.$$

On the other hand, $(A\natural_\alpha B)^r$ and $A^r\natural_\alpha B^r$ have no relation under the usual order. As an application of Theorem 3.3, we have the following result.

THEOREM 4.1. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A,B \leqslant M$ for some scalars $m < M$. Let $0 \leqslant \alpha \leqslant 1$.*

(i) *If $0 < r \leqslant 1$, then*

$$\frac{1}{K_+\left(h^r,\frac{1}{r}\right)^r M_h(r)}(A\natural_\alpha B)^r \leqslant A^r\natural_\alpha B^r \leqslant M_h(1)^r(A\natural_\alpha B)^r. \tag{1}$$

(ii) *If $1 \leqslant r \leqslant 2$, then*

$$\frac{1}{K_+(h,r)M_h(r)}(A\natural_\alpha B)^r \leqslant A^r\natural_\alpha B^r \leqslant K_+(h,r)^2 M_h(1)^r(A\natural_\alpha B)^r. \tag{2}$$

(iii) *If $r \geqslant 2$, then*

$$\frac{1}{K_+(h,r)^2 M_h(r)}(A\natural_\alpha B)^r \leqslant A^r\natural_\alpha B^r \leqslant K_+(h,r)^2 M_h(1)^r(A\natural_\alpha B)^r \tag{3}$$

where $h = \frac{M}{m}$, $K_+(h,r)$ is defined as (6) and $M_h(1)$ is defined as (3).

Proof. Suppose that $0 < r \leqslant 1$. By (4) and (5) we have

$$M_h(1)A\sharp_\alpha B \geqslant A\nabla_\alpha B \geqslant A\sharp_\alpha B. \tag{4}$$

By raising all terms of (4) to the power r, it follows from Löwner-Heinz Theorem, operator concavity of t^r $(0 < r \leqslant 1)$ and the arithmetic-geometric mean inequality that

$$M_h(1)^r(A\sharp_\alpha B)^r \geqslant (A\nabla_\alpha B)^r \geqslant A^r\nabla_\alpha B^r \geqslant A^r\sharp_\alpha B^r.$$

Applying (4) to A^r and B^r, we have

$$M_h(r)A^r\sharp_\alpha B^r \geqslant A^r\nabla_\alpha B^r \geqslant K_+\left(h^r,\frac{1}{r}\right)^{-r}(A\nabla_\alpha B)^r \geqslant K_+\left(h^r,\frac{1}{r}\right)^{-r}(A\sharp_\alpha B)^r,$$

because $M_{h^r}(1) = M_h(r)$ and the second inequality holds by (i) of Theorem 3.3. Hence we have

$$A^r\sharp_\alpha B^r \geqslant \frac{1}{K_+\left(h^r,\frac{1}{r}\right)^r M_h(r)}(A\sharp_\alpha B)^r.$$

Suppose that $r \geqslant 1$. By raising all terms of the first inequality of (4) to the power r $(\geqslant 1)$, it follows from Theorem C that

$$K_+(h,r)M_h(1)^r(A\sharp_\alpha B)^r \geqslant (A\nabla_\alpha B)^r.$$

By Theorem 3.3 we have

$$(A\nabla_\alpha B)^r \geqslant \frac{1}{K_+(h,r)}A^r\nabla_\alpha B^r \geqslant \frac{1}{K_+(h,r)}A^r\sharp_\alpha B^r.$$

Applying the first inequality of (4) to A^r and B^r, operator convexity of t^r, we have

$$M_h(r)A^r\sharp_\alpha B^r \geqslant A^r\nabla_\alpha B^r \geqslant (A\nabla_\alpha B)^r \geqslant K_+(h,r)^{-1}(A\sharp_\alpha B)^r \quad \text{if} \quad 1 \leqslant r \leqslant 2,$$

and

$$M_h(r)A^r\sharp_\alpha B^r \geqslant A^r\nabla_\alpha B^r \geqslant K_+(h,r)^{-1}(A\nabla_\alpha B)^r \geqslant K_+(h,r)^{-2}(A\sharp_\alpha B)^r \quad \text{if} \quad r \geqslant 2.$$

□

COROLLARY 4.2. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A, B \leqslant M$ for some scalars $m < M$. Let $0 < r \leqslant s$.*

(*i*) *If* $0 < r \leqslant 1$, *then*

$$\begin{aligned} \frac{1}{K_+\left(h^r,\frac{1}{r}\right)K_+\left(h^r,\frac{s}{r}\right)^{1/s}M_h(r)^{1/r}}(A^s\sharp_\alpha B^s)^{1/s} &\leqslant (A^r\sharp_\alpha B^r)^{1/r} \\ &\leqslant K_+\left(h^r,\frac{1}{r}\right)M_h(s)^{1/s}(A^s\sharp_\alpha B^s)^{1/s}. \end{aligned} \tag{5}$$

(*ii*) *If* $r \geqslant 1$, *then*

$$\frac{1}{K_+\left(h^r,\frac{s}{r}\right)^{1/s}M_h(r)^{1/r}}(A^s\sharp_\alpha B^s)^{1/s} \leqslant (A^r\sharp_\alpha B^r)^{1/r} \leqslant M_h(s)^{1/s}(A^s\sharp_\alpha B^s)^{1/s}. \tag{6}$$

Proof. Since $0 < \frac{r}{s} \leqslant 1$, by (i) of Theorem 4.1 we have

$$\frac{1}{K_+(h^{\frac{r}{s}}, \frac{s}{r})^{r/s} M_h(\frac{r}{s})}(A\sharp_\alpha B)^{r/s} \leqslant A^{\frac{r}{s}}\sharp_\alpha B^{\frac{r}{s}} \leqslant M_h(1)^{r/s}(A\sharp_\alpha B)^{r/s}.$$

Replacing A and B by A^s and B^s respectively, we have

$$\frac{1}{K_+\left(h^r, \frac{s}{r}\right)^{r/s} M_h(r)}(A^s\sharp_\alpha B^s)^{r/s} \leqslant A^r\sharp_\alpha B^r \leqslant M_h(s)^{r/s}(A^s\sharp_\alpha B^s)^{r/s}. \tag{7}$$

If $r \geqslant 1$, then by raising all terms of (7) to the power $\frac{1}{r} \leqslant 1$, we have (6) from Löwner-Heinz Theorem.

If $0 < r \leqslant 1$, then by raising all terms of (7) to the power $\frac{1}{r} \geqslant 1$, we have (5) from Theorem C. □

5. Comparison between arithmetic and chaotically geometric means

Nakamoto and one of the authors showed some properties of the chaotically geometric mean $A\diamondsuit_\alpha B$. The operator function $F(r)$ is monotone increasing under the chaotic order and $F(r)$ converges to $A\diamondsuit_\alpha B$ as $r \to +0$ in the strong operator topology. Moreover, by using it, they showed that both $(A^r \,!_\alpha\, B^r)^{1/r}$ and $(A^r\sharp_\alpha B^r)^{1/r}$ converge to $A\diamondsuit_\alpha B$ as $r \to +0$ in the strong operator topology.

In this section, we shall consider some order relations between the chaotically geometric mean and the geometric one, which are applications of the results in the previous sections 3 and 4. First, by Corollary 3.5 we obtain an order relation between the chaotically geometric mean and arithmetic one. The obtained inequality $A\nabla_\alpha B \leqslant M_h(1)A\diamondsuit_\alpha B$ is understood as a variant of the reverse Young inequality $A\nabla_\alpha B \leqslant M_h(1)A\sharp_\alpha B$ due to Tominaga [14].

THEOREM 5.1. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A$, $B \leqslant M$ for some scalars $m < M$. Then for each $0 \leqslant \alpha \leqslant 1$*

$$\frac{1}{M_h(1)}A\nabla_\alpha B \leqslant A\diamondsuit_\alpha B \leqslant M_h(1)A\nabla_\alpha B$$

where $h = \frac{M}{m}$.

Proof. Replacing A and B by $\log A$ and $\log B$ respectively in Corollary 3.5, we have, for $h = e^{\log M - \log m} = \frac{M}{m}$,

$$\frac{1}{M_h(1)}e^{(1-\alpha)\log A + \alpha \log B} \leqslant (1-\alpha)e^{\log A} + \alpha e^{\log B} \leqslant M_h(1)e^{(1-\alpha)\log A + \alpha\log B},$$

which imply the desired inequalities. □

The operator function $F(s)$ is not generally monotone increasing on $(0, 1]$ under the usual order. However, we have the following result.

THEOREM 5.2. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A,\ B \leqslant M$ for some scalars $m < M$. Then for each $0 \leqslant \alpha \leqslant 1$*

$$\frac{1}{M_h(1)M_h(s)^{1/s}}F(s) \leqslant A\diamondsuit_\alpha B \leqslant M_h(1)F(s) \quad \textit{for} \quad s > 0.$$

Proof. By (i) of Corollary 3.4, if $0 < r \leqslant s$ and $0 < r < 1$, then we have

$$K_+\left(h^r, \frac{1}{r}\right)^{-1} K_+(h^r, \frac{s}{r})^{-1/s}F(s) \leqslant F(r) \leqslant K_+\left(h^r, \frac{1}{r}\right)F(s). \tag{1}$$

Since $\lim_{r\to+0} K_+\left(h^r, \frac{s}{r}\right) = M_h(s)$ which is shown in [15, Proposition 14], we have the desired result as $r \to +0$ in (1). $\square$

Next, we see an order relation between the chaotically geometric mean and the geometric one.

THEOREM 5.3. *Let A and B be positive operators on a Hilbert space H satisfying $0 < m \leqslant A,\ B \leqslant M$ for some scalars $m < M$. Then for each $0 \leqslant \alpha \leqslant 1$*

$$\frac{1}{M_h(1)M_h(s)^{1/s}}(A^s\sharp_\alpha B^s)^{1/s} \leqslant A\diamondsuit_\alpha B \leqslant M_h(1)M_h(s)^{1/s}(A^s\sharp_\alpha B^s)^{1/s} \quad \textit{for} \quad s > 0,$$

where $h = \frac{M}{m}$.

Proof. By Corollary 4.2, if $0 < r \leqslant s$ and $0 < r \leqslant 1$, then we have

$$\begin{aligned} \frac{1}{K_+\left(h^r, \frac{1}{r}\right)K_+\left(h^r, \frac{s}{r}\right)^{1/s}M_h(r)^{1/r}}(A^s\sharp_\alpha B^s)^{1/s} &\leqslant (A^r\sharp_\alpha B^r)^{1/r} \qquad (2)\\ &\leqslant K_+\left(h^r, \frac{1}{r}\right)M_h(s)^{1/s}(A^s\sharp_\alpha B^s)^{1/s}. \end{aligned}$$

Since $\lim_{r\to+0} K_+\left(h^r, \frac{s}{r}\right) = M_h(s)$ and $\lim_{r\to+0} M_h(r)^{1/r} = 1$, we have the desired result as $r \to +0$ in (2). $\square$

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