Mathematical Inequalities & Applications
Volume 6, Number 4 (2003), 659–666

ON EXTENSIONS AND REFINEMENTS OF HERMITE–HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

LIANGCHENG WANG

(*communicated by J. Pečarić*)

Abstract. In this paper, we obtain two theorems: Theorem 1 is extensions and infinite refinements of Hermite-Hadamard inequalities; Theorem 2 is extensions from Theorem 1.

1. Introduction

The inequalities

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1}$$

which hold for all convex functions on a closed interval $[a,b]$, it is called Hermite-Hadamard inequalities[1]. In this paper, we establish new extensions and refinements for (1). The result from other inequalities connected with (1) can be seen in [1–7], where further references are given.

2. Main results

THEOREM 1. *Let f be a continuous convex function on $[a,b], 0<t<1, u = ta+(1-t)b$, and let A, B and C be defined by*

$$\begin{aligned}
A &= \frac{1}{t(1-t)(b-a)^2}\int_a^u \left[\int_u^b f\left(tx+(1-t)y\right)dy\right]dx,\\
B &= \frac{1}{(1-t)(b-a)^2}\int_a^u \left[\int_u^b f\left(((b-y)x+(y-u)u)/(t(b-a))\right)dy\right]dx\\
&\quad+\frac{1}{t(b-a)^2}\int_a^u \left[\int_u^b f\left(((u-x)u+(x-a)y)/((1-t)(b-a))\right)dy\right]dx,\\
C &= \frac{t}{(1-t)(b-a)}\int_a^u f(x)dx+\frac{1-t}{t(b-a)}\int_u^b f(x)dx.
\end{aligned}$$

Mathematics subject classification (2000): 26D15.
Key words and phrases: Hermite-Hadamard inequalities, convex function, extension, refinement.

For any positive integer n*, then we have*

$$\begin{aligned} f\,(ta+(1-t)b) &\leqslant A \leqslant B \leqslant \frac{1}{2}\,(f\,(u)+C) \leqslant \frac{1}{2}\,(A+C) \leqslant \frac{1}{2}\,(B+C) \leqslant \dots \\ &\leqslant \frac{1}{2^n}f\,(u)+\frac{2^n-1}{2^n}C \leqslant \frac{1}{2^n}A+\frac{2^n-1}{2^n}C \leqslant \frac{1}{2^n}B+\frac{2^n-1}{2^n}C \\ &\leqslant \frac{1}{2^{n+1}}f\,(u)+\frac{2^{n+1}-1}{2^{n+1}}C \\ &\leqslant \dots \leqslant C \leqslant tf\,(a)+(1-t)f\,(b). \end{aligned} \tag{2}$$

Proof. Let $D=[a,u]\times[u,b]$, then $|D|=t(1-t)(b-a)^2$. According to Theorem 112 of [8], for u, then there is a λ so that

$$f\,(tx+(1-t)y) \geqslant f\,(u)+\lambda\,(tx+(1-t)y-u)\,, x,y\in[a,b]. \tag{3}$$

For $(x,y)\in D$, integrating (3) yields

$$\begin{aligned} &\int_a^u \left[\int_u^b f\,(tx+(1-t)y)\,dy\right]dx \\ &\quad = \iint\limits_D f\,(tx+(1-t)y)dxdy \\ &\quad \geqslant \iint\limits_D [f\,(u)+\lambda\,(tx+(1-t)y-u)]dxdy \\ &\quad = f\,(u)\,|D|+\lambda\left[\iint\limits_D (tx+(1-t)y)\,dxdy-u\,|D|\right]. \end{aligned} \tag{4}$$

Combining (4) with the equally

$$\iint\limits_D (tx+(1-t)y)dxdy = t\int_a^u xdx\int_u^b dy+(1-t)\int_a^u dx\int_u^b ydy = u\,|D|\,,$$

we obtain

$$f\,(ta+(1-t)b)=f\,(u) \leqslant \frac{1}{|D|}\iint\limits_D f\,(tx+(1-t)y)dxdy = A. \tag{5}$$

From the convexity of f and the property of integration, we obtain

$$\begin{aligned} B &= \frac{1}{|D|}\iint\limits_D \left[tf\left(\frac{(b-y)x+(y-u)u}{t(b-a)}\right)+(1-t)f\left(\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}\right)\right]dxdy \\ &\geqslant \frac{1}{|D|}\iint\limits_D f\left(t\cdot\frac{(b-y)x+(y-u)u}{t(b-a)}+(1-t)\cdot\frac{(u-x)u+(x-a)y}{(1-t)(b-a)}\right)dxdy \\ &= \frac{1}{|D|}\iint\limits_D f\,(tx+(1-t)y)dxdy = A, \end{aligned} \tag{6}$$

and

$$\begin{aligned} B &\leqslant \frac{1}{|D|}\iint\limits_{D}\left[t\left(\frac{(b-y)f(x)+(y-u)f(u)}{t(b-a)}\right)\right. \\ &\quad \left. +(1-t)\left(\frac{(u-x)f(u)+(x-a)f(y)}{(1-t)(b-a)}\right)\right]dxdy \\ &= \left\{f(u)\int_a^u\left[\int_u^b (y-x)dy\right]dx+\int_a^u f(x)dx\int_u^b (b-y)dy\right. \\ &\quad \left.+\int_a^u (x-a)dx\int_u^b f(y)dy\right\}\div t(1-t)(b-a)^3 \\ &= \frac{1}{2}\left(f(u)+C\right). \end{aligned} \tag{7}$$

From (5) and (6), we have

$$\frac{1}{2^n}f(u)+\frac{2^n-1}{2^n}C\leqslant\frac{1}{2^n}A+\frac{2^n-1}{2^n}C\leqslant\frac{1}{2^n}B+\frac{2^n-1}{2^n}C, \tag{8}$$

where $n=1,2,\ldots$.

From (7) we have

$$\frac{1}{2^n}B+\frac{2^n-1}{2^n}C\leqslant\frac{1}{2^n}\left(\frac{1}{2}\left(f(u)+C\right)\right)+\frac{2^n-1}{2^n}C=\frac{1}{2^{n+1}}f(u)+\frac{2^{n+1}-1}{2^{n+1}}C, \tag{9}$$

where $n=1,2,\ldots$.

From the convexity of f, we obtain

$$\begin{aligned} A &\leqslant \frac{1}{t(1-t)(b-a)^2}\int_a^u\left[\int_u^b \left(tf(x)+(1-t)f(y)\right)dy\right]dx \\ &= \frac{1}{t(1-t)(b-a)^2}\left[t\int_a^u f(x)dx\int_u^b dy+(1-t)\int_a^u dx\int_u^b f(y)dy\right] \\ &= \frac{t}{(1-t)(b-a)}\int_a^u f(x)dx+\frac{1-t}{t(b-a)}\int_u^b f(x)dx=C. \end{aligned} \tag{10}$$

For any nonnegative integers n, from (5) and (10) we get

$$\frac{1}{2^{n+1}}f(u)+\frac{2^{n+1}-1}{2^{n+1}}C\leqslant\frac{1}{2^{n+1}}C+\frac{2^{n+1}-1}{2^{n+1}}C=C. \tag{11}$$

From the convexity of f and the property of definite integral, we obtain

$$\begin{aligned} \frac{1}{1-t}\int_a^u f(x)dx &= \int_a^b f\left(ta+(1-t)x\right)dx \\ &\leqslant \int_a^b \left[tf(a)+(1-t)f(x)\right]dx \\ &= t(b-a)f(a)+(1-t)\int_a^b f(x)dx, \end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{1}{t}\int_u^b f(x)dx &= \int_a^b f\,(tx+(1-t)b)\,dx \\ &\leqslant \int_a^b [tf(x)+(1-t)f(b)]\,dx \\ &= t\int_a^b f(x)dx+(1-t)(b-a)f(b). \end{aligned} \tag{13}$$

Expression (12) plus (13), with a simple manipulation we get

$$C \leqslant tf(a)+(1-t)f(b). \tag{14}$$

Combination of (5), (6), (7), (8), (9), (11) and (14) yields (2).

REMARK. If we choose $t=1/2$ in Theorem 1, then

$$f\,(ta+(1-t)b) \leqslant C \leqslant tf(a)+(1-t)f(b)$$

reduce to the inequalities (1).

COROLLARY. *Let f be a continuous convex function on $[a,b]$, and let $x_i = a+(b-a)i/k$ $(i=0,1,2,\ldots,k;\ k\geqslant 1)$, $u_1=(a+b)/2$, then for any positive integer n, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\sum_{i=0}^{k} x_i \Big/ (k+1)\right) \leqslant A_1 \leqslant B_1 \leqslant \frac{1}{2}\left(f(u_1)+C_1\right) \\ &\leqslant \frac{1}{2}(A_1+C_1) \leqslant \frac{1}{2}(B_1+C_1) \leqslant \ldots \leqslant \frac{1}{2^n}f(u_1)+\frac{2^n-1}{2^n}C_1 \\ &\leqslant \frac{1}{2^n}A_1+\frac{2^n-1}{2^n}C_1 \leqslant \frac{1}{2^n}B_1+\frac{2^n-1}{2^n}C_1 \\ &\leqslant \frac{1}{2^{n+1}}f(u_1)+\frac{2^{n+1}-1}{2^{n+1}}C_1 \leqslant \ldots \leqslant \frac{1}{b-a}\int_a^b f(x)dx \\ &\leqslant \sum_{i=1}^{k}\frac{f(x_{i-1})+f(x_i)}{2}\Big/ k \\ &\leqslant \sum_{i=0}^{k} f(x_i)\Big/(k+1) \leqslant \frac{f(a)+f(b)}{2}, \end{aligned} \tag{15}$$

where

$$A_1 = \frac{4}{(b-a)^2}\int_a^{u_1}\left[\int_{u_1}^b f\,((x+y)/2)\,dy\right]dx,$$

$$B_1 = \frac{2}{(b-a)^2}\int_a^{u_1}\left[\int_{u_1}^b f\,(2\,((b-y)x+(y-u_1)u_1)/(b-a))\,dy\right]dx$$

$$
+\frac{2}{(b-a)^2}\int_a^{u_1}\left[\int_{u_1}^b f\left(2\left((u_1-x)u_1+(x-a)y\right)/(b-a)\right)dy\right]dx,
$$

$$
C_1=\frac{1}{b-a}\int_a^b f(x)dx.
$$

Proof. For $0<h\leqslant u_1$, define two functions G and H, by

$$
G(x)=((f(b)-f(a))(x-a))/(b-a)+f(a),
$$
$$
H(x)=((f(u_1+h)-f(u_1-h))(x-u_1+h))/(2h)+f(u_1-h).
$$

From the convexity of f we obtain

$$
f(a)+f(b)=2G(u_1)\geqslant 2H(u_1)=f(u_1+h)+f(u_1-h). \tag{16}
$$

By (16) and (1), we obtain

$$
\begin{aligned}
\frac{1}{k+1}\sum_{i=0}^{k}f(x_i)-\frac{1}{k}\sum_{i=1}^{k}\frac{f(x_{i-1})+f(x_i)}{2}
&=\frac{1}{k(k+1)}\left[(k-1)\frac{f(x_0)+f(x_k)}{2}-\sum_{i=1}^{k-1}f(x_i)\right]\\
&=\frac{1}{k(k+1)}\sum_{r=1}^{\frac{k-1}{2}}[f(a)+f(b)-(f(x_r)+f(x_{k-r}))]\geqslant 0
\end{aligned}
$$

(k is an odd number), or

$$
=\frac{1}{k(k+1)}\left\{\sum_{r=1}^{\frac{k-2}{2}}[f(a)+f(b)-(f(x_r)+f(x_{k-r}))]+\frac{f(a)+f(b)}{2}-f(u_1)\right\}\geqslant 0 \tag{17}
$$

(k is an even number).

The second inequality of the right in (15) is proved.

By (17), we obtain

$$
\frac{f(a)+f(b)}{2}-\frac{1}{k}\sum_{i=1}^{k}\frac{f(x_{i-1})+f(x_i)}{2}=\frac{1}{2k}\left[(k-1)\frac{f(a)+f(b)}{2}-\sum_{i=1}^{k-1}f(x_i)\right]\geqslant 0.
$$

The first inequality of the right in (15) is proved.

By (1), we obtain

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx&=\frac{1}{b-a}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}f(x)dx\leqslant\frac{x_i-x_{i-1}}{b-a}\sum_{i=1}^{k}\frac{f(x_{i-1})+f(x_i)}{2}\\
&=\frac{1}{k}\sum_{i=1}^{k}\frac{f(x_{i-1})+f(x_i)}{2}.
\end{aligned}
$$

The third inequality of the right in (15) is also proved.

Since $\sum_{i=0}^{k} x_i \Big/ (k+1) = (a+b)/2 = u_1$, then choosing $t = 1/2$ in Theorem 1, we obtain the rest in (15).

REMARK. The (15) is refinements of (1).

THEOREM 2. *Let f be a continuous convex function on $[a,b]$, and let any $x_i \in [a,b]$, $t_i > 0$ $(i = 1,2,\ldots,k;\ k \geqslant 2)$, $T_j = \sum_{i=1}^{j} t_i$ $(j = 1,2,\ldots,k)$, $u_0 = \sum_{i=1}^{k} t_i x_i \Big/ T_k$. If $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k$ and a natural number m $(1 \leqslant m \leqslant k-1)$ exists so that $x_m < x_{m+1}$, then there is $[a_0,b_0] \subseteq [a,b]$ $(a_0 < b_0)$ and for any positive integer n, the following inequalities*

$$
\begin{aligned}
f\left(\sum_{i=1}^{k} t_i x_i \Big/ T_k\right) &\leqslant A_0 \leqslant B_0 \leqslant \frac{1}{2}\left(f(u_0) + C_0\right) \leqslant \frac{1}{2}\left(A_0 + C_0\right) \leqslant \frac{1}{2}\left(B_0 + C_0\right) \\
&\leqslant \ldots \leqslant \frac{1}{2^n} f(u_0) + \frac{2^n - 1}{2^n} C_0 \leqslant \frac{1}{2^n} A_0 + \frac{2^n - 1}{2^n} C_0 \\
&\leqslant \frac{1}{2^n} B_0 + \frac{2^n - 1}{2^n} C_0 \leqslant \frac{1}{2^{n+1}} f(u_0) + \frac{2^{n+1} - 1}{2^{n+1}} C_0 \\
&\leqslant \ldots \leqslant C_0 \leqslant \sum_{i=1}^{k} t_i f(x_i) \Big/ T_k \qquad (18)
\end{aligned}
$$

hold, where

$$
A_0 = \frac{T_k^2}{(T_k - T_m) T_m (b_0 - a_0)^2} \int_{a_0}^{u_0} \left[\int_{u_0}^{b_0} f\left((T_m x + (T_k - T_m) y)/T_k \right) dy \right] dx,
$$

$$
\begin{aligned}
B_0 = {} & \frac{T_k}{(T_k - T_m)(b_0 - a_0)^2} \int_{a_0}^{u_0} \left[\int_{u_0}^{b_0} f\left(T_k \left((b_0 - y) x + (y - u_0) u_0 \right) / (T_m (b_0 - a_0)) \right) dy \right] dx \\
& + \frac{T_k}{T_m (b_0 - a_0)^2} \int_{a_0}^{u_0} \left[\int_{u_0}^{b_0} f\left(T_k \left((u_0 - x) u_0 + (x - a_0) y \right) / ((T_k - T_m)(b_0 - a_0)) \right) dy \right] dx,
\end{aligned}
$$

$$
C_0 = \frac{T_m}{(T_k - T_m)(b_0 - a_0)} \int_{a_0}^{u_0} f(x) dx + \frac{T_k - T_m}{T_m (b_0 - a_0)} \int_{u_0}^{b_0} f(x) dx.
$$

Proof. We define a_0 and b_0 by

$$
a_0 \overset{\text{def}}{=} \sum_{i=1}^{m} t_i x_i \Big/ T_m \leqslant x_m < x_{m+1} \leqslant \sum_{i=m+1}^{k} t_i x_i \Big/ (T_k - T_m) \overset{\text{def}}{=} b_0,
$$

then $[a_0, b_0] \subseteq [a,b]$ and $a_0 < b_0$. Let $t = T_m/T_k$, then $0 < t < 1$ and

$$
t a_0 + (1-t) b_0 = u_0.
$$

Thus applying to (2), we obtain

$$
\begin{aligned}
f\left(\sum_{i=1}^{k} t_i x_i \Big/ T_k\right) &= f\left(ta_0+(1-t)b_0\right) \leqslant A_0 \leqslant B_0 \leqslant \frac{1}{2}\left(f\left(u_0\right)+C_0\right) \\
&\leqslant \frac{1}{2}\left(A_0+C_0\right) \leqslant \frac{1}{2}\left(B_0+C_0\right) \leqslant \ldots \leqslant \frac{1}{2^n} f\left(u_0\right)+\frac{2^n-1}{2^n} C_0 \\
&\leqslant \frac{1}{2^n} A_0+\frac{2^n-1}{2^n} C_0 \leqslant \frac{1}{2^n} B_0+\frac{2^n-1}{2^n} C_0 \\
&\leqslant \frac{1}{2^{n+1}} f\left(u_0\right)+\frac{2^{n+1}-1}{2^{n+1}} C_0 \leqslant \ldots \leqslant C_0 \\
&\leqslant \frac{T_m}{T_k} f\left(a_0\right)+\frac{T_k-T_m}{T_k} f\left(b_0\right). \qquad (19)
\end{aligned}
$$

Combining (19) with Jensen inequalities of convex function [8, 9]

$$
f\left(a_0\right)=f\left(\sum_{i=1}^{m} t_i x_i \Big/ T_m\right) \leqslant \sum_{i=1}^{m} t_i f\left(x_i\right) \Big/ T_m
$$

and

$$
f\left(b_0\right)=f\left(\sum_{i=m+1}^{k} t_i x_i \Big/ \left(T_k-T_m\right)\right) \leqslant \sum_{i=m+1}^{k} t_i f\left(x_i\right) \Big/ \left(T_k-T_m\right)
$$

we get (18).

REMARK. If we choose $k=2$ in Theorem 2, then Theorem 2 reduce to Theorem 1.

REFERENCES

[1] G. S. YANG AND K. L. TSENG, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl. **239** (1999), 180–187.

[2] C. L. WANG AND X. H. WANG, *On an extention of Hadamard inequalities for convex functions*, Chin. Ann. of Math. **3** No. 5 (1982), 567–569.

[3] C. H. FENG, *On Hadamard inequalities for convex functions*, Chin. Ann. of Math. **6A** No. 4 (1985), 443–446. (Chinese)

[4] J. L. BRENNER AND H. ALZER, *Integral inequalities for concave functions with applications to special functions*, Proc. Roy. Soc. Edinburgh Sect. *A* **118** (1991), 137–192.

[5] S. S. DRAGOMIR, *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl. **167** (1992), 49–56.

[6] G. S. YANG AND M. C. HONG, *A note on Hadamard's inequality*, Tamkang J. Math. **28**, No. 1 (1997), 33–37.

[7] G. S. YANG AND K. L. TSENG, *Inequalities of Hadamard's type for Lipschitzian mappings*, J. Math. Anal. Appl. **260** (2001), 230–238.

[8] G. H. HARDY, J. E. LITTLEWOOD AND G. PÒLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

[9] L. C. WANG, *Convex functions and their inequalities*, Sichuan Univ. Press, Sichuan, China, 2001. (Chinese)

(Received June 1, 2002)

Liangcheng Wang
Department of Mathematics
Daxian Teachers' College
Dazhou Sichuan 635000
The People's Republic of China
e-mail: wangliangcheng@163.com