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ON KY FAN'S INEQUALITY AND ITS ADDITIVE ANALOGUES

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Abstract. In this paper we give a discrete proof of Ky Fan's inequality. Also, we extend the additive analogues of Ky Fan's inequality to a very general case and give some applications.

1. Introduction

Throughout this paper, given n arbitrary nonnegative real numbers $x_1, \dots, x_n$, we denote by A_n and G_n, the unweighted arithmetic and geometric means of $x_1, \dots, x_n$ respectively, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{\frac{1}{n}}, \tag{1}$$

and moreover, if $x_i \in [0, \frac{1}{2}]$, we denote by A'_n and G'_n, the unweighted arithmetic and geometric means of $1 - x_1, \dots, 1 - x_n$ respectively, i.e.

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - x_i), \qquad G'_n = \prod_{i=1}^{n}(1 - x_i)^{\frac{1}{n}}. \tag{2}$$

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [5, p. 5]:

If $x_i \in (0, \frac{1}{2}]$, *then*

$$\frac{A'_n}{G'_n} \leqslant \frac{A_n}{G_n}. \tag{3}$$

Equalities holds in (3) if and only if $x_1 = \dots = x_n$.

For some new proofs and refinements of (3) see e.g. [7–10].

Inequality (3) has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [3] and the references therein.

In 1988, H. Alzer [4] proved an additive analogue of (3) as follows:

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If $x_i \in [0, \frac{1}{2}]$ $(i = 1, 2, \ldots, n)$, *then*

$$A'_n - G'_n \leqslant A_n - G_n, \tag{4}$$

which equality holding if and only if $x_1 = x_2 = \ldots = x_n$.

We remark that, just as for (3), inequality (4) was originally established for unweighted means. Proofs of (3) and (4) for weighted means can be found in [3].

Also, in 1995, J. E. Pečarić and H. Alzer [9], using the Dinghas Identity [6], proved that:

If $x_i \in [0, \frac{1}{2}]$ $(i = 1, 2, \ldots, n)$, *then*

$$A_n^n - G_n^n \leqslant {A'_n}^n - {G'_n}^n, \tag{5}$$

in which if $n = 1, 2$ *equality always holds in (5), and if* $n \geqslant 3$, *equality is valid if and only if* $x_1 = \ldots = x_n$.

The aim of this paper is to give a discrete proof of (3), and by studying the behavior of the function F defined in (11), generalize the additive analogues (4) and (5) for other powers except than 1 and n, and give some applications of it.

2. A discrete proof of Ky Fan's inequality

In this section, we suppose that x_1, x_2, ..., x_n are nonnegative real numbers. First, we prove the following useful lemma, and then using it, we give a discrete proof of Ky Fan's inequality (3).

LEMMA 2.1. *If* $a,\ b \geqslant 0$, *then*

$$\left(\frac{ka+b}{k+1}\right)^{k+1} - a^k b = \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k} (k-m+1) \left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m}.$$

Proof.

$$\begin{aligned}
\left(\frac{ka+b}{k+1}\right)^{k+1} - a^k b &= \left(\frac{ka+b}{k+1}\right)^{k+1} - a^{k+1} + a^{k+1} - a^k b \\
&= \sum_{l=0}^{k} \left(\frac{b-a}{k+1}\right) \left(\frac{ka+b}{k+1}\right)^{l} a^{k-l} - a^k(b-a) \\
&= \frac{b-a}{k+1} \left[\sum_{l=0}^{k} \left(\frac{ka+b}{k+1}\right)^{l} a^{k-l} - \sum_{l=0}^{k} a^l a^{k-l}\right] \\
&= \frac{b-a}{k+1} \sum_{l=1}^{k} \left[\left(\frac{ka+b}{k+1}\right)^{l} - a^l\right] a^{k-l} \\
&= \left(\frac{b-a}{k+1}\right)^2 \sum_{l=1}^{k} \sum_{m=1}^{l} \left(\frac{ka+b}{k+1}\right)^{m-1} a^{l-m} a^{k-l}
\end{aligned}$$

$$= \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k} \sum_{l=m}^{k} \left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m}$$

$$= \left(\frac{b-a}{k+1}\right)^2 \sum_{m=1}^{k} (k-m+1) \left(\frac{ka+b}{k+1}\right)^{m-1} a^{k-m}.$$

COROLLARY 2.2. (Dinghas Identity)

$$A_n^n - G_n^n = \sum_{k=1}^{n-1} \sum_{m=1}^{k} \left(\frac{x_{k+1} - A_k}{k+1}\right)^2 (k-m+1) A_{k+1}^{m-1} A_k^{k-m} x_{k+2} \dots x_n. \tag{6}$$

This is, of course, a constructive proof of Dinghas identity which is different from one given in [6], based only on the mathematical induction.

Proof. We have

$$A_{k+1}^{k+1} - A_k^k x_{k+1} = \left(\frac{kA_k + x_{k+1}}{k+1}\right)^{k+1} - A_k^k x_{k+1},$$

and so, by Lemma 2.1,

$$A_n^n - G_n^n = \sum_{k=1}^{n-1} (A_{k+1}^{k+1} - A_k^k x_{k+1}) x_{k+2} \dots x_n$$
$$= \sum_{k=1}^{n-1} \sum_{m=1}^{k} \left(\frac{x_{k+1} - A_k}{k+1}\right)^2 (k-m+1) A_{k+1}^{m-1} A_k^{k-m} x_{k+2} \dots x_n.$$

THEOREM 2.3. (Ky Fan's inequality) *If* $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n)$*, then*

$$\frac{A_n'}{G_n'} \leqslant \frac{A_n}{G_n}.$$

Equality holds if and only if $x_1 = \dots = x_n$.

Proof. There is nothing to prove if $n = 1$, and so we suppose that $n \geqslant 2$. Without losing generality, suppose that $x_1 \leqslant x_2 \leqslant \dots \leqslant x_n$. We show equivalently

$$\frac{A_n^n}{A_n'^n} \geqslant \frac{G_n^n}{G_n'^n}. \tag{7}$$

We have

$$\frac{A_n^n}{A_n'^n} - \frac{G_n^n}{G_n'^n} = \sum_{k=1}^{n-1} \left[\left(\frac{A_{k+1}}{A_{k+1}'}\right)^{k+1} - \left(\frac{A_k}{A_k'}\right)^k \frac{x_{k+1}}{1-x_{k+1}} \right] \frac{x_{k+2}}{1-x_{k+2}} \cdots \frac{x_n}{1-x_n}$$
$$= \sum_{k=1}^{n-1} \frac{A_k^k x_{k+1}}{A_{k+1}'^{k+1}} \left[\frac{A_{k+1}^{k+1}}{A_k^k x_{k+1}} - \frac{A_{k+1}'^{k+1}}{A_k'^k (1-x_{k+1})} \right] \frac{x_{k+2}}{1-x_{k+2}} \cdots \frac{x_n}{1-x_n}.$$

So, it is sufficient to show that the expressions in the brackets are nonnegative for $k = 1, \ldots, n-1$. But

$$\frac{A_{k+1}^{k+1}}{A_k^k x_{k+1}} - \frac{{A'_{k+1}}^{k+1}}{{A'_k}^k (1-x_{k+1})} = \frac{A_{k+1}^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} - \frac{{A'_{k+1}}^{k+1} - {A'_k}^k (1-x_{k+1})}{{A'_k}^k (1-x_{k+1})}.$$

Using Lemma 2.1 and the binomial expansion, we have

$$\begin{aligned} \frac{A_{k+1}^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} &= \frac{\left(\frac{kA_k + x_{k+1}}{k+1}\right)^{k+1} - A_k^k x_{k+1}}{A_k^k x_{k+1}} \qquad (8) \\ &= \frac{\left(\frac{x_{k+1} - A_k}{k+1}\right)^2 \sum_{m=1}^{k} (k-m+1) \left(\frac{kA_k + x_{k+1}}{k+1}\right)^{m-1} A_k^{k-m}}{A_k^k x_{k+1}} \\ &= (x_{k+1} - A_k)^2 \sum_{m=1}^{k} \sum_{p=0}^{m-1} \frac{k-m+1}{(k+1)^{m+1}} \binom{m-1}{p} k^p \left(\frac{x_{k+1}}{A_k}\right)^{m-p-2} \frac{1}{A_k^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{{A'}_{k+1}^{k+1} - {A'}_k^k (1-x_{k+1})}{{A'}_k^k (1-x_{k+1})} \qquad (9) \\ &\quad = (x_{k+1} - A_k)^2 \sum_{m=1}^{k} \sum_{p=0}^{m-1} \frac{k-m+1}{(k+1)^{m+1}} \binom{m-1}{p} k^p \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2} \frac{1}{{A'}_k^2}. \end{aligned}$$

So, it is sufficient to show that

$$\left(\frac{x_{k+1}}{A_k}\right)^{m-p-2} \frac{1}{A_k^2} \geqslant \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2} \frac{1}{{A'}_k^2} \qquad (10)$$

$$(k = 1, \ldots, n-1;\ m = 1, \ldots, k;\ p = 0, \ldots, m-1).$$

If $p = m-1$, (10) is equivalent to $1 - A_k - x_{k+1} \geqslant 0$ which is valid since A_k, $x_{k+1} \leqslant \frac{1}{2}$.

If $0 \leqslant p \leqslant m-2$, because of $\frac{x_{k+1}}{A_k} \geqslant \frac{1-x_{k+1}}{1-A_k}$, we have $\left(\frac{x_{k+1}}{A_k}\right)^{m-p-2} \geqslant \left(\frac{1-x_{k+1}}{A'_k}\right)^{m-p-2}$, which together with $\frac{1}{A_k} \geqslant \frac{1}{A'_k}$, we get (10), and so, (7) is obtained.

Clearly, equality holds in Ky Fan's inequality if $x_1 = \cdots = x_n$.

Conversely, if $x_1, \ldots, x_n$ are not all equal, there exists a k with $1 \leqslant k \leqslant n-1$, such that $A_k \neq x_{k+1}$. Therefore, taking $m = k$ and $p = k-1$, we get an strict inequality in (10) and so, considering (8) and (9), the strict inequality holds in (7), and the proof is completed.

3. Extension of additive analogues

In this section, we extend the inequalities (4) and (5) for arbitrary powers. Throughout this section we assume that $n \geqslant 2$ is an integer and $x_1, x_2, \ldots, x_n$ are n given real numbers in $(0, \frac{1}{2}]$ not all equal. Consider the continuous real-valued function F defined by

$$F(x) = (A'^x_n - G'^x_n) - (A^x_n - G^x_n) \qquad (-\infty < x < +\infty). \tag{11}$$

Clearly $F(0) = 0$. Also, it is proved in [1] that $F(-1) > 0$. By (4) and (5), $F(1) < 0$ and $F(n) \geqslant 0$. So, there exists an $\alpha \in (1, n]$ such that $F(\alpha) = 0$. In Theorem 3.2, we study the main behaviors of the function F and show that α is the unique nonzero root of F.

First, we prove the following lemma which is used in the proof of (iv) of Theorem 3.2.

LEMMA 3.1. *If $a > b \geqslant c > d > 0$, then*

$$f(x) = \frac{a^x - b^x}{c^x - d^x} \qquad (-\infty < x < +\infty)$$

is an strictly increasing function on the real line. Moreover, $f(x) \to 0$ $(x \to -\infty)$ and $f(x) \to +\infty$ $(x \to +\infty)$.

Proof. Let x and y with $x < y < 0$ or $0 < x < y$ be two arbitrary real numbers. We have $f(x) < f(y)$ if and only if

$$\frac{c^y - d^y}{c^x - d^x} < \frac{a^y - b^y}{a^x - b^x}. \tag{12}$$

But, by the Cauchy's mean value theorem, there are ξ and η with $d < \xi < c$ and $b < \eta < a$, such that

$$\frac{c^y - d^y}{c^x - d^x} = \left(\frac{y}{x}\right) \xi^{y-x},$$

and

$$\frac{a^y - b^y}{a^x - b^x} = \left(\frac{y}{x}\right) \eta^{y-x}.$$

Now, since $0 < \xi < \eta$, $\frac{y}{x} > 0$ and $y - x > 0$, we obtain (12), and so, f is strictly increasing on the real line.

To other assertions follows from

$$\frac{a^x - b^x}{c^x - d^x} = \left(\frac{b}{d}\right)^x \frac{\left(\frac{a}{b}\right)^x - 1}{\left(\frac{c}{d}\right)^x - 1} \to 0 \qquad (x \to -\infty),$$

and

$$\frac{a^x - b^x}{c^x - d^x} = \left(\frac{a}{c}\right)^x \frac{1 - \left(\frac{b}{a}\right)^x}{1 - \left(\frac{d}{c}\right)^x} \to +\infty \qquad (x \to +\infty).$$

THEOREM 3.2. *With the above notations, we have*

(i) $F(x) > 0$ *for all* $x < 0$, *and* $F(x) < 0$ *for all* $0 < x \leqslant 1$.

(ii) F *is strictly convex and strictly decreasing on* $(-\infty, 0]$, *and we have* $\lim_{x\to-\infty} F(x) = +\infty$.

(iii) $F(x) > 0$ *for all* $x > n$, *and* $\lim_{x\to+\infty} F(x) = 0$.

(iv) F *has exactly two distinct roots; one zero and the other* $\alpha \in (1, n]$.

Proof. (i) Given $x \in \mathbb{R}$, by the mean value theorem, we have

$$F(x) = (A'_n - G'_n)x\xi'^{x-1} - (A_n - G_n)x\xi^{x-1},$$

where $G'_n < \xi' < A'_n$ and $G_n < \xi < A_n$. Now, let $x \leqslant 1$. Then, since $0 < \xi < A_n < \frac{1}{2} < G'_n < \xi'$, we have $\xi'^{x-1} \leqslant \xi^{x-1}$. So, by (4), $F(x) < 0$ for all $0 < x \leqslant 1$, and $F(x) > 0$ for all $x < 0$.

(ii) We have

$$\begin{aligned} F''(x) &= [A'^x_n(\ln A'_n)^2 - G'^x_n(\ln G'_n)^2] - [A^x_n(\ln A_n)^2 - G^x_n(\ln G_n)^2] \\ &= A'^x_n\left(\ln\frac{A'_n}{G'_n}\right)\ln(A'_nG'_n) + (A'^x_n - G'^x_n)(\ln G'_n)^2 \\ &\quad - A^x_n\left(\ln\frac{A_n}{G_n}\right)\ln(A_nG_n) - (A^x_n - G^x_n)(\ln G_n)^2. \end{aligned}$$

Now, since for $x < 0$,

$$\begin{aligned} &0 < A'^x_n < A^x_n, \\ &0 < -(A'^x_n - G'^x_n) < -(A^x_n - G^x_n), \end{aligned}$$

and by (3) and $0 < G_n < A_n < G'_n < A'_n < 1$,

$$\begin{aligned} &0 < \ln\frac{A'_n}{G'_n} < \ln\frac{A_n}{G_n}, \\ &0 < -\ln(A'_nG'_n) < -\ln(A_nG_n), \\ &0 < (\ln G'_n)^2 < (\ln G_n)^2, \end{aligned}$$

we get $F''(x) > 0 \ \ (x < 0)$, and so F is strictly convex on $(-\infty, 0]$.

Since F' is strictly increasing on $(-\infty, 0]$, by (3), we have

$$F'(x) < F'(0) = \ln\frac{A'_n}{G'_n} - \ln\frac{A_n}{G_n} < 0 \qquad (x < 0),$$

and so F is strictly decreasing on $(-\infty, 0]$.

Let $L = \lim_{x\to-\infty} F(x)$. We have $L > 0$. Since $F(0) = 0$ and F is convex on $(-\infty, 0]$,

$$F\left(\frac{x}{2}\right) \leqslant \frac{1}{2}F(x) + \frac{1}{2}F(0) = \frac{1}{2}F(x) \qquad (x < 0).$$

Now, if $x \to -\infty$, we obtain $L \leqslant \frac{1}{2}L$, which implies that $L = +\infty$.

(iii) By the mean value theorem, we have

$$F(x) = \left[(A_n'^{\,n})^{\frac{x}{n}} - (G_n'^{\,n})^{\frac{x}{n}}\right] - \left[(A_n^n)^{\frac{x}{n}} - (G_n^n)^{\frac{x}{n}}\right]$$
$$= (A_n'^{\,n} - G_n'^{\,n})\frac{x}{n}\eta'^{\frac{x}{n}-1} - (A_n^n - G_n^n)\frac{x}{n}\eta^{\frac{x}{n}-1},$$

where $G_n'^{\,n} < \eta' < A_n'^{\,n}$ and $G_n^n < \eta < A_n^n$. Now, if $x > n$, then $\eta'^{\frac{x}{n}-1} > \eta^{\frac{x}{n}-1}$, which by (5), we get $F(x) > 0$.

Since, A_n, A_n', G_n, and G_n' belong to $(0,1)$, it follows that $F(x) \to 0$ as $x \to +\infty$.

(iv) For $x \neq 0$, we have $F(x) = 0$ iff $f(x) = \frac{A_n'^x - G_n'^x}{A_n^x - G_n^x} = 1$. Now, since $A_n' > G_n' > A_n > G_n > 0$, it follows from Lemma 3.1 that f is strictly increasing on the real line, and so, there is a unique α such that $f(\alpha) = 1$. Clearly, we have $\alpha \in (1, n]$ and the proof is completed.

REMARK 3.3. (i) It must be noted that the inequality (4) is stronger than (3), see [2, 3]. So, if it is possible, it is better to use (3) rather than (4). For example, for the proof of $F(x) > 0$ $(x < 0)$ in (i) of Theorem 3.2, we may use (3) instead of (4) in the following manner:

$$A_n'^x - G_n'^x = -A_n'^x\left[\left(\frac{A_n'}{G_n'}\right)^{-x} - 1\right] = -A_n'^x\sum_{k=1}^{\infty}\frac{\left(-x\ln\frac{A_n'}{G_n'}\right)^k}{k!}$$
$$> -A_n^x\sum_{k=1}^{\infty}\frac{\left(-x\ln\frac{A_n}{G_n}\right)^k}{k!} = A_n^x - G_n^x \qquad (x<0).$$

(ii) Since F has two distinct roots and $\lim_{x\to+\infty} F(x) = 0$, F' has at least two distinct roots. It will be interesting to show that whether F' has exactly two distinct roots?

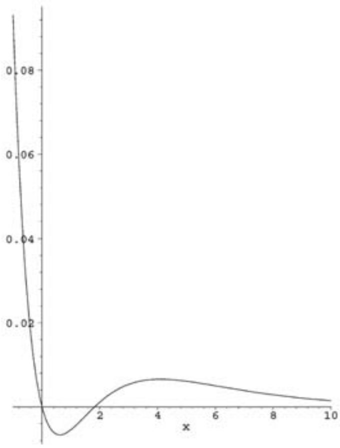

Figure 1: $y = F(x)$

Figure 1 shows the behavior of the function F drawn for the special case $n = 3$; $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{3}$ and $x_3 = \frac{1}{4}$:

4. Applications

APPLICATION 1. If $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n)$, then for any $x > 0$,

$$\left(\frac{A'_n}{G'_n}\right)^{\frac{A'^x_n(A^x_n - G^x_n)}{A^x_n(A'^x_n - G'^x_n)}} \leqslant \frac{A_n}{G_n} \leqslant \left(\frac{A'_n}{G'_n}\right)^{\frac{G'^x_n(A^x_n - G^x_n)}{G^x_n(A'^x_n - G'^x_n)}}. \tag{13}$$

The inequalities in (13) become sharper as x decreases and when $x \to 0+$, equality holds in each of them.

Also, we have

$$\left(\frac{A'_n}{G'_n}\right)^{\frac{A'_n}{A_n}} \leqslant \frac{A_n}{G_n} \leqslant \left(\frac{A'_n}{G'_n}\right)^{\left(\frac{G'_n}{G_n}\right)^n}. \tag{14}$$

Equality holds in each inequality if and only if $x_1 = \cdots = x_n$.

Since the exponents are greather than or equal to one (here, in the case of $x_1 = \cdots = x_n$, the expression $\frac{0}{0}$ is understood as one), the left-hand inequalities in (13) and (14) sharpen (3), whereas the right-hand ones give some inverses of it.

The proof of (13) in the nontrivial case, follows immediately from the following lemma, taking $a = \frac{A_n}{G_n}$ and $b = \frac{A'_n}{G'_n}$.

Finally, (14) follows from (13) by taking $x = 1$ in the left and $x = n$ in the right, and considering (4) and (5).

LEMMA 4.1. *If $a > b > 1$, then*

$$b^{\frac{a^{-x}-1}{b^{-x}-1}} < a < b^{\frac{a^x-1}{b^x-1}} \qquad (x > 0). \tag{15}$$

The inequalities become sharper as x decreases and when $x \to 0+$, equality holds in each of them.

Proof. We can prove (15) by the usual differentiation method, but we prefer to establish it by integration only.

Fix a $x > 0$. Integrating both sides of the trivial inequality

$$b^{-xt} > a^{-xt} \qquad (t > 0)$$

with respect to t from zero to one, we get

$$\frac{b^{-x} - 1}{-x \ln b} > \frac{a^{-x} - 1}{-x \ln a},$$

which gives the first inequality in (15).

Similarly, the second inequality in (15) is achieved by integrating both sides of the trivial inequality

$$b^{xt} < a^{xt} \qquad (t > 0)$$

with respect to t from zero to one.

Since, $a > b > 1$, by Lemma 3.1, the functions

$$\frac{a^{-x}-1}{b^{-x}-1} = 1 + \frac{\left(\frac{1}{b}\right)^x - \left(\frac{1}{a}\right)^x}{1-\left(\frac{1}{b}\right)^x},$$

and

$$\frac{a^x-1}{b^x-1} = 1 + \frac{a^x-b^x}{b^x-1},$$

are strictly decreasing and strictly increasing respectively. Therefore, the inequalities in (15) become sharper as x decreases; the best ones, actually equality, are obtained when $x \to 0+$, and the worst ones, actually $b < a < +\infty$, are obtained when $x \to +\infty$.

APPLICATION 2. If $x_i \in (0, \frac{1}{2}]$ $(i = 1, \ldots, n)$, then

$$\frac{A'^x_n}{(-\ln A'_n)^k} - \frac{G'^x_n}{(-\ln G'_n)^k} \geqslant \frac{A^x_n}{(-\ln A_n)^k} - \frac{G^x_n}{(-\ln G_n)^k}, \tag{16}$$

$$(x \geqslant n;\ k = 0, 1, \ldots).$$

In particular, when $x = k = n$,

$$\left(\frac{A'_n}{-\ln A'_n}\right)^n - \left(\frac{G'_n}{-\ln G'_n}\right)^n \geqslant \left(\frac{A_n}{-\ln A_n}\right)^n - \left(\frac{G_n}{-\ln G_n}\right)^n. \tag{17}$$

Except than the trivial case $k = 0$ and $x = n = 2$, equality holds if and only if $x_1 = \cdots = x_n$.

Taking $x = n$ and $k = 0$, it is clear that the inequality (16) is an extension of (5).

Proof. Clearly equality holds if $x_1 = \ldots = x_n$. Suppose that x_i $(i = 1, \ldots, n)$ are not all equal. By (iii) of Theorem 3.2, we have

$$A'^x_n - G'^x_n > A^x_n - G^x_n \qquad (x > n).$$

Integrating both sides of this inequality from x to $+\infty$, we get

$$\frac{A'^x_n}{-\ln A'_n} - \frac{G'^x_n}{-\ln G'_n} > \frac{A^x_n}{-\ln A_n} - \frac{G^x_n}{-\ln G_n} \qquad (x \geqslant n).$$

Now, (16) follows by induction on k.

We conclude the paper by proposing the following open problem:

OPEN PROBLEM. As you know, the arithmetic and geometric means are members of the family of power means, which is defined (in the unweighted case) by

$$M_r = M_r(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^n x_i^r\right)^{\frac{1}{r}} \qquad (r \neq 0),$$

$$M_0 = M_0(x_1, \dots, x_n) = \lim_{r \to 0} M_r(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i^{\frac{1}{n}}.$$

This leads to the following question: Is it possible to extend the given inequalities to power means? In particular, determine all real parameters r and s such that

$$\frac{(M'_r)^x}{(-\ln M'_r)^k} - \frac{(M'_s)^x}{(-\ln M'_s)^k} \geqslant \frac{(M_r)^x}{(-\ln M_r)^k} - \frac{(M_s)^x}{(-\ln M_s)^k} \tag{18}$$

is valid for $x_i \in (0, \frac{1}{2}]$ $(i = 1, \dots, n)$, $x \geqslant n$ and $k = 0, 1, 2, \dots$, where $M'_r(x_1, \dots, x_n) = M_r(1 - x_1, \dots, 1 - x_n)$.

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