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GENERALIZATION OF SOME THEOREMS ON CLASSES OF NUMERICAL SEQUENCES

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(*communicated by J. Pečarić*)

Abstract. The theorem proved here is a generalization of some earlier results due to L. Leindler and to the present author regarding embedding relations among classes of Fourier coefficients.

1. Introduction

Several authors have studied the problems of L^1 convergence of Fourier series ([1]–[5], [14]–[21]). In connection with this topic many classes of coefficients have been defined. For example Telyakovskiĭ [18] introduced the following very applicable definition of a class of sequence $\{a_n\}$ and denoted by S.

A null-sequence $\{a_n\}$ belongs to the class S if there exists a monotonically decreasing sequences $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad |\Delta a_n| \leqslant A_n \quad \text{for all} \quad n.$$

Some further classes are listed as follows. Here and later $\mathbf{a} := \{a_n\}$ denotes always a null-sequence $(a_n \to 0)$ and furthermore $p \geqslant 1$.

1. A sequence $\mathbf{a}$ belongs to the class F_p if

$$\sum_{n=1}^{\infty} n^{-1/p} \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} < \infty \tag{1.1}$$

(Fomin [2]).

2. $\mathbf{a} \in F_p^*$ if

$$\sum_{m=0}^{\infty} 2^{m\left(1-\frac{1}{p}\right)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} < \infty \tag{1.2}$$

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(Fomin [2], Leindler [9]).

3. $\mathbf{a} \in S_p$ if there exists a monotonically decreasing sequence $\mathbf{A} := \{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1) \tag{1.3}$$

(Č. V. Stanojevič and V. B. Stanojevič [17]).

4. $\mathbf{a} \in S_p(\delta)$ if there exists a δ-quasi-monotone sequence $\mathbf{A}$ (i.e. $A_n > 0$ and $\Delta A_n \geqslant -\delta_n$ and $\delta_n > 0$) satisfying $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} n\,\delta_n < \infty$ and (1.3) (Tomovski [19]).

5. $\mathbf{a} \in S_p(A)$ if there exists a null-sequence $\{A_n\}$ satisfying

$$\sum_{n=1}^{\infty} n|\Delta A_n| < \infty \tag{1.4}$$

and (1.3) (Leindler [9]).

Many authors investigated the embedding relations among the classes above mentioned. See for example [2], [4], [8], [9], [10], [12], [13], [16], [20], [21]. The strongest result is due to Leindler [8], [9]. Namely he proved

THEOREM A. [9] *If* $p > 1$ *then*

$$F_p \equiv S_p \equiv F_p^* \equiv S_p(\delta) \equiv S_p(A). \tag{1.5}$$

Later in [10] L. Leindler generalized this result using a certain positive monotonic sequence ρ_n instead of $n^{-1/p}$ (see for example (1.1)). Namely he considered the classes $F_p(\rho), F_p^*(\rho), S_p(\rho), S_p(\delta,\rho)$ and $S_p(A,\rho)$ obtained from $F_p, F_p^*, S_p, S_p(\delta)$ and $S_p(A)$ by replacing (1.1) by

$$\sum_{n=1}^{\infty} \rho_n \left(\sum_{k=n}^{\infty} |\Delta a_k|^p\right)^{1/p} < \infty, \tag{1.6}$$

(1.2) by

$$\sum_{m=0}^{\infty} 2^m \rho_{2^m} \left\{\sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p\right\}^{1/p} < \infty, \tag{1.7}$$

and (1.3) by

$$\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(\rho_n^{-p}), \tag{1.8}$$

respectively.

Before formulating Leindler's result concerning the embedding relations of classes defined above we recall some definitions.

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is quasi β-power-monotone increasing (decreasing) if there exist a natural number $N := N(\beta,\gamma)$ and a constant $K := K(\beta,\gamma) \geqslant 1$ such that

$$K n^{\beta} \gamma_n \geqslant m^{\beta} \gamma_m \quad (n^{\beta} \gamma_n \leqslant K m^{\beta} \gamma_m) \tag{1.9}$$

holds for any $n \geqslant m \geqslant N$.

[In (1.9) and later in the sequel K and K_i denote positive constants, not necessarily the same on any two occurrences].

Now we formulate the result of Leindler.

THEOREM B. [10] *Assume that $p \geqslant 1$ and a given positive sequence $\rho := \{\rho_n\}$, for a certain positive β is quasi β-power-monotone decreasing and simultaneously quasi $(1-\beta)$-power monotone increasing. Then the following embedding relations*

$$F_p(\rho) \subseteq F_p^*(\rho) \subseteq S_p(\rho) \subseteq S_p(\delta,p) \subseteq S_p(A,p) \subseteq F_p(\rho) \tag{1.10}$$

hold, i.e. these classes are identical.

Very recently we generalized the result of Leindler being in Theorem A by using functions more general than the power functions. Namely, following M. Mateljevič and M. Pavlovič [11], we redefined and defined the following classes of functions.

$\Delta(q,p)$ $(q \geqslant p > 0)$ denotes the family of the nonnegative real functions $\varphi(x)$ defined on $[0,\infty)$ with the following properties: $\varphi(0) = 0$, and there exist $q \geqslant p > 0$ such that $\frac{\varphi(t)}{t^q}$ is nonincreasing and $\frac{\varphi(t)}{t^p}$ is nondecreasing on $(0,\infty)$. Δ will denote the set of the functions $\varphi(x)$ belonging to $\Delta(q,p)$ for some $q \geqslant p > 0$. We need some subclasses of Δ. $\Delta^{(1)}$ and $\Delta^{(2)}$ denote the families of functions $\varphi(x)$ belonging to $\Delta(q,p)$ for some $q \geqslant p \geqslant 1$ and $q \geqslant p > 1$, respectively. And $\Delta^{(3)}$ is the collection of functions $\varphi(x)$ belonging to $\Delta(q,1)$ for some $q > 1$ such that for all positive A there exists $1 < p(A) = p$ satisfying that $\frac{\varphi(x)}{x^p}$ is nondecreasing on $(0,A)$. Using these classes of functions we introduced the classes $F_\varphi, F_\varphi^*, S_\varphi, S_\varphi(\delta), S_\varphi(A)$ obtained from $F_p, F_p^*, S_p, S_p(\delta), S_p(A)$ by replacing (1.1) by

$$\sum_{n=1}^{\infty} \overline{\varphi}\left(\frac{\sum\limits_{k=n}^{\infty} \varphi(|\Delta a_k|)}{n}\right) < \infty, \tag{1.11}$$

where $\overline{\varphi}$ denotes the inverse of φ; (1.2) by

$$\sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{n=2^m+1}^{2^{m+1}} \varphi(|\Delta a_n|)}{2^m}\right) < \infty, \tag{1.12}$$

and (1.3) by

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(1), \tag{1.13}$$

respectively.

The embedding relations of the above mentioned classes are formulated in the next statement.

THEOREM C. [13] *If* $\varphi \in \Delta^{(3)}$ *then the following relations*

$$F_\varphi \subseteq S_\varphi \subseteq S_\varphi(\delta) \subseteq S_\varphi(A) \subseteq F_\varphi^* \subseteq F_\varphi \tag{1.14}$$

hold, i.e. these classes are identical.

The aim of the present paper is to combine the two directions of generalizations stated in Theorem B and C. Before formulating our result we need the definition of new classes of sequences. Let $\eta := \{\eta_n\}$ be a positive monotonically increasing sequence. Then the new classes

$$F_\varphi(\eta),\ F_\varphi^*(\eta),\ S_\varphi(\eta),\ S_\varphi(\delta,\eta),\ S_\varphi(A,\eta)$$

can be defined from

$$F_p,\ F_p^*,\ S_p,\ S_p(\delta),\ S_p(A)$$

by changing (1.1) to

$$\sum_{n=1}^{\infty} \overline{\varphi}\left(\frac{\sum_{k=n}^{\infty} \varphi(|\Delta a_k|)}{\eta_n}\right) < \infty, \tag{1.15}$$

(1.2) to

$$\sum_{m=0}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum_{k=2^m+1}^{2^{m+1}} \varphi(|\Delta a_k|)}{\eta_{2^m}}\right) < \infty, \tag{1.16}$$

and (1.3) to

$$\sum_{k=1}^{\infty} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(\eta_n). \tag{1.17}$$

2. Result

We now proceed to formulate our new result.

THEOREM. *Assume that* $\varphi \in \Delta(q,p)$ $(p \geqslant 1)$ *and a given positive sequence* $\eta := \{\eta_n\}$ *for certain negative* γ *is quasi* γ*-power-monotone increasing and for certain positive* β *is* $p(\beta-1)$*-power-monotone decreasing. Then the following embedding relations*

$$F_\varphi(\eta) \subseteq F_\varphi^*(\eta) \subseteq S_\varphi(\eta) \subseteq S_\varphi(\delta,\eta) \subseteq S_\varphi(A,\eta) \subseteq F_\varphi(\eta) \tag{2.1}$$

hold, i.e. these classes are identical.

It is easy to verify that if $\varphi(x) = x^p$, $p > 1$ and $\eta_n = n$ then Theorem A follows from our Theorem, furthermore if $\eta_n = n$ then Theorem C is a consequence of the Theorem for $\varphi \in \Delta^{(2)}(\subset \Delta^{(3)})$ and finally Theorem B can be obtained from our Theorem if $\varphi(x) = x^p$, $p \geqslant 1$ and $\eta_n = \rho_n^{-p}$ where $\{\rho_n\}$ is satisfying the conditions required in Theorem B.

Furthermore it should be noted that if $\varphi(x) = x(\in \Delta^{(1)})$ then in the Theorem η_n cannot be equal to n, but it follows from our Theorem C that if for example $\varphi(x) = x\log(1+x)$ then $\eta_n = n$ can be taken. It is obvious that this function is belonging to $\Delta^{(3)}$ which is narrower than $\Delta^{(1)}$ but wider than $\Delta^{(2)}$.

3. Lemmas

LEMMA 1. *If* $\varphi \in \Delta(q,p)$ $(q \geqslant p > 0)$ *and* $0 \leqslant \Theta \leqslant 1$, $1 \leqslant \eta$ *then*

$$\Theta^q \varphi(t) \leqslant \varphi(\Theta t) \leqslant \Theta^p \varphi(t) \tag{3.1}$$

and

$$\eta^p \varphi(t) \leqslant \varphi(\eta t) \leqslant \eta^q \varphi(t) \quad \textit{for} \quad t \geqslant 0. \tag{3.2}$$

Result (3.1) is a part of Lemma 1 in [11] and (3.2) is an obvious consequence of (3.1).

LEMMA 2. [13] *If* $\varphi \in \Delta^{(1)}$ *then*

$$\overline{\varphi}\left(\sum_{i=1}^{\infty} a_i\right) \leqslant \sum_{i=1}^{\infty} \overline{\varphi}(a_i), \tag{3.3}$$

where $a_i \geqslant 0$ *for all* i.

LEMMA 3. [21] *Let* $\{c_n\}$ *be a* δ*-quasi-monotone sequence with*

$$\sum_{n=1}^{\infty} n\,\delta_n < \infty. \tag{3.4}$$

If $\sum_{n=1}^{\infty} c_n$ *converges, then* $\sum_{n=1}^{\infty} (n+1)|\Delta c_n| < \infty$.

LEMMA 4. [10] *If* $\{R_n\}$ *is a nonnegative monotonically decreasing sequence such that*

$$\sum_{n=1}^{\infty} R_n < \infty \tag{3.5}$$

then there exists a monotone decreasing sequence $\{A_n\}$ *such that for any* $n \geqslant 1$

$$R_n \leqslant A_n, \tag{3.6}$$

$$A_n \leqslant K A_{2n}, \tag{3.7}$$

and

$$\sum_{k=1}^{\infty} A_k < \infty. \tag{3.8}$$

LEMMA 5. *If* $\{A_n\}$ *is a null-sequence such that* $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$ *and* $D_m := \sum_{n=2^m}^{2^{m+1}} |\Delta A_n|$, *moreover* $C_m := A_{2^m} + D_m$, *then the following relations hold:*

$$\sum_{m=1}^{\infty} 2^m D_m < \infty, \tag{3.9}$$

$$\sum_{m=1}^{\infty} 2^m A_{2^m} < \infty, \tag{3.10}$$

and

$$A_n \leqslant C_m, \quad \textit{if} \quad 2^m < n \leqslant 2^{m+1}. \tag{3.11}$$

This statement can be found in the proof of Theorem 1 of L. Leindler in [9].

Before formulating the following lemmas we need two more definitions due to L. Leindler.

We shall say that a sequence $\mathbf{c} := \{c_m\}$ of positive terms is quasi geometrically increasing if there exist numbers $\mu := \mu(\mathbf{c}), N := N(\mathbf{c})$ and a constant $K := K(\mathbf{c}) \geqslant 1$ such that

$$c_{m+\mu} \geqslant 2c_m \tag{3.12}$$

and

$$c_m \leqslant K\, c_{m+1} \tag{3.13}$$

hold for all $m \geqslant N$.

A sequence $\gamma := \{\gamma_n\}$ will be called bounded by blocks if the inequalities

$$\alpha_1 \Gamma_m^{(k)} \leqslant \gamma_n \leqslant \alpha_2 \Gamma_M^{(k)}, \quad 0 < \alpha_1 < \alpha_2 < \infty$$

hold for any $2^k \leqslant n \leqslant 2^{k+1},\ k = 1, 2, \ldots$ where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

LEMMA 6. [7] *A positive sequence* $\gamma := \{\gamma_n\}$ *bounded by blocks is quasi* β*-power monotone increasing with certain negative exponent* β *if and only if the sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically increasing.*

LEMMA 7. [6] *For any positive sequence* $\gamma := \{\gamma_n\}$ *the inequality*

$$\sum_{n=1}^{m} \gamma_n \leqslant K\, \gamma_m \quad (m = 1, 2, \ldots; K \geqslant 1)$$

holds if and only if the sequence γ *is quasi geometrically increasing.*

LEMMA 8. *If* $\varphi \in \Delta(q,p)$ $(p \geqslant 1)$ *and* $k \geqslant 1$ *then*

$$k^{1/p}\overline{\varphi}(t) \geqslant \overline{\varphi}(kt) \tag{3.14}$$

holds for all $t \in [0,\infty)$.

Proof. Using (3.1) we have that for all $x \in [0,\infty)$

$$\varphi(x) = \varphi(k^{-1/p}k^{1/p}x) \leqslant k^{-1}\varphi(k^{1/p}x). \tag{3.15}$$

From (3.15) it follows that

$$\overline{\varphi}(k\varphi(x)) \leqslant k^{1/p}x. \tag{3.16}$$

Taking $t = \varphi(x)$ we get from (3.16) that (3.14) holds. Thus Lemma 8 is proved.

LEMMA 9. *If* $\varphi \in \Delta(q,p)$ $(p \geqslant 1)$ *and the sequence* $\{\eta_n\}$ *has the same properties as in the Theorem and* $\{b_n\}$ *is an arbitrary sequence of positive real numbers then for any* $n \geqslant 1$

$$\sum_{m=0}^{n} 2^m\overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) \leqslant K2^n\overline{\varphi}\left(\frac{b_n}{\eta_{2^n}}\right), \tag{3.17}$$

where K *is independent of* $\{b_n\}$ *and* n.

Proof. First we show that the sequence $\mathbf{c} := \{c_m\} := \left\{2^m\overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right)\right\}$ is quasi geometrically increasing for any fix b_n such that μ, N and K being in the definition do not depend on $\{b_n\}$.

Since η_n is a $p(\beta-1)$-power-monotone decreasing for some $\beta > 0$, there exist $K_1 \geqslant 1$ and N_1 such that

$$\eta_n n^{p(\beta-1)} \leqslant K_1\eta_m m^{p(\beta-1)} \tag{3.18}$$

holds if $n \geqslant m \geqslant N_1$.

Using (3.18) for arbitrary number μ and for $m \geqslant \log_2 N_1$ we get that

$$\begin{aligned} 2^{m+\mu}\overline{\varphi}\left(\frac{b_n}{\eta_{2^{m+\mu}}}\right) &= 2^{m+\mu}\overline{\varphi}\left(\frac{b_n\,2^{(m+\mu)p(\beta-1)}}{\eta_{2^{m+\mu}}\,2^{(m+\mu)p(\beta-1)}}\right) \\ &\geqslant 2^{m+\mu}\overline{\varphi}\left(\frac{b_n\,2^{(m+\mu)p(\beta-1)}}{K_1\eta_{2^m}2^{m(\beta-1)p}}\right) \\ &= 2^{m+\mu}\overline{\varphi}\left(\frac{b_n\,2^{\mu(\beta-1)p}}{K_1\eta_{2^m}}\right) = I. \end{aligned} \tag{3.19}$$

Since $0 < \beta < 1$ therefore there exists μ such that

$$k := \frac{K_1}{2^{\mu(\beta-1)p}} \geqslant 1. \tag{3.20}$$

Applying (3.14) for this number k with $t = \frac{b_n}{\eta_{2^m}}$ we have that

$$I \geqslant 2^{m+\mu}\left(\frac{2^{\mu(\beta-1)p}}{K_1}\right)^{1/p}\overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) = II. \tag{3.21}$$

Since $\beta > 0$, μ can also be choosen such that

$$\frac{2^{\mu\beta}}{K_1^{1/p}} \geqslant 2 \tag{3.22}$$

be satisfied besides (3.20), and it is obvious that (3.22) is equivalent to

$$\left(\frac{2^{\mu(\beta-1)p}}{K_1}\right)^{1/p} \geqslant 2^{1-\mu} \tag{3.23}$$

from which

$$II \geqslant 2 \cdot 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right). \tag{3.24}$$

follows.

Taking into account (3.19), (3.20), (3.21) and (3.24) we have that (3.12) is fulfilled for μ choosen above, if $m \geqslant N := [\log_2 N_1] + 1$.

Similar calculation gives (3.13) with $K = 2^{-\beta} K_1^{1/p}$ and for the same N as obtained before, that is $\mathbf{c}$ is quasi geometrically increasing, indeed.

It is very important to emphasize that the numbers μ, K and N just obtained are independent of $\{b_n\}$.

The following part of the proof is based on the idea used by Leindler in Lemma 7.

Let n arbitrary and $(i-1)\mu + N < n \leqslant i\mu + N$, furthermore let $d_N \leqslant 1$ such that $\eta_{2^m} \geqslant d_N \eta_{2^N}$ for all $m = 1, 2, \ldots, N-1$.

Then, for any fix n and for any b_n using (3.14), we have that

$$\sum_{m=0}^{n} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) \leqslant \left(\frac{2}{d_N}\right)^{1/p} \sum_{m=N}^{n} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) = J. \tag{3.25}$$

Applying (3.12) and (3.13) with $c_m := 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right)$ and with μ, K, N obtained above, we get that

$$\begin{aligned} J &\leqslant \frac{2}{d_N} \sum_{k=1}^{i-1} \sum_{m=\mu(k-1)+N}^{\mu k+N} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) + \frac{2}{d_N} \sum_{m=(i-1)\mu+N+1}^{n} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) \\ &\leqslant \frac{4}{d_N} \sum_{m=(i-2)\mu+N+1}^{(i-1)\mu+N} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) + \frac{2}{d_N} \sum_{m=(i-1)\mu+N+1}^{n} 2^m \overline{\varphi}\left(\frac{b_n}{\eta_{2^m}}\right) \\ &\leqslant \frac{4}{d_N} \sum_{m=0}^{2\mu} K^m 2^n \overline{\varphi}\left(\frac{b_n}{\eta_{2^n}}\right) = \hat{K} 2^n \overline{\varphi}\left(\frac{b_n}{\eta_{2^n}}\right), \end{aligned} \tag{3.26}$$

where $\hat{K}$ is depending only on d_N, K, μ, but it is independent of $\{b_n\}$.

Taking (3.25) and (3.26) we get (3.17), thus Lemma 9 is proved.

4. Proofs

First we prove the relation $F_\varphi(\eta) \subseteq F^*_\varphi(\eta)$. Let $\mathbf{a} \in F_\varphi(\eta)$. Since η_n is monotonically increasing thus an easy computation gives that

$$\sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{k=2^m+1}^{2^{m+1}} \varphi(|\Delta a_n|}{\eta_{2^m}}\right) \leqslant K \sum_{m=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} \overline{\varphi}\left(\frac{\sum\limits_{k=n}^{\infty} \varphi(|\Delta a_k|)}{\eta_n}\right)$$
$$= K \sum_{n=1}^{\infty} \overline{\varphi}\left(\frac{\sum\limits_{k=n}^{\infty} \varphi(|\Delta a_k|)}{\eta_n}\right) < \infty,$$

and this was to be proved.

Next we prove the statement $F^*_\varphi(\eta) \subseteq F_\varphi(\eta)$. Suppose that $\mathbf{a} \in F^*_\varphi(\eta)$. Since $\eta_n \uparrow$ thus we have that

$$\sum_{n=2}^{\infty} \overline{\varphi}\left(\frac{\sum\limits_{k=n}^{\infty} \varphi(|\Delta a_k|)}{\eta_n}\right) \leqslant K \sum_{m=0}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{k=2^m+1}^{\infty} \varphi(|\Delta a_k|)}{\eta_{2^m}}\right) = I. \tag{4.1}$$

Using Lemma 2 and in the last step Lemma 9 we get

$$I \leqslant K \sum_{m=0}^{\infty} 2^m \sum_{n=m}^{\infty} \overline{\varphi}\left(\frac{\sum\limits_{k=2^n+1}^{2^{n+1}} \varphi(|\Delta a_k|)}{\eta_{2^m}}\right)$$
$$= K \sum_{n=0}^{\infty} \sum_{m=0}^{n} 2^m \overline{\varphi}\left(\frac{1}{\eta_{2^m}} \sum_{k=2^n+1}^{2^{n+1}} \varphi(|\Delta a_k|)\right) \tag{4.2}$$
$$\leqslant K \sum_{n=0}^{\infty} 2^n \overline{\varphi}\left(\frac{\sum\limits_{k=2^n+1}^{2^{n+1}} \varphi(|\Delta a_k|)}{\eta_{2^n}}\right) < \infty.$$

Combining (4.1) and (4.2) proves the statement.

In the following step we prove that $F_\varphi(\eta) \subseteq S_\varphi(\eta)$. Suppose that $\mathbf{a} \in F_\varphi(\eta)$ and set

$$R_n := \overline{\varphi}\left(\frac{\sum\limits_{k=n}^{\infty} \varphi(|\Delta a_k|)}{\eta_n}\right).$$

Since $\{R_n\}$ is monotone decreasing and (by (1.15))

$$\sum_{n=1}^{\infty} R_n < \infty,$$

thus by Lemma 4 we know that there exists a monotone decreasing $\{A_n\}$ such that

$$R_n \leqslant A_n \tag{4.3}$$

$$A_n \leqslant 2A_{2n} \tag{4.4}$$

and

$$\sum_{n=1}^{\infty} A_n < \infty$$

hold.

To prove $\mathbf{a} \in S_\varphi(\eta)$ we have to show that (see (1.17))

$$\sum_{k=1}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} = O(\eta_n). \tag{4.5}$$

Let $2^i \leqslant n < 2^{i+1}$. Then using (4.3), (4.4) and Lemma 1 we get

$$\begin{aligned} \sum_{k=2}^{n} \frac{\varphi(|\Delta a_k|)}{\varphi(A_k)} &= \sum_{k=1}^{n} \frac{\eta_k \varphi(R_k) - \eta_{k+1}\varphi(R_{k+1})}{\varphi(A_k)} \\ &\leqslant \sum_{m=0}^{i} \sum_{k=2^m}^{2^{m+1}-1} [\eta_k \varphi(R_k) - \eta_{k+1}\varphi(R_{k+1})] \frac{1}{\varphi(A_k)} \\ &\leqslant K \sum_{m=0}^{i} \eta_{2^m} \varphi(R_{2^m}) \frac{1}{\varphi(A_{2^{m+1}})} \leqslant K_1 \sum_{m=0}^{i} \eta_{2^m} = \sigma_i. \end{aligned} \tag{4.6}$$

Since σ_i can be estimated by Lemma 7, using that $\{\eta_{2^m}\}$ is quasi geometrically increasing because of Lemma 6, thus we get that

$$\sigma_i \leqslant K_1 \eta_{2^i} \leqslant K\eta_n. \tag{4.7}$$

(4.6) and (4.7) give (4.5), that is the embedding $F_\varphi(\eta) \subseteq S_\varphi(\eta)$ is proved.

The embedding relation

$$S_\varphi(\eta) \subseteq S_\varphi(\delta, \eta)$$

obviously holds without any additional condition, it is enough to take $\delta_n := n^{-3}$.

Now we will prove the relation

$$S_\varphi(\delta, \eta) \subseteq S_\varphi(A, \eta).$$

Since $\mathbf{a} \in S_\varphi(\delta, \eta)$ there exists a δ-quasi monoton sequence $\{A_n\}$ with $\sum_{n=1}^{\infty} n\delta_n < \infty$. Applying Lemma 3 we get that $\sum_{n=1}^{\infty} n|\Delta A_n| < \infty$. At the same time (1.17) is automatically satisfied by $\mathbf{a} \in S_\varphi(\delta, \eta)$. Thus $S_\varphi(\delta, \eta) \subseteq S_\varphi(A, \eta)$ is proved.

Finally we verify

$$S_\varphi(A,\eta) \subseteq F^*_\varphi(\eta).$$

Suppose that $\mathbf{a} \in S_\varphi(A,\eta)$. By using Lemma 5 we can get that

$$\begin{aligned}\sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{n=2^m+1}^{2^{m+1}} \varphi(|\Delta a_n|)}{\eta_{2^m}}\right) &= \sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{n=2^m+1}^{2^{m+1}} \frac{\varphi(|\Delta a_n|)}{\varphi(A_n)}\varphi(A_n)}{\eta_{2^m}}\right) \\ &\leqslant \sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{\sum\limits_{n=2^m+1}^{2^{m+1}} \frac{\varphi(|\Delta a_n|)}{\varphi(A_n)}\varphi(C_m)}{\eta_{2^m}}\right) = I.\end{aligned} \tag{4.8}$$

Since $\mathbf{a} \in S_\varphi(A,\eta)$ we get by (1.17) that

$$I \leqslant \sum_{m=1}^{\infty} 2^m \overline{\varphi}\left(\frac{K\eta_{2^{m+1}}\varphi(C_m)}{\eta_{2^m}}\right) = II. \tag{4.9}$$

Using that η_n is quasi $p(\beta-1)$-power-monotone decreasing, from Lemma 1 and Lemma 5 we have that

$$II \leqslant K_1 \sum_{m=1}^{\infty} 2^m C_m < \infty \tag{4.10}$$

from which it follows that the first sum of (4.8) is also finite, that is $\mathbf{a} \in F^*_\varphi(\eta)$, which gives $S_\varphi(A,\eta) \subseteq F^*_\varphi(\eta)$. Taking into account that $F_\varphi(\eta) \equiv F^*_\varphi(\eta)$, the proof of Theorem is completed.

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