

## ON WEIGHTED EXTENSIONS OF CARLEMAN'S INEQUALITY AND HARDY'S INEQUALITY

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*Abstract.* Recently K. S. Kedlaya gives a weighted extension of Carleman's inequality under a certain weight condition. However this condition is not essential. In fact we show that the Kedlaya type extension holds without the weight condition by considering a certain weighted extension of Hardy's inequality. Moreover we shall consider the other weighted extension of Carleman's inequality.

### 1. Introduction

T. Carleman presented the following inequality in [2]: for a sequence  $\{x_n\}$  of positive real numbers with  $\sum_{n=1}^{\infty} x_n < +\infty$ ,

$$\sum_{n=1}^{\infty} (x_1 x_2 \dots x_n)^{1/n} < e \sum_{n=1}^{\infty} x_n.$$

Recently, K. S. Kedlaya proved the following weighted extension in [4]:

**THEOREM (Kedlaya).** *Let  $\{\alpha_n\}$  be a sequence of positive real numbers satisfying*

$$\frac{\alpha_1}{\alpha_1} \geq \frac{\alpha_2}{\alpha_1 + \alpha_2} \geq \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \geq \dots \quad (*)$$

*If a sequence  $\{x_n\}$  of positive real numbers satisfies  $\sum_{n=1}^{\infty} \alpha_n x_n < +\infty$ , then*

$$\sum_{n=1}^{\infty} \alpha_n x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} < e \sum_{n=1}^{\infty} \alpha_n x_n.$$

Actually he proves that a certain weighted mixed mean inequality holds under the condition (\*) and shows that the above extension is deduced from this. However we

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note that the condition (\*) is not essential for proving the Kedlaya type extension of Carleman's inequality.

In Section 2 we first investigate a certain weighted extension of Hardy's inequality. As an application of our extension we show that the Kedlaya type extension holds without the condition (\*). In Section 3, we shall consider the other weighted extension of Carleman's inequality using H. Alzer's technique observed in [1].

## 2. A weighted extension of Hardy's inequality

According to the book [3], Carleman's inequality can be obtained from Hardy's inequality. In order to prove the Kedlaya type extension we shall consider a weighted extension of Hardy's inequality.

**THEOREM 2.1.** *Let  $\{\alpha_n\}$  be a sequence of positive real numbers. If  $p > 1$  and a sequence  $\{x_n\}$  of positive real numbers satisfies  $\sum_{n=1}^{\infty} \alpha_n x_n^p < +\infty$ , then for any  $1 \leq N \leq \infty$*

$$\sum_{n=1}^N \alpha_n \left( \frac{\alpha_1 x_1 + \dots + \alpha_n x_n}{\alpha_1 + \dots + \alpha_n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^N \alpha_n x_n^p. \quad (1)$$

*Proof.* We first consider the case  $N < \infty$ . Let  $q$  be the positive real number with  $1/p + 1/q = 1$ . Consider

$$s_n = \alpha_1 + \dots + \alpha_n, \quad A_n = \frac{\alpha_1 x_1 + \dots + \alpha_n x_n}{s_n},$$

then for any  $n \geq 2$

$$\begin{aligned} A_n^p - \frac{p}{p-1} A_n^{p-1} x_n &= A_n^p - \frac{p}{\alpha_n(p-1)} A_n^{p-1} \alpha_n x_n \\ &= A_n^p - \frac{p}{\alpha_n(p-1)} A_n^{p-1} \{s_n A_n - s_{n-1} A_{n-1}\} \\ &= A_n^p \left\{ 1 - \frac{p s_n}{\alpha_n(p-1)} \right\} + \frac{s_{n-1}}{\alpha_n(p-1)} p A_n^{p-1} A_{n-1}. \end{aligned}$$

By Young's inequality we have

$$A_n^{p-1} A_{n-1} \leq \frac{1}{q} A_n^{q(p-1)} + \frac{1}{p} A_{n-1}^p = \frac{1}{q} A_n^p + \frac{1}{p} A_{n-1}^p,$$

and so

$$\begin{aligned} A_n^p - \frac{p}{p-1} A_n^{p-1} x_n &\leq A_n^p \left\{ 1 - \frac{p s_n}{\alpha_n(p-1)} \right\} + \frac{s_{n-1}}{\alpha_n(p-1)} \{(p-1)A_n^p + A_{n-1}^p\} \\ &= \frac{1}{\alpha_n(p-1)} A_n^p \{\alpha_n(p-1) - p s_n + s_{n-1}(p-1)\} + \frac{s_{n-1}}{\alpha_n(p-1)} A_{n-1}^p \\ &= \frac{1}{\alpha_n(p-1)} (s_{n-1} A_{n-1}^p - s_n A_n^p). \end{aligned}$$

Hence

$$\alpha_n A_n^p - \frac{p}{p-1} A_n^{p-1} \alpha_n x_n \leq \frac{1}{p-1} (s_{n-1} A_{n-1}^p - s_n A_n^p),$$

which implies

$$\begin{aligned} \sum_{n=1}^N \alpha_n A_n^p - \frac{p}{p-1} \sum_{n=1}^N A_n^{p-1} \alpha_n x_n &\leq \alpha_1 A_1^p - \frac{p}{p-1} A_1^{p-1} \alpha_1 x_1 + \frac{1}{p-1} \sum_{n=2}^N (s_{n-1} A_{n-1}^p - s_n A_n^p) \\ &= -\frac{1}{p-1} s_N A_N^p < 0. \end{aligned}$$

According to Hölder's inequality

$$\begin{aligned} \sum_{n=1}^N \alpha_n A_n^p &< \frac{p}{p-1} \sum_{n=1}^N A_n^{p-1} \alpha_n x_n \\ &= \frac{p}{p-1} \sum_{n=1}^N \alpha_n^{1/q} A_n^{p-1} \cdot \alpha_n^{1/p} x_n \\ &\leq \frac{p}{p-1} \left( \sum_{n=1}^N \alpha_n A_n^p \right)^{1/q} \left( \sum_{n=1}^N \alpha_n x_n^p \right)^{1/p}. \end{aligned} \tag{2}$$

Dividing both sides by  $\left(\sum_{n=1}^N \alpha_n A_n^p\right)^{1/q}$  and raising the result to the  $p$ -th power, we obtain the inequality (1) in the case  $N < \infty$ .

In order to show the case  $N = \infty$ , make  $N \rightarrow \infty$ , we can obtain

$$\sum_{n=1}^{\infty} \alpha_n \left( \frac{\alpha_1 x_1 + \dots + \alpha_n x_n}{\alpha_1 + \dots + \alpha_n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \alpha_n x_n^p. \tag{3}$$

So it will be sufficient to show that the equality does not occur. Make  $N \rightarrow \infty$  in (2), we have

$$\sum_{n=1}^{\infty} \alpha_n A_n^p \leq \frac{p}{p-1} \sum_{n=1}^{\infty} \alpha_n^{1/q} A_n^{p-1} \cdot \alpha_n^{1/p} x_n \leq \frac{p}{p-1} \left( \sum_{n=1}^{\infty} \alpha_n A_n^p \right)^{1/q} \left( \sum_{n=1}^{\infty} \alpha_n x_n^p \right)^{1/p}.$$

Assume that the equality occurs in (3), then the second term is equal to the third one. According to the equality condition for Hölder's inequality,  $\{\alpha_n A_n^p\}$  and  $\{\alpha_n x_n^p\}$  are proportional, i.e. there exists a constant  $c$  such that  $\alpha_n A_n^p = c \cdot \alpha_n x_n^p$  ( $n = 1, 2, \dots$ ). Hence  $x_1 = x_2 = x_3 = \dots$ , and so  $1 = \{p/(p-1)\}^p$ . This is a contradiction.  $\square$

Using Theorem 2.1 we can obtain a weighted extension of Carleman's inequality.

**THEOREM 2.2.** *Let  $\{\alpha_n\}$  be a sequence of positive real numbers. If a sequence  $\{x_n\}$  of positive real numbers satisfies  $\sum_{n=1}^{\infty} \alpha_n x_n < +\infty$ , then for any  $1 \leq N \leq \infty$*

$$\sum_{n=1}^N \alpha_n x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \leq e \sum_{n=1}^N \alpha_n x_n. \tag{4}$$

*Proof.* We first consider the case  $N < \infty$ . Let  $p > 1$  and replace  $x_n$  by  $x_n^{1/p}$  in the inequality (1), we can obtain

$$\sum_{n=1}^N \alpha_n \left( \frac{\alpha_1 x_1^{1/p} + \dots + \alpha_n x_n^{1/p}}{\alpha_1 + \dots + \alpha_n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^N \alpha_n x_n. \tag{5}$$

Since

$$\begin{aligned} \lim_{p \rightarrow \infty} \left( \frac{\alpha_1 x_1^{1/p} + \dots + \alpha_n x_n^{1/p}}{\alpha_1 + \dots + \alpha_n} \right)^p &= \lim_{t \rightarrow +0} \left( \frac{\alpha_1 x_1^t + \dots + \alpha_n x_n^t}{\alpha_1 + \dots + \alpha_n} \right)^{1/t} \\ &= x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \end{aligned}$$

and  $\lim_{p \rightarrow \infty} \{p/(p-1)\}^p = e$ , making  $p \rightarrow \infty$  in (5),

$$\sum_{n=1}^N \alpha_n x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \leq e \sum_{n=1}^N \alpha_n x_n$$

is valid. Moreover, making  $N \rightarrow \infty$ , we can obtain the case  $N = \infty$ .  $\square$

The inequality (4) is the Kedlaya type extension of Carleman’s inequality. However we don’t need any condition for a weight  $\{\alpha_n\}$ .

### 3. Another type weighted extension of Carleman’s inequality

In this section we consider the other weighted extension of Carleman’s inequality. The first result is a weighted extension of finite type.

**THEOREM 3.1.** *Let  $\{\alpha_n\}$  and  $\{x_n\}$  be sequences of positive real numbers. For any  $1 \leq N < \infty$ ,*

$$\lambda_N \mu_N \sum_{n=1}^N x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} < \sum_{n=1}^N x_n$$

is valid, where

$$\begin{aligned} \lambda_N &= \min\{\alpha_2, \dots, \alpha_{N+1}\}, \\ \mu_N &= \min_{1 \leq n \leq N} \beta_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \beta_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)}, \\ \beta_k &= \frac{1}{\alpha_k} \left( \frac{\alpha_1 + \dots + \alpha_k}{\alpha_1 + \dots + \alpha_{k+1}} \right)^{(\alpha_1+\dots+\alpha_k)/\alpha_k}. \end{aligned}$$

*Proof.* We shall apply H. Alzer's technique observed in [1]. Consider the weighted arithmetic mean and geometric mean inequality:

$$(y_1^{\alpha_1} \dots y_n^{\alpha_n})^{1/(\alpha_1+\dots+\alpha_n)} \leq \frac{\alpha_1 y_1 + \dots + \alpha_n y_n}{\alpha_1 + \dots + \alpha_n},$$

and replace  $y_k = (\alpha_1 + \dots + \alpha_k)x_k/\alpha_k$  for each  $1 \leq k \leq n$ . Moreover divide both sides of the obtained inequality by  $\alpha_1 + \dots + \alpha_{n+1}$  and sum from  $n = 1$  to  $n = N$ . Let  $L_N$  and  $R_N$  denote the left side and right side of resulting inequality, respectively. Then

$$\begin{aligned} R_N &= \sum_{n=1}^N \frac{\alpha_1 x_1 + (\alpha_1 + \alpha_2)x_2 + \dots + (\alpha_1 + \dots + \alpha_n)x_n}{(\alpha_1 + \dots + \alpha_n)(\alpha_1 + \dots + \alpha_{n+1})} \\ &= x_1 \sum_{n=1}^N \frac{\alpha_1}{(\alpha_1 + \dots + \alpha_n)(\alpha_1 + \dots + \alpha_{n+1})} \\ &\quad + x_2 \sum_{n=2}^N \frac{\alpha_1 + \alpha_2}{(\alpha_1 + \dots + \alpha_n)(\alpha_1 + \dots + \alpha_{n+1})} \\ &\quad + \dots + x_N \frac{\alpha_1 + \dots + \alpha_N}{(\alpha_1 + \dots + \alpha_N)(\alpha_1 + \dots + \alpha_{N+1})} \\ &\leq \frac{\alpha_1}{\lambda_N} x_1 \left\{ \frac{1}{\alpha_1} - \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \right\} \\ &\quad + \frac{\alpha_1 + \alpha_2}{\lambda_N} x_2 \left\{ \frac{1}{\alpha_1 + \alpha_2} - \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \right\} \\ &\quad + \dots + \frac{\alpha_1 + \dots + \alpha_N}{\lambda_N} x_N \left\{ \frac{1}{\alpha_1 + \dots + \alpha_N} - \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \right\} \\ &= \frac{1}{\lambda_N} \left\{ \sum_{n=1}^N x_n - \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \sum_{n=1}^N (\alpha_1 + \dots + \alpha_n)x_n \right\}, \end{aligned}$$

removing the term  $(\alpha_1 + \dots + \alpha_{N+1})^{-1} \sum_{n=1}^N (\alpha_1 + \dots + \alpha_n)x_n$ , we can obtain  $\lambda_N R_N <$

$\sum_{n=1}^N x_n$ . And put

$$v_N = \min_{1 \leq n \leq N} \frac{\left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\dots+\alpha_n)} \dots \left(\frac{\alpha_1+\dots+\alpha_n}{\alpha_n}\right)^{\alpha_n/(\alpha_1+\dots+\alpha_n)}}{\alpha_1 + \dots + \alpha_{n+1}},$$

then

$$\begin{aligned} L_N &= \sum_{n=1}^N \frac{\left(\frac{\alpha_1}{\alpha_1} x_1\right)^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \left(\frac{\alpha_1+\dots+\alpha_n}{\alpha_n} x_n\right)^{\alpha_n/(\alpha_1+\dots+\alpha_n)}}{\alpha_1 + \dots + \alpha_{n+1}} \\ &\geq v_N \sum_{n=1}^N x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)}. \end{aligned}$$

Hence we can obtain

$$\lambda_N v_N \sum_{n=1}^N x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} < \sum_{n=1}^N x_n,$$

so it only remains to show  $\mu_N = v_N$ . But it will be sufficient to show that

$$\begin{aligned} & \beta_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \beta_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \\ &= \frac{\left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\dots+\alpha_n)} \dots \left(\frac{\alpha_1+\dots+\alpha_n}{\alpha_n}\right)^{\alpha_n/(\alpha_1+\dots+\alpha_n)}}{\alpha_1 + \dots + \alpha_{n+1}} \end{aligned} \tag{6}$$

for each  $n$ . In fact,

$$\begin{aligned} \beta_1^{\alpha_1} \dots \beta_n^{\alpha_n} &= \left(\frac{1}{\alpha_1}\right)^{\alpha_1} \left(\frac{1}{\alpha_2}\right)^{\alpha_2} \dots \left(\frac{1}{\alpha_n}\right)^{\alpha_n} \\ &\quad \cdot \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^{\alpha_1} \left(\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}\right)^{\alpha_1+\alpha_2} \dots \left(\frac{\alpha_1 + \dots + \alpha_n}{\alpha_1 + \dots + \alpha_{n+1}}\right)^{\alpha_1+\dots+\alpha_n} \\ &= \left(\frac{1}{\alpha_1}\right)^{\alpha_1} \left(\frac{1}{\alpha_2}\right)^{\alpha_2} \dots \left(\frac{1}{\alpha_n}\right)^{\alpha_n} \\ &\quad \cdot \alpha_1^{\alpha_1} (\alpha_1 + \alpha_2)^{\alpha_2} \dots (\alpha_1 + \dots + \alpha_n)^{\alpha_n} \frac{1}{(\alpha_1 + \dots + \alpha_{n+1})^{\alpha_1+\dots+\alpha_n}} \\ &= \left(\frac{\alpha_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{\alpha_1 + \alpha_2}{\alpha_2}\right)^{\alpha_2} \dots \left(\frac{\alpha_1 + \dots + \alpha_n}{\alpha_n}\right)^{\alpha_n} \frac{1}{(\alpha_1 + \dots + \alpha_{n+1})^{\alpha_1+\dots+\alpha_n}}, \end{aligned}$$

raising to the power  $1/(\alpha_1 + \dots + \alpha_n)$ , (6) is valid.  $\square$

In the proof of Theorem 3.1 we remove the term  $(\alpha_1 + \dots + \alpha_{N+1})^{-1} \sum_{n=1}^N (\alpha_1 + \dots + \alpha_n)x_n$ . If  $\sum_{n=1}^\infty \alpha_n = +\infty$  and  $\sum_{n=1}^\infty x_n < +\infty$ , then this term is sufficiently small, that is,

$$\lim_{N \rightarrow \infty} \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \sum_{n=1}^N (\alpha_1 + \dots + \alpha_n)x_n = 0.$$

To see this, let  $\varepsilon$  be an arbitrary positive number. Since  $\sum_{n=1}^\infty x_n < +\infty$ , there exists a number  $N_0$  such that  $\sum_{n=N_0+1}^\infty x_n < \varepsilon$ . For any  $N \geq N_0$ ,

$$\begin{aligned} & \frac{1}{\alpha_1 + \dots + \alpha_{N+1}} \sum_{n=1}^N (\alpha_1 + \dots + \alpha_n)x_n \\ & \leq \frac{(\max_n x_n) \sum_{n=1}^{N_0} (\alpha_1 + \dots + \alpha_n)}{\alpha_1 + \dots + \alpha_{N+1}} + \sum_{n=N_0+1}^N \frac{\alpha_1 + \dots + \alpha_n}{\alpha_1 + \dots + \alpha_{N+1}} x_n \end{aligned}$$

$$\leq \frac{(\max_n x_n) \sum_{n=1}^{N_0} (\alpha_1 + \dots + \alpha_n)}{\alpha_1 + \dots + \alpha_{N+1}} + \varepsilon.$$

By taking the upper limit with respect to  $N$  and letting  $\varepsilon \downarrow 0$ , we obtain the desired equality.

Next we shall consider the infinite case. By Theorem 3.1, put  $\gamma = \inf_N \lambda_N \mu_N$ , then we have

$$\gamma \sum_{n=1}^{\infty} x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \leq \sum_{n=1}^{\infty} x_n, \tag{7}$$

whenever  $\sum_{n=1}^{\infty} x_n < +\infty$ . When  $\inf_n \alpha_n = 0$ ,  $\gamma = 0$  and the inequality (7) is trivial. If  $\lim_{n \rightarrow \infty} \alpha_n = +\infty$ , then

$$\begin{aligned} \beta_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \beta_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} &\leq \left(\frac{1}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \left(\frac{1}{\alpha_n}\right)^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \\ &\leq \frac{\alpha_1}{\alpha_1 + \dots + \alpha_n} \cdot \frac{1}{\alpha_1} + \dots + \frac{\alpha_n}{\alpha_1 + \dots + \alpha_n} \cdot \frac{1}{\alpha_n} \\ &= \frac{n}{\alpha_1 + \dots + \alpha_n} \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

which implies  $\inf_n \beta_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \beta_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} = 0$ , and so  $\gamma = 0$ . Consequently, when  $\lim_{n \rightarrow \infty} \alpha_n = +\infty$ , the inequality (7) is also trivial. We shall consider the case when  $\{\alpha_n\}$  is bounded.

**THEOREM 3.2.** *Let  $\{\alpha_n\}$  be a sequence of positive real numbers satisfying  $0 < m \leq \alpha_n \leq M < +\infty$  ( $n = 1, 2, \dots$ ). If a sequence  $\{x_n\}$  of positive real numbers satisfies  $\sum_{n=1}^{\infty} x_n < +\infty$  then*

$$\frac{m}{M} \left(\frac{1}{e}\right)^{M/m} \sum_{n=1}^{\infty} x_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots x_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} \leq \sum_{n=1}^{\infty} x_n. \tag{8}$$

*Proof.* Put  $t_k = (\alpha_1 + \dots + \alpha_k)/\alpha_{k+1}$ , then for any  $k \in \mathbb{N}$

$$\begin{aligned} \beta_k &= \frac{1}{\alpha_k} \left(\frac{\alpha_1 + \dots + \alpha_k}{\alpha_1 + \dots + \alpha_{k+1}}\right)^{(\alpha_1+\dots+\alpha_k)/\alpha_k} \\ &\geq \frac{1}{M} \left(\frac{1}{1 + 1/t_k}\right)^{\alpha_{k+1}t_k/\alpha_k} \\ &> \frac{1}{M} \left(\frac{1}{e}\right)^{M/m}. \end{aligned}$$

Since  $\lambda_N = \inf_{2 \leq n \leq N+1} \alpha_n \geq m$  and

$$\mu_N = \inf_{1 \leq n \leq N} \beta_1^{\alpha_1/(\alpha_1+\dots+\alpha_n)} \dots \beta_n^{\alpha_n/(\alpha_1+\dots+\alpha_n)} > \frac{1}{M} \left(\frac{1}{e}\right)^{M/m},$$

the assertion is proved by Theorem 3.1.  $\square$

The inequality (8) reduces to Carleman's inequality in case of  $m = M$ .

#### REFERENCES

- [1] H. ALZER, *A refinement of Carleman's inequality*, J. Approx. Theory **95** (1998), 497–499.
- [2] T. CARLEMAN, *Sur les fonctions quasi-analytiques*, Conférences faites au cinquième congrès des mathématiciens scandinaves (Helsingfors, 1923), pp. 181–196.
- [3] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities, second edition*, Cambridge University Press, Cambridge, 1951.
- [4] K. S. KEDLAYA, *A weighted mixed-mean inequality*, Amer. Math. Monthly **106** (1999), 355–358.

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